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SOME RESULTS ON DUPLICATION SELF VERTEX SWITCHINGS

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Abstract: A vertex $v \in V(G)$ is said to be a *self vertex switching* of G if G is isomorphic to G^v, where G^v is the graph obtained from G by deleting all edges of G incident to v and adding all edges incident to v which are not in G. Duplication of a vertex v of graph G produces a new graph $G^{'}$ by adding a new vertex $v^{'}$ such that $N(v^{'}) = N(v)$. In other words a vertex v' is said to be duplication of v if all the vertices which are adjacent to v in G are also adjacent to $v^{'}$ in $G^{'}$. A vertex v is called a *duplication self vertex switching* of a graph G if the resultant graph obtained after duplication of v has v as a self vertex switching. In this paper, we give some properties of duplication self vertex switching.

AMS Subject Classification: 05C60
Key Words: switching, self vertex switching, duplication self vertex switching, $dss_1(G)$

1. Introduction

For a finite undirected graph $G(V, E)$ with $|V| = p$ and a set $\sigma \subseteq V$, Seidel [8] defined the *switching* of G by σ as the graph $G^\sigma(V, E^{'})$, which is obtained from G by removing all edges between σ and its compliment $V - \sigma$ and adding as edges all non edges between σ and $V - \sigma$. When $\sigma = \{v\} \subseteq V$, the corresponding switching $G^{\{v\}}$ is called a *vertex switching* and is denoted by G^v [9].

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Switching is an equivalence relation and the associated equivalence classes are called *switching classes*. For a survey of switching classes of graphs we refer to Seidel [8]. A subset σ of $V(G)$ to be a *self switching* of G if $G \cong G^{\sigma}$. The set of all self switchings of G with cardinality k is denoted by $SS_k(G)$ and its cardinality by $ss_k(G)$. If $k = 1$, then we call the corresponding self switching as *self vertex switching* [11]. Duplication of a vertex v of graph G produces a new graph $G^{'}$ by adding a new vertex $v^{'}$ such that $N(v^{'}) = N(v)$. In other words a vertex v' is said to be duplication of v if all the vertices which are adjacent to v in G are also adjacent to $v^{'}$ in $G^{'}$. The concept of duplication self vertex switching was introduced by C. Jayasekaran and V. Prabavathy [5]. A vertex v is called a *duplication self vertex switching* of a graph G if the resultant graph obtained after duplication of v has v as a self vertex switching. The number of duplication self vertex switching is denoted by $dss_1(G)$. For any vertex $v \in V(G)$, the *open neighbourhood* $N(V)$ of v is the set of all vertices adjacent to v. That is $N(V) = \{u \in V(G)/\ uv \in E(G)\}$. The *closed neighbourhood* of v is defined by $N[v] = N(v) \cup \{v\}$. Two vertices u and v in G are said to be *interchange similar* if there is an automorphism α of G such that $\alpha(u) = v$ and $\alpha(v) = u$ [7]. In [10], a characterization is given for a cut vertex in G to be a self vertex switching where G is a connected graph such that any two self vertex switchings if exists , are interchange similar. The existance of graphs with given number of self vertex switchings were discussed in [1]. The trees [3] and unicyclic graphs [2], [4] are characterized for self vertex switchings.

In this paper, we give some properties of duplication self vertex switching and we prove that if v is a duplication self vertex switching of G then there exists a duplication self vertex switching $u \in N(v)$ such that u and v are interchange similar in G, $N(u) \cup N(v) = V(G)$ and $N(u) \cap N(v) = \phi$. We consider simple graphs only.

We consider the following results which are required in the subsequent section.

Theorem 1.1. **[5]** If v is a duplication self vertex switching of a graph G of order p then p is even and $d_G(v) = p/2$.

Theorem 1.2. **[5]** Let G be a graph and let v be any vertex of G. Then v is a duplication self vertex switching of G iff there exists an automorphism on G which maps elements of $N(v)$ onto elements of $[N(v)]^c$.

Theorem 1.3. **[6]** If $dss_1(G) > 0$ then the number of duplication self vertex switchings in G is even.

