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CONFORMALLY INVARIANT GRAVITATIONAL WAVES IN A ZELDOVICH FLUID DISTRIBUTION

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Abstract: Interest in modified theories of gravity is attracting many due to numerous applications in different branches of physics. Among the possible alternatives to the Einstein theory of gravity especially $f(R)$ theories of gravity have received immense importance owing to its applicability in providing interesting solutions to cosmological and astrophysical problems. We have obtained a wave like solution of field equations of an $f(R)$ theory of gravity motivated by gravitational waves to obtain them conformally invariant in a Zeldovich fluid distribution. Some important physical features of the solution is then discussed.

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1. Introduction

The first purely geometric theory of gravity which incorporates one of the revolutionary conceptions of modern science, the idea that gravity is geometry of four dimensional curved spacetime, is the general theory of relativity. This rel-

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ativistic theory of gravity predicts possibility of existence of gravitational waves which are meaningfully comparable with electromagnetic waves except for their non-conformal invariance. This is because the gravitational wave equations in the background of Friedman universe can be reduced to [1, 6]

$$\mu'' + \mu\left(n^2 - \frac{a''}{a}\right) = 0, \tag{1}$$

where $a(\eta)$ is the scale factor of Friedman universe and η is the wave number such that the physical wave length is given by $\left(\frac{2\pi a}{n}\right)$. The effective potential $\frac{a''}{a}$ in (1) where prime denotes derivatives with respect to η and is related to cosmic time t by $cdt = a(\eta)\, d\eta$ distinguishes this equation from the ordinary wave equations in the Minkowski world. The fact that $\frac{a''}{a}$ is non-zero except for $a = const$, and $a = a_0\eta$ is a manifestation of the so-called conformal non-invariance of gravitational wave equations.

Hence gravitational waves are an inevitable consequence of Einstein theory of gravitation derivable from Hilbert Lagrangian. It is, therefore, necessary to change Hilbert Lagrangian to modify Einstein field equations to obtain conformally invariant gravitational waves equations.

Over the years many alternative theories have been developed. For example, Weyl [3] suggested the invariant R^2 to make the field action scale invariant (and to unify gravitation with electromagnetism). Another attractive alternative suggested is $R^{\frac{3}{2}}$ so that the coupling constant in matter Lagrangian is dimensionless.

Because of non-conformal invariance of gravitational wave equations, to nullify the manifestation of gravitation as evident from equation (1) without any special choice of scale factor, Pandey [4] gave an $f(R)$ theory of gravity considering Lagrangian in the form [5, 6]

$$L = R + \sum_{n=2}^{N} C_n \frac{\left(l^2 R\right)^n}{6l^2} \text{ or, equivalently, } L = R + \sum_{n=2}^{N} a_n R_n, \tag{2}$$

where l is the characteristic length and C_n are the dimensionless arbitrary coefficients corresponding to the values of n. They are introduced to nullify the manifestation of gravitation. The values $n = 0$ and $n = 1$ result in Hilbert Lagrangian, that is, Einstein theory. Therefore n begins from $n = 2$ onwards.

By applying variational principle to this action, Pandey [7] obtained the following field equations:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \sum_{n=2}^{N} na_n R^{n-1}\left[R_{\mu\nu} - \frac{Rg_{\mu\nu}}{2n} - \frac{n(n-1)}{R}\left(R_{;\mu;\nu} - g_{\mu\nu}\Box R\right)\right.$$
$$\left. - \left\{\frac{(n-1)(n-2)}{R^2}\right\}\left(R_{;\mu}R_{;\nu} - g_{\mu\nu}R_{;\alpha}R^{;\alpha}\right)\right] = kT_{\mu\nu}. \quad (3)$$

Here $T_{\mu\nu} = (-g)^{\frac{1}{2}}\left(\frac{\delta L_s}{\delta g^{\mu\nu}}\right)$ stands for the energy momentum tensor responsible for the production of the gravitational potential $g_{\mu\nu}$. It can be seen that $T^{\nu}_{\mu;\nu} = 0$ holds for these field equations as it is in case of Einstein general relativity. The gravitational field surrounding spherically symmetric mass [7], Gödel Type Solution [8], have already been studied in this $f R)$ theory of gravity.

