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QUASITOPOLOGICAL GROUPS VIA ω-OPEN SETS

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Abstract: In this paper, we introduce and study a class of topologized groups called ω-topological groups.

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1. Introduction

If $(G, \star)$ is a group, and τ and τ_2 are topologies on G, then we say that $(G, \star, \tau)$ is a bitopologized group. Given a topologized group G, a question arises about interactions and relations between algebraic and topological structures: which topological properties are satisfied by the multiplication mapping $m : G \times G \to G$, $(x, y) \to x \star y$, and the inverse mapping $i : G \to G$, $x \to x^{-1}$. In this paper, we introduce and study a class of topologized groups called ω-topological groups.

2. Preliminaries

Throughout this paper $(G, \star, \tau)$, or simply G, will denote a group $(G, \star)$ en-

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dowed with the topologies τ and τ_2 on G. The identity element of G is denoted by e, or e_G when it is necessary, the operation $\star : G \times G \to G$, $(x, y) \to x \star y$, is called the multiplication mapping and sometimes denoted by m, and the inverse mapping $i : G \to G$, $x \to x^{-1}$ is denoted by i. For a subset A of a topological space (X, τ), $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ denote the closure of A and the interior of A in (X, τ), respectively.

Definition 2.1. A subset S of a topological space (X, τ) is said to be semiopen [1] if $S \subset \mathrm{Cl}(\mathrm{Int}(S))$.

Definition 2.2. [4] A subset A of a topological space (X, τ) is called an ω-closed set if $\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and U is semiopen in (X, τ). The complement of an ω-closed set is called an ω-open set. The family of all ω-open sets of (X, τ) is denoted by τ_ω.

Definition 2.3. [5] The intersection of all ω-closed sets containing $A \subset X$ is called the ω-closure of A and is denoted by $\omega\,\mathrm{Cl}(A)$. The union of all ω-open sets contained in $A \subset X$ is called the ω-interior of A and is denoted by $\omega\,\mathrm{Int}(A)$.

Definition 2.4. Let X be a topological space the family β of ω-open sets is called an ω-base if and only if for each ω-open set a union of members of a family β.

Definition 2.5. [5] A subset $M(x)$ of a topological space (X, τ) is called an ω-neighbourhood of a point $x \in X$ if there exists an ω-open set S such that $x \in S \subset M(x)$.

Definition 2.6. [5] A function $f : (X, \tau) \to (Y, \sigma)$ is said to be ω-continuous if $f^{-1}(V) \in \omega O(X)$ for every $V \in \sigma$.

Definition 2.7. A subset U of a topological space (X, τ) is called an ω-neighbourhood of a point $x \in X$ if there exists an ω-open set V of X such that $x \in V \subset U$.

3. On ω-Topological Groups

Definition 3.1. $(G, \circ, \tau)$ is said to be ω-topological group if $(G, \circ)$ is a group, (G, τ) is a topological space and left translation $L_x : G \to G$ for all $x \in G$ and right translation $R_x : G \to G$ for all $x \in G$ are ω-continuous and the mapping of inversion $i : G \to G$ defined by $i(x) = x^{-1}$ is ω-continuous on G.

Theorem 3.2. *Let $(G, \circ, \tau)$ be an ω-topological group and β_e be the base at identity element e of G. Then:*

1. *for every $U \in \beta_e$, there is an element $V \in \omega O(G,e)$ such that $V^{-1} \subset U$.*

2. *for every $U \in \beta_e$, there is an element $V \circ x \subset U$, and $x \circ V \subset U$ for each $x \in U$.*

Proof. (1) Since $(G, \circ, \tau)$ is an ω-topological group, for every $U \in \beta_e$ there exists $V \in \omega O(G,e)$ such that $i(V) = V^{-1} \in U$ because the inverse mapping $i : G \to G$ is ω-continuous.
(2) Since $(G, \circ, \tau)$ is an ω-topological group, for each $U \in \tau$ containing x, there exists $V \in \omega O(G,e)$ such that $R_x(V) = V \circ x \subset U$. □

Lemma 3.3. *Let A be a subset of an ω-topological group $(G, \circ, \tau)$. Then $\omega \operatorname{Cl}(A^{-1}) \subset \operatorname{Cl}(A^{-1})$.*

Proof. Let $x \in (\omega \operatorname{Cl}(A))^{-1}$ and let $U \in \tau$ containing x. Then, U^{-1} is an ω-open neighbourhood of x^{-1}. Since $x^{-1} \in \omega \operatorname{Cl}(A)$, $U^{-1} \cap A \neq \emptyset$. This implies that $U \cap A^{-1} \neq \emptyset$. That is, $x \in \operatorname{Cl}(A^{-1})$ and so $(\omega \operatorname{Cl}(A))^{-1} \subset \operatorname{Cl}(A^{-1})$. □