2. Main Results

Theorem 2.1. *Let v be a duplication self vertex switching of a graph G of order p. Then there exists a duplication self vertex switching $u \in N(v)$ such that u and v are interchange similar in G.*

Proof. We may assume that $p > 2$ as the statement holds trivially when $p = 2$. Since v is a duplication self vertex switching of G, by Theorem 1.2, there exists an automorphism α on G which maps elements of $N(v)$ onto $[N(v)]^c$ and vice versa. This implies that $\alpha(v) = u$ for some $u \in N(v)$ and so u is a duplication self vertex switching of G. If $\alpha(u) = v$, then there is nothing to prove. Suppose that $\alpha(u) = v_1 (\neq v)$ for some $v_1 \in [N(v)]^c$. Then $d(v_1) = p/2$, since $d(u) = p/2$.
Claim. $N(v) = N(v_1)$.

To prove this, it is sufficient to show that v_1 is non adjacent to any vertex of $[N(v)]^c$ in G. Suppose that v_1 is adjacent to a vertex of $[N(v)]^c$ in G. Then there exists a vertex $v_2 \in [N(v)]^c$ such that v_1 is adjacent to v_2 in G. This implies that $\alpha(u)$ is adjacent to v_2 in G. This implies that $\alpha(u)$ is adjacent to a vertex of $[N(v)]^c$ in G. This implies that u is adjacent to a vertex of $N(v)$ in G. This implies that $\alpha(v)$ is adjacent to a vertex of $N(v)$ in G. This implies that v is adjacent to a vertex of $[N(v)]^c$ in G, which is a contradiction since v is non adjacent to any vertex of $[N(v)]^c$ in G and hence the vertex v_1 is non adjacent to any vertex of $[N(v)]^c$ in G. Thus we conclude that v_1 is adjacent to every vertex of $N(v)$ in G, completing the claim that $N(v) = N(v_1)$.

Define a map $\beta : V(G) \to V(G)$ such that $\beta(v) = v_1$, $\beta(v_1) = v$ and $\beta(w) = w$ for all $w \in V(G) - \{v, v_1\}$. Clearly β is an automorphism of G which interchanges the vertices v and v_1.

Further define a map $\gamma : V(G) \to V(G)$ such that $\gamma = \beta \circ \alpha$. Clearly γ is an automorphism of G which maps elements of $N(v)$ onto $[N(v)]^c$ and vice versa. In particular, $\gamma(v) = u$ *and* $\gamma(u) = v$ and so the vertices u and v are interchange similar in G. □

Note 2.2. Any two interchange similar vertices are need not be duplication self vertex switchings. For example, consider the graph G given in Figure 2.1. For $1 \leq i \leq 3$, define a map $\alpha : V(G) \to V(G)$ such that $\alpha(u) = v, \alpha(v) = u, \alpha(u_i) = v_i$ and $\alpha(v_i) = u_i$. Clearly α is an automorphism on G such that $\alpha(u) = v$ and $\alpha(v) = u$. This implies that u and v are interchange similar in G. Also G is of order 8 and has only two vertices u and v such that $d(u) = d(v) = 4$. In $N(v)$, there is no vertex with degree 2 but in $[N(v)]^c$, there are two vertices u_2 and v_2 such that $d(u_2) = d(v_2) = 2$. This implies that there is no automorphism on G which maps elements of $N(v)$ onto $[N(v)]^c$ and vice versa. By Theorem 1.2, v is not a duplication self vertex switching of G. Similarly, u is not a duplication self vertex switching of G.

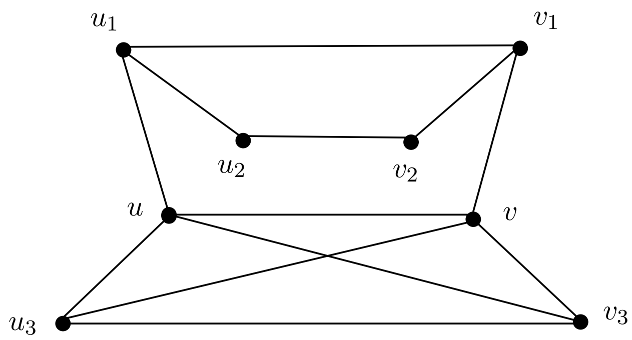


Figure 2.1

Theorem 2.3. *Let v be a duplication self vertex switching of a graph G. Then there exists a duplication self vertex switching $u \in N(v)$ such that v is a self vertex switching of $G - u$.*