2. Zeldovich Fluid

Zeldovich fluid can be regarded as perfect fluid having the energy momentum tensor

$$T_{\mu\nu} = (p+\rho)V_\mu V_\nu - pg_{\mu\nu}, \quad (4)$$

characterized by equation of state

$$\rho = p, \quad (5)$$

where ρ is the density and p is the pressure of the distribution. The astrophysical importance of this equation of state is that it describes several important cases, e.g. radiation, relativistic degenerate Fermi gas and possibly very dense matter (Zeldovich [10], Walecka [11]). The casual limit for ideal gas has also the form $p = \rho$.

Higher Order field equation (3) given by Pandey [7] containing the energy momentum tensor $T_{\mu\nu}$ defined by (4) are given by

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \sum_{n=2}^{N} na_n R^{n-1}\left[R_{\mu\nu} - \frac{Rg_{\mu\nu}}{2n} - \frac{n(n-1)}{R}\left(R_{;\mu;\nu} - g_{\mu\nu}\Box R\right)\right.$$
$$\left. - \left\{\frac{(n-1)(n-2)}{R^2}\right\}\left(R_{;\mu}R_{;\nu} - g_{\mu\nu}R_{;\alpha}R^{;\alpha}\right)\right] = kT_{\mu\nu}. \quad (6)$$

where $R_{\mu\nu}, R$ is the Ricci tensor and Ricci scalar of the spacetime metric, viz.

$$ds^2 = (1+E)dt^2 - dx^2 - dy^2 - (1-E)dz^2 - 2Edzdt, \quad (7)$$

where $E = E(t, x, y, z)$, which is easily obtainable from generalized Peres spacetime taken by Pandey [4] for $A = B = 1$. This spacetime is comparable with Peres [12] metric

$$ds^2 = -dx^2 - dy^2 - dz^2 + dt^2 - 2f(x, y, z - t)(dz - dt)^2, \tag{8}$$

which can also be put in the form

$$ds^2 = -dx^2 - dy^2 - (1 - E)dz^2 - 2Edzdt + (1 + E)dt^2, \tag{9}$$

by replacing $-2f$ by E. Evidently E is a function of x, y and $z - t$. The spacetime of Peres belong to second class of Petrov's classification and the source of f (or E), as interpreted by Peres, is a null electromagnetic field. This spacetime has been studied by Takeno [14] in detail concerning its phase velocity, coordinate conditions, energy momentum pseudo tensor, etc., and also the solutions of various field equations obtained in this space-time have been called plane wave-like.

Using co-moving coordinates and the metric (7) we have

$$\begin{aligned} V^1 &= \sqrt{1 - E}, \\ V^2 &= 0, \\ V^3 &= 0, \\ V^4 &= \frac{-E}{\sqrt{1 - E}}. \end{aligned} \tag{10}$$

Then (4) gives

$$\begin{aligned} T_{11} &= -(1 + E)p + \frac{(p + \rho)}{(1 - E)}, \\ T_{14} &= T_{41} = pE, \\ T_{22} &= T_{33} = p, \\ T_{44} &= -(-1 + E)p. \end{aligned} \tag{11}$$