Theorem 3.4. *Let $(G, \circ, \tau)$ be an ω-topological group. If U is ω-open set in $(G, \circ, \tau)$, then U^{-1} is ω-open in $(G, \circ, \tau^{-1})$.*

Proof. The proof follows from the respective definitions. □

Theorem 3.5. *If $(G, \circ, \tau)$ is an ω-topological group, then $(G, \circ, \tau^{-1})$ is also an ω-topological group.*

Proof. Since $(G, \circ)$ is a group and (G, τ) is a topological space, $(G, \circ, \tau^{-1})$ is a topological group. We need to prove that: $i : (G, \circ, \tau^{-1}) \to (G, \circ, \tau^{-1})$, and $L_x : (G, \circ, \tau^{-1}) \to (G, \circ, \tau^{-1})$ and $R_x : (G, \tau^{-1}) \to (G, \tau^{-1})$ are ω-continuous mappings. First, we show that L_x is ω-continuous. For this, let $V \in \tau^{-1}$. Then $V^{-1} = U \in \tau$. Since $(G, \circ, \tau)$ is ω-topological group, the left (right) translation is ω-continuous. Hence $L_x^{\leftarrow}(U) \in \omega O(G, \tau)$, that is, $(U \circ x^{-1})^{-1} \in \omega O(G, \tau^{-1})$, that is, $U \circ x^{-1} = V^{-1} \circ x^{-1} = (x \circ V)^{-1} = L_x^{\leftarrow}(V) \in \omega O(G, \tau^{-1})$. This proves that $L_x : (G, \circ, \tau^{-1}) \to (G, \circ, \tau^{-1})$ is ω-continuous for every $x \in G$. Similarly, we can prove that right translation $R_x : (G, \circ, \tau^{-1}) \to (G, \circ, \tau^{-1})$ is ω-continuous. Trivially $i : (G, \circ, \tau^{-1}) \to (G, \circ, \tau^{-1})$ is continuous and hence ω-continuous. Hence $(G, \circ, \tau^{-1})$ is also an ω-topological group. □

Theorem 3.6. *If H is a discrete subgroup of an ω-topological group $(G, \circ, \tau^{-1})$, then $\omega \operatorname{Cl}(H)$ is a subgroup of G.*

Proof. Let $x, y \in \omega\operatorname{Cl}(H)$. If U and V are respective τ^{-1}-open neighbourhoods of x and y, then $L_{x^{-1}}(U) = x^{-1} \circ U$ and $L_{y^{-1}}(U) = y^{-1} \circ U$ are ω-open neighbourhoods of e. Since H is a discrete subgroup of an ω-topological group G, $x^{-1} \circ U \cap H \neq \emptyset$ and $y^{-1} \circ U \cap H \neq \emptyset$. Therefore, $(x \circ y^{-1} \circ x^{-1} \circ U \cap x \circ y^{-1} \circ H) \cup (x \circ y^{-1} \circ y^{-1} \circ V \cap x \circ y^{-1} \circ H) \neq \emptyset$. That is, $W \cap x^{-1} \circ y^{-1} \circ H \neq \emptyset$, where $W = x \circ y^{-1} \circ x^{-1} \circ U \cup x \circ y^{-1} \circ y^{-1} \circ V$ is an ω-open neighbourhood of $x \circ y^{-1}$. Thus, for each $x, y \in \omega\operatorname{Cl}(H)$ implies that $x \circ y^{-1} \in \omega\operatorname{Cl}(H)$. Hence $\omega\operatorname{Cl}(H)$ is a subgroup of G. □

Corollary 3.7. *If H is a discrete subgroup of an ω-topological group $(G, \circ, \tau)$, then $\operatorname{Cl}(H)$ is a subgroup of G.*

Theorem 3.8. *Let $(G, \circ, \tau)$ be an ω-topological group. If A is open in G, then $A \circ B$ and $B \circ A$ are ω-open in $(G, \circ, \tau)$ for any subset B of G.*

Proof. Let $x \in B$ and $z \in A \circ x$ we show that z is ω-interior point of $A \circ x$. Let $z = y \circ x$ for some $y \in A = A \circ x \circ x^{-1}$. This implies that $y = z \circ x^{-1}$. Now $R_{x^{-1}} : G \to G$ is ω-continuous, that is, for every open set containing $R_{x^{-1}}(z) = z \circ x^{-1} = y$, there exists an ω-open set M_z containing z such that $R_{x^{-1}}(M_z) \subset A$. This implies $M_z \circ x^{-1} \subset A$ or $M_z \subset A \circ x$. This implies z is ω-interior point of $A \circ x$. Thus $A \circ x$ is ω-open. This implies $A \circ B = \bigcup_{x \in B} A \circ x$ is ω-open in $(G, \circ, \tau)$. Similarly we can prove that for every open set A of G and arbitrary subset B of G, $B \circ A$ is ω-open in an ω-topological group $(G, \circ, \tau)$. □

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