Proof. Let v be a duplication self vertex switching of a graph G. Then by Theorem 2.1, there exists a duplication self vertex switching $u \in N(v)$ such that u and v are interchange similar in G. Also by Theorem 2.1, we can find an automorphism γ on G such that $\gamma(v) = u$, $\gamma(u) = v$ and γ maps elements of $N(v)$ on to $[N(v)]^c$ and vice versa. Now let us denote the graph $G - u = G_\circ$ and we have to prove that v is a self vertex switching of $G_\circ$. It is enough to prove that $G_\circ \cong G_\circ^v$. Define $f : V(G_\circ) \to V(G_\circ^v)$ such that $f(v) = v$ and $f(w) = \gamma(w)$ for all $w \in V(G_\circ) - \{v\}$. Clearly f is a bijection. To prove f is an isomorphism between $G_\circ$ and $G_\circ^v$, it is enough to check the adjacency property. Let $v_1,\ v_2 \in V(G_\circ)$.

1. v and v_1 are adjacent in $G_\circ$

$\Leftrightarrow$ v and v_1 are adjacent in G

$\Leftrightarrow$ $v_1 \in N(v)$ in G

$\Leftrightarrow$ $\gamma(v_1) \in [N(v)]^c$ in G

$\Leftrightarrow$ v and $\gamma(v_1)$ are non adjacent in G

$\Leftrightarrow$ v and $\gamma(v_1)$ are non adjacent in $G_\circ$

$\Leftrightarrow$ v and $\gamma(v_1)$ are adjacent in $G_\circ^v$

$\Leftrightarrow$ $f(v)$ and $f(v_1)$ are adjacent in $G_\circ^v$
2. v_1 and v_2 are adjacent in $G_\circ$
$\Leftrightarrow$ v_1 and v_2 are adjacent in G
$\Leftrightarrow$ $\gamma(v_1)$ and $\gamma(v_2)$ are adjacent in G
$\Leftrightarrow$ $\gamma(v_1)$ and $\gamma(v_2)$ are adjacent in $G_\circ$
$\Leftrightarrow$ $f(v_1)$ and $f(v_2)$ are adjacent in $G_\circ^v$. Hence v is a self vertex switching of $G_\circ$. This completes the proof of the theorem. □

Theorem 2.4. *If v is a duplication self vertex switching of a graph G, then there exists a vertex $u \in N(v)$ such that $G - u \cong G - v$.*

Proof. Let v be a duplication self vertex switching of a graph G. Then there exists an automorphism on G which maps elements of $N(v)$ onto $[N(v)]^c$ and vice versa. Also by Theorem 2.1, there exists a duplication self vertex switching $u \in N(v)$ such that u and v are interchange similar in G. Hence we can find an automorphism α on G such that $\alpha(v) = u$, $\alpha(u) = v$ and α maps elements of $N(v)$ onto $[N(v)]^c$ and vice versa. Define $f : V(G-u) \to V(G-v)$ such that $f(w) = \alpha(w)$ for all $w \in V(G-u)$. Clearly f is a bijection. To complete the proof we have to check the adjacency property. Let $v_1, v_2 \in V(G-u)$ such that v_1 and v_2 are different from v.
1. v and v_1 are adjacent in $G-u$
$\Leftrightarrow$ v and v_1 are adjacent in G
$\Leftrightarrow$ $\alpha(v)$ and $\alpha(v_1)$ are adjacent in G
$\Leftrightarrow$ u and $\alpha(v_1)$ are adjacent in G
$\Leftrightarrow$ u and $\alpha(v_1)$ are adjacent in $G-v$
$\Leftrightarrow$ $\alpha(v)$ and $\alpha(v_1)$ are adjacent in $G-v$
$\Leftrightarrow$ $f(v)$ and $f(v_1)$ are adjacent in $G-v$
2. v_1 and v_2 are adjacent in $G-u$
$\Leftrightarrow$ v_1 and v_2 are adjacent in G
$\Leftrightarrow$ $\alpha(v_1)$ and $\alpha(v_2)$ are adjacent in G
$\Leftrightarrow$ $\alpha(v_1)$ and $\alpha(v_2)$ are adjacent in $G-v$
$\Leftrightarrow$ $f(v_1)$ and $f(v_2)$ are adjacent in $G-v$. Hence f is an isomorphism between $G-u$ and $G-v$. This completes the proof of the theorem. □