3. Field Equations

The field equations (6) on the background metric (7), for $n = 2$, when the components of $T_{\mu\nu}$ are used from (11) yields

$$\frac{1}{2}\Big[-\partial_y^2 E - \partial_x^2 E - 4a_2 U\Big(\frac{1}{4}(1+E)U + \frac{1}{2}\big(-\partial_y^2 E - \partial_x^2 E - (1+E)U\big) + \big[-\big((1+E)\partial_z E + (2+E)\partial_t E\big)(-\partial_z U) - 2\partial_t^2 U + \partial_y E \partial_y U + \partial_x E \partial_x U + \big(\partial_t E + E(\partial_z E + \partial_t E)\big)(\partial_t U) + 2(1+E)\big(\partial_y^2 U + \partial_x^2 U + (\partial_z^2 - \partial_t^2)U\big)\big]/U\Big)\Big] = k\left(-(1+E)p + \frac{(p+\rho)}{(1-E)}\right), \tag{12}$$

$$\frac{1}{2}\Big[-\partial_x\partial_z E - \partial_x\partial_t E + 2a_2\Big(-2\partial_z^3 E\partial_x E + 2(\partial_x\partial_z E)(\partial_t\partial_z E) - 6(\partial_x E)(\partial_t\partial_z^2 E) + 2(\partial_t\partial_z E)(\partial_t\partial_x E) + (\partial_z^2 E)(\partial_x\partial_z E + \partial_x\partial_t E) + 4\partial_t\partial_x\partial_z^2 E + (\partial_x\partial_z E)(\partial_t^2 E) + (\partial_t\partial_x E)(\partial_t^2 E) - 6(\partial_x E)(\partial_z\partial_t^2 E) + 8\partial_x\partial_z\partial_t^2 E - 2(\partial_x E)(\partial_t^3 E) + 4\partial_t^3\partial_x E\Big)\Big] = 0, \tag{13}$$

$$\frac{1}{2}(-U)\Big(1 + a_2\big(-U - 8(\partial_y^2 U + 2\partial_x^2 U + (\partial_z^2 - \partial_t^2)U)/U\big)\Big) = kp, \tag{14}$$

$$\frac{1}{2}\Big[-\partial_y\partial_z E - \partial_y\partial_t E + 2a_2\Big(-2\partial_z^3 E\partial_y E + 2(\partial_y\partial_z E)(\partial_t\partial_z E) - 6(\partial_y E)(\partial_t\partial_z^2 E) + 2(\partial_t\partial_z E)(\partial_t\partial_y E) + (\partial_z^2 E)(\partial_y\partial_z E + \partial_y\partial_t E) + 4\partial_t\partial_y\partial_z^2 E + (\partial_y\partial_z E)(\partial_t^2 E) + (\partial_t\partial_y E)(\partial_t^2 E) - 6(\partial_y E)(\partial_z\partial_t^2 E) + 8\partial_y\partial_z\partial_t^2 E - 2(\partial_y E)(\partial_t^3 E) + 4\partial_t^3\partial_y E\Big)\Big] = 0, \tag{15}$$

$$4a_2\Big[\partial_x\partial_y\partial_z^2 E + 2\partial_t\partial_x\partial_y\partial_z E + \partial_t^2\partial_x\partial_y E\Big] = 0, \tag{16}$$

$$\frac{1}{2}(-U)\Big(1 + a_2\big(-U - 8(2\partial_y^2 U + \partial_x^2 U + (\partial_z^2 - \partial_t^2)U)/U\big)\Big) = kp, \tag{17}$$

$$\frac{1}{2}\Bigg[\partial_y^2 E + \partial_x^2 E - a_2 U\Bigg(-EU + 2(\partial_y^2 E + \partial_x^2 E + EU) + \Bigg[4\Bigg(-2\partial_t\partial_z U \\ -\big(\partial_t E + E(\partial_z E + \partial_t E)\big)\partial_z U - \partial_y E\partial_y U - \partial_x E\partial_x U - \big((-1+E) \\ \partial_z E + E\partial_t E\big)\partial_t U - 2E\big(\partial_y^2 U + \partial_x^2 U + (\partial_z^2 - \partial_t^2)U\big)\Bigg)\Bigg]/U\Bigg)\Bigg] = kpE, \tag{18}$$

$$\frac{1}{2}\Bigg[\partial_x\partial_z E + \partial_x\partial_t E + 2a_2\Bigg(2\partial_z^3 E\partial_x E + 4\partial_z^3\partial_x E - 2(\partial_x\partial_z E)(\partial_t\partial_z E) \\ 6(\partial_x E)(\partial_t\partial_z^2 E) - 2(\partial_t\partial_z E)(\partial_t\partial_x E) - (\partial_z^2 E)(\partial_x\partial_z E + \partial_x\partial_t E) \\ +8\partial_t\partial_x\partial_z^2 E - (\partial_x\partial_z E)(\partial_t^2 E) - (\partial_t\partial_x E)(\partial_t^2 E) + 6(\partial_x E) \\ (\partial_z\partial_t^2 E) + 4\partial_x\partial_z\partial_t^2 E + 2(\partial_x E)(\partial_t^3 E)\Bigg)\Bigg] = 0, \tag{19}$$

$$\frac{1}{2}\Bigg[\partial_y\partial_z E + \partial_y\partial_t E + 2a_2\Bigg(2\partial_z^3 E\partial_y E + 4\partial_z^3\partial_y E - 2(\partial_y\partial_z E)(\partial_t\partial_z E) \\ 6(\partial_y E)(\partial_t\partial_z^2 E) - 2(\partial_t\partial_z E)(\partial_t\partial_y E) - (\partial_z^2 E)(\partial_y\partial_z E + \partial_y\partial_t E) \\ +8\partial_t\partial_y\partial_z^2 E - (\partial_y\partial_z E)(\partial_t^2 E) - (\partial_t\partial_y E)(\partial_t^2 E) + 6(\partial_y E) \\ (\partial_z\partial_t^2 E) + 4\partial_y\partial_z\partial_t^2 E + 2(\partial_y E)(\partial_t^3 E)\Bigg)\Bigg] = 0, \tag{20}$$

$$\frac{1}{2}\Bigg[-\partial_y^2 E - \partial_x^2 E - 4a_2 U\Bigg(\frac{1}{4}(-1+E)U + \frac{1}{2}\big(-\partial_y^2 E - \partial_x^2 E - (-1+E) \\ U\big) + \Bigg[-2\partial_z^2 U - \big((-1+E)\partial_z E + E\partial_t E\big)(-\partial_z U) + \partial_y E\partial_y U \\ +\partial_x E\partial_x U - \big((-2+E)\partial_z E + (-1+E)\partial_t E\big)(-\partial_t U) + 2 \\ (-1+E)\Bigg(\partial_y^2 U + \partial_x^2 U + \big(\partial_z^2 - \partial_t^2\big)U\Bigg)\Bigg]/U\Bigg)\Bigg] = -kp(-1+E), \tag{21}$$

where $U = (\partial_z^2 + 2\partial_z\partial_t + \partial_t^2)E$, $\partial_z = \frac{\partial}{\partial z}$, $\partial_z^2 = \partial_{zz} = \frac{\partial^2}{\partial z^2}$, $\partial_t^3 = \partial_{ttt} = \frac{\partial^3}{\partial t^3}$ etc.

We now determine E such that it satisfies all the above field equations.
We now determine E such that it satisfies all the above field equations. Subtracting equation (17) from equation (14), since $a_2 \neq 0$ we obtain

$$\big(\partial_x^2 - \partial_y^2\big)U = 0. \tag{22}$$

Also equation (16), since $a_2 \neq 0$ can be written as

$$\partial_x \partial_y U = 0. \tag{23}$$

Adding equation (19) and (13), since $a_2 \neq 0$ we obtain

$$\partial_x \big(\partial_z + \partial_t\big)^3 E = 0. \tag{24}$$

Adding equations (20) and (15), since $a_2 \neq 0$ we obtain

$$\partial_y \big(\partial_z + \partial_t\big)^3 E = 0. \tag{25}$$

Equation (24) and (25) yields on integration form of E as

$$E = f_1(x,y,Z) + z f_2(x,y,Z) + z^2 f_3(x,y,Z) + f(z,t) \quad (Z = z - t). \tag{26}$$