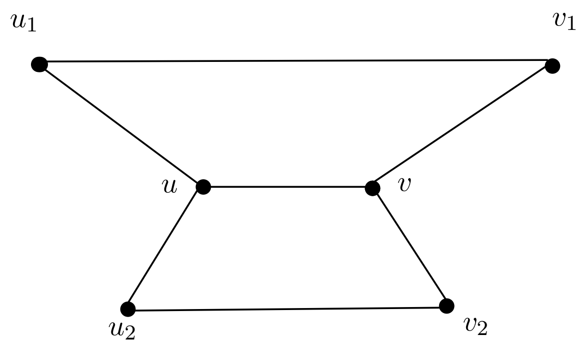
Theorem 2.5. *If v is a duplication self vertex switching of G then there exists a duplication self vertex switching $u \in N(v)$ such that $N(u) \cup N(v) = V(G)$ and $N(u) \cap N(v) = \phi$.*

Proof. Let G be a graph of order p and let v be a duplication self vertex switching of G. If $p = 2$ then the statement is obviously true and so we proceed by assuming $p > 2$. Since v is a duplication self vertex switching of G and $v \in [N(v)]^c$, by Theorem 1.2, there exists an automorphism α on G which maps v into some vertex $u \in N(v)$ and hence u will be a duplication self vertex switching in G. By Theorem 2.1, u and v are interchange similar in G. To complete the proof, first we show that u is adjacent to every vertex of $[N(v)]^c$. Suppose there exists a vertex $v_1 \in [N(v)]^c$ such that v_1 is non adjacent to u in G. Then $\alpha(v_1)$ is non adjacent to $\alpha(u)$ in G and so $\alpha(v_1)$ must be non adjacent to v in G, because u and v are interchange similar vertices. This implies that v is non adjacent to a vertex of $N(v)$ in G, since $\alpha(v_1) \in N(v)$. But this contradicts the fact that every vertex of $N(v)$ is adjacent to v and hence we conclude that u is adjacent to every vertex of $[N(v)]^c$. This implies that $N(u) \subseteq [N(v)]^c$. Since $|N(u)| = |[N(v)]^c| = p/2$, $N(u) = [N(v)]^c$. The result follows from $N(v) \cup N[(v)]^c = V(G)$ and $N(v) \cap N[(v)]^c = \phi$. □

Theorem 2.6. *Let u and v be any two adjacent duplication self vertex switchings of a graph G of order p such that $N(u) \cup N(v) = V(G)$. If uv is a cut edge of G then $dss_1(G) = 2$.*

Proof. By assumption, $dss_1(G) \geq 2$. Let uv be a cut edge of G. If u and v are the only duplication self vertex switchings then there is nothing to prove. Assume that $dss_1(G) > 2$. Let w be a duplication self vertex switching of G other than u and v. Since $N(u) \cup N(v) = V(G)$, $N(u) \cap N(v) = \phi$ and hence w is adjacent to either u *or* v but not both. Suppose that w is adjacent to u. Now $|N(v) \cup \{v\}| = (p/2) + 1$ and $d(w) = p/2$. This implies that w is adjacent to atleast two vertices of $N(v) \cup \{v\}$. Since w is non adjacent to v, it must be adjacent to atleast two vertices of $N(v)$. Let $v_1 \in N(v)$ such that v_1 is adjacent to w. Then uvv_1wu is a cycle in G. This implies that the edge uv lies on a cycle of G, which is a contradictoin to uv is a cut edge of G. Similarly we get a contradiction if w is adjacent to v. Thus we conclude that there is no duplication self vertex switching in G other than u and v. This completes the proof. □

Remark 2.7. *In the above theorem, the converse need not to be true. For example, the vertices u and v of the graph G given in Figure 2.2 are the two adjacent duplication self vertex switchings of G such that $N(u) \cup N(v) = V(G)$. But the edge uv is not a cut edge.*



G

Figure 2.2.