Adding two times equation (18) to the equations (12) and (21), we obtain

$$4a_2(\partial_z + \partial_t)^2 U = k\Big[\frac{-4E(1-E)+2}{1-E}\Big]p. \tag{27}$$

Equation (27) when E given by (26) is substituted becomes

$$\begin{aligned} 4a_2(1 - f_1(x,y,Z) + z f_2(x,y,Z) + z^2 f_3(x,y,Z) \\ + f(z,t))(\partial_z + \partial_t)^4 f = q, \end{aligned} \tag{28}$$

where q is

$$\begin{aligned} q = k\Big[-4(f_1(x,y,Z) + z f_2(x,y,Z) + z^2 f_3(x,y,Z) + f(z,t))(1 \\ - f_1(x,y,Z) + z f_2(x,y,Z) + z^2 f_3(x,y,Z) + f(z,t)) + 2\Big]p. \end{aligned} \tag{29}$$

Equation (22) when E given by (26) is substituted becomes

$$\big(\partial_x^2 - \partial_y^2\big) f_3 = 0. \tag{30}$$

Also equation (23) when E given by (26) is substituted becomes

$$\partial_x \partial_y f_3 = 0. \tag{31}$$

Therefore when E is given by (26) and the conditions (28),(30) and (31) holds, then the field equations (12)-(21) are identically satisfied.

Also considering (14), viz.

$$\frac{1}{2}(-U)\Big(1+a_2\big(-U-8(\partial_y^2U+2\partial_x^2U+(\partial_z^2-\partial_t^2)U)/U\big)\Big)=kp, \tag{32}$$

when E given by (26) is substituted yields

$$p=\rho=\Big(f_3+\frac{1}{2}r\Big)\frac{w}{k}, \tag{33}$$

where

$$r=\big(\partial_z+\partial_t\big)^2 f, \tag{34}$$

and

$$w=1+a_2\Big(-2f_3-r-8\frac{2\partial_y^2f_3+2\partial_x^2f_3+4f_3'+\big(\partial_z^2-\partial_t^2\big)r}{2f_3+r}\Big). \tag{35}$$

where $f_3' = \frac{\partial f_3}{\partial Z}$, $(Z=z-t), etc.$

Hence we have:

A necessary and sufficient condition that $g_{\mu\nu}$ given by (7) where E having the form given by (26) constitute the solutions of the field equations (12)-(21) in $f(R)$ theory of gravity is that f_1, f_2, f_3 and f satisfies (28),(30) and (31).

4. A Particular Solution

Adding twice equation (18) to equations (12) and (21), we obtain

$$4a_2(\partial_z+\partial_t)^4E=k\Big[\frac{-4E(1-E)+2}{1-E}\Big]p. \tag{36}$$

Adding equations (20) and (15), since $a_2\neq 0$ we obtain

$$\partial_y\big(\partial_z+\partial_t\big)^3E=0. \tag{37}$$

Adding equation (19) and (13), since $a_2\neq 0$ we obtain

$$\partial_x\big(\partial_z+\partial_t\big)^3E=0. \tag{38}$$

Equation (36) and (37) yields on integration form of E as

$$E=f_1(x,y,Z)+zf_2(x,y,Z)+z^2f_3(x,y,Z)+f(z,t),\quad (Z=z-t). \tag{39}$$

If we restrict f_3 from its dependence on x and y then the form of E given by (39) becomes

$$E = f_1(x, y, Z) + zf_2(x, y, Z) + z^2 f_3(Z) + f(z, t). \tag{40}$$

. Equation (16), since $a_2 \neq 0$ can be written as

$$\partial_x \partial_y U = 0. \tag{41}$$

Also, subtracting equation (17) from equation (14), since $a_2 \neq 0$ we obtain

$$\big(\partial_x^2 - \partial_y^2\big)U = 0. \tag{42}$$

When E is given by (40) then (41) and (42) are identically satisfied. Also (36) when E given by (40) is substituted becomes

$$\begin{aligned} 4a_2(1 - f_1(x, y, Z) + zf_2(x, y, Z) + z^2 f_3(Z) \\ + f(z, t))(\partial_z + \partial_t)^4 f = v, \end{aligned} \tag{43}$$

where v is given by

$$\begin{aligned} v = k\Big[-4(f_1(x, y, Z) + zf_2(x, y, Z) + z^2 f_3(Z) + f(z, t))(1 \\ - f_1(x, y, Z) + zf_2(x, y, Z) + z^2 f_3(Z) + f(z, t)) + 2\Big]p. \end{aligned} \tag{44}$$

Also considering (14), viz.

$$\frac{1}{2}(-U)\bigg(1 + a_2\big(-U - 8(\partial_y^2 U + 2\partial_x^2 U + (\partial_z^2 - \partial_t^2)U)/U\big)\bigg) = kp, \tag{45}$$

when E given by (40) is substituted yields

$$p = \rho = \Big(f_3 + \frac{1}{2}r\Big)\frac{s}{k}, \tag{46}$$

where

$$r = \big(\partial_z + \partial_t\big)^2 f, \tag{47}$$

and

$$s = 1 + a_2\Big(-2f_3 - r - 8\frac{4f_3'' + \big(\partial_z^2 - \partial_t^2\big)r}{2f_3 + r}\Big). \tag{48}$$

where $f_3'' = \frac{\partial f_3}{\partial Z}$, $(Z = z - t), etc.$

Therefore, when E if given by (40) and the condition (43) hold, then the field equations (12)-(21) are identically satisfied, a similar form which has been studied in [9], [13].

Hence we have:

A necessary and sufficient condition that $g_{\mu\nu}$ given by (7) where E having the form given by (40), p, ρ given by (45), constitute the solutions of the field equations (12)-(21) in $f(R)$ theory of gravity is that f_1, f_2, f_3 and f satisfies (43).

5. Concluding Remarks

Equation (27) and (36) is obtained assuming that $U = \left(\partial_z^2 + 2\partial_z\partial_t + \partial_t^2\right)E$ is non zero. It is evident that the form of E given by (26) or (40) satisfies this condition.

Taking the trace of (6) in vacuum [7] for $n = 2$, gives $\Box R - \frac{R}{6a_2} = 0$, which is a wave equation comparable with the massless scalar field equation $\Box\phi + \frac{R}{6}\phi = 0$. Also the term $\Box R$ is non vanishing for all values of $n \geq 2$, showing that R has a wave nature in higher order theory of gravity which we have investigated for the spacetime metric (7). The form of spacetime metric obtained in our case (40) contains extra terms in the form of $z^2 f_3(Z)$ and $f(z,t)$ which appeared due to the correction term introduced in the Hilbert Lagrangian.

The form of E given by (26) is a solution of the field equations (12)-(21) subject to conditions (28), (30) and (31) viz.

$$\begin{aligned} 4a_2(1 - f_1(x,y,Z) + zf_2(x,y,Z) + z^2 f_3(x,y,Z) \\ + f(z,t))(\partial_z + \partial_t)^4 f = q, \end{aligned} \tag{49}$$

$$\left(\partial_x^2 - \partial_y^2\right) f_3 = 0, \tag{50}$$

$$\partial_x \partial_y f_3 = 0, \tag{51}$$

whereas the form of E given by (40) identically satisfies (50) and (51), satisfying the field equations (12)-(21) subject to condition (43) only.

Also it should be noted that the equation of state connecting p and ρ obtained in (33) is different from that obtained in (46), since $w \neq s$.

We saw that as some of the metric coefficients do depend on x and y the solutions can be called plane wave-like following Takeno [14].

Lastly taking the form of E as given by (40) and assume the functions f_2, f_3, f to be zero; E will reduce to Peres function and the solution will reduce to those obtained by Takeno [15].

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