Theorem 2.8. *If $dss_1(G) > 2$ then G has no end vertices.*

Proof. Let G be a graph of order p and let $dss_1(G) > 2$. Then by Theorem 1.3, $dss_1(G) \geq 4$. The conclusion clearly holds if $p = 4$, since each vertex in G is a duplication self vertex switching and by Theorem 1.1, $d(v) = 2$ for all $v \in V(G)$. So assume that $p > 4$. First we consider one of the duplication self vertex switchings, say v_1. In view of Theorem 2.5, there exists a duplication self vertex switching v_2 in G such that $N(v_1) \cup N(v_2) = V(G)$ and $N(v_1) \cap N(v_2) = \phi$. Furthermore, consider a duplication self vertex switching v_3 in G which is different from v_1 and v_2. Again by Theorem 2.5, we can find a duplication self vertex switching v_4 in G such that $N(v_3) \cup N(v_4) = V(G)$ and $N(v_3) \cap N(v_4) = \phi$. Since $N(v_1) \cup N(v_2) = V(G)$ and $N(v_1) \cap N(v_2) = \phi$, v_3 may either adjacent to v_1 or v_2. Without loss of generality, we may assume that v_3 is adjacent to v_1. If v_4 is different from v_1 and v_2 then there is nothing to prove, because each vertex of G must be adjacent to atleast two of the vertices $v_1,\ v_2,\ v_3$ *and* v_4. Clearly v_4 is different from v_2, otherwise v_1 should be adjacent to both v_3 and v_4. But this is impossible, because $N(v_3) \cap N(v_4) = \phi$.

Now the vertex v_4 is different from either v_1 or different from v_1. If v_4 is different from v_1 then there is nothing to prove. Otherwise, the fact that $N(v_3) \cup N(v_4) = V(G)$ can be written as $N(v_3) \cup N(v_1) = V(G)$. That is v_3 is adjacent to every vertex of $[N(v_1)]^c$. Since $N(v_1) \cup N(v_2) = V(G)$ and$N(v_1) \cap N(v_2) = \phi$, the vertex v_2 must be adjacent to every vertex of $[N(v_1)]^c$ and hence it is clear that $N(v_3) = N(v_2)$. This shows that each vertex of $[N(v_1)]^c$ has degree atleast two in G. Since there is an automorphism of G which maps elements of $N(v_1)$ onto $[N(v_1)]^c$ and vice versa, each vertex of $N(v_1)$ also has degree atleast two in G. Consequently, there is no end vertices in G and hence the theorem follows. □

Remark 2.9. *The converse of the above theorem need not to be true. For example, the graph G given in Figure 2.3 has no end vertices but it has only two duplication self vertex switchings u and v.*

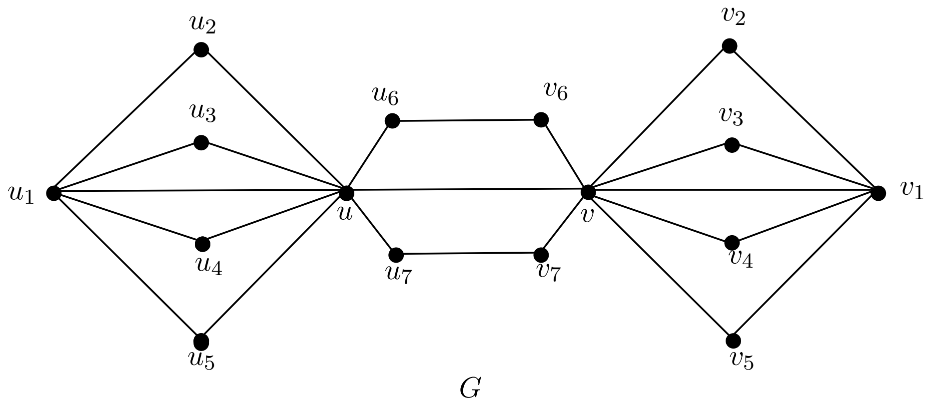


Figure 2.3.

Definition 2.10. *A shortest $u - v$ path is often called a geodesic.*

Definition 2.11. *The diameter $d(G)$ of a connected graph G is the length of any longest geodesic.*

Theorem 2.12. *If $dss_1(G) \geq 2$ then $d(G) \leq 3$.*

Proof. Let v be a duplication self vertex switching of G. Then by Theorem 2.5, there exists a duplication self vertex switching $u \in N(v)$ such that every vertex of G is adjacent to u *or* v. Since u and v are adjacent, for any two vertices v_1 and v_2 in G, the length of a shortest path joining v_1 and v_2 is atmost 3. This completes the proof. □

3. Conclusion

In this paper we have discussed some properties of duplication self vertex switchings in graphs. Also we have proved that if v is a duplication self vertex switching of G then there exists a duplication self vertex switching $u \in N(v)$ such that u and v are interchange similar in G, $N(u) \cup N(v) = V(G)$ and $N(u) \cap N(v) = \phi$.

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