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ON $\delta\theta$-$\mathcal{I}$-CONTINUOUS FUNCTIONS

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Abstract: In this paper we use the Kuratowski closure operator $\delta Cl^{\star}$ in order to introduce, investigate and characterize the notions of $\delta\theta$-$\mathcal{I}$-continuous functions. Also, we investigate the relationship with another types of related functions in ideal topological spaces.

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1. Introduction

The notion of ideals on topological spaces was first studied in the classic text

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of Kuratowski [7]. Since then many studies have been carried out related to the theory of ideals on topological spaces. One of the most important was the work done by Janković and Hamlett [6], where they investigated many of the properties and concepts related to this theory. Recently, Hatir et. al [5] introduced the notion of δ-local function and with this also a Kuratowski closure operator $\delta Cl^\star$. In this paper, we use the operator $\delta Cl^\star$ to introduce a new class of functions, namely $\delta\theta$-$\mathcal{I}$-continuous functions, obtain several characterizations and finally, we investigate some interesting properties of this class of functions and its relationship with another types of related functions.

2. Preliminaries

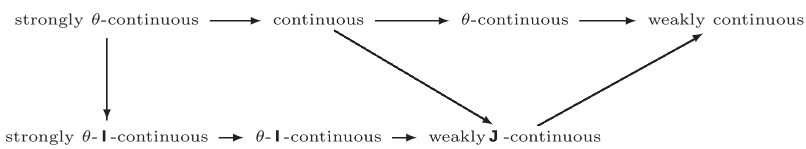
Throughout this paper, (X, τ) always means a topological space on which no separation axioms are assumed unless explicitly stated. If A is a subset of X, we denote the closure of A and the interior of A by $Cl(A)$ and $Int(A)$, respectively. A point $x \in X$ is called a *δ-cluster point* of A if $Int(Cl(U)) \cap A \neq \emptyset$ for each open set U containing x (see [14]). The set of all δ-cluster points of A is called the *δ-closure* of A and is denoted by $\delta Cl(A)$. A subset A of X is said to be *δ-closed* if $A = \delta Cl(A)$. The complement of a δ-closed set is said to be a *δ-open* set. It follows from [14] that the collection of all δ-open sets in a topological space (X, τ) forms a topology on X which is denoted by τ_δ. From the definitions it follows that $\tau_\delta \subset \tau$.

An *ideal* $\mathcal{I}$ on a nonempty set X is a nonempty collection of subsets of X which satisfies the following two properties: (1) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$; (2) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. A topological space (X, τ) with an ideal $\mathcal{I}$ on X is called an *ideal topological space* and is denoted by $(X, \tau, \mathcal{I})$. Given an ideal topological space $(X, \tau, \mathcal{I})$, a set operator $(.)^\star : P(X) \to P(X)$, called the *local function* [7] of A with respect to τ and $\mathcal{I}$, is defined as follows: for $A \subset X$, $A^\star(\mathcal{I}, \tau) = \{x \in X : U \cap A \notin \mathcal{I} \text{ for every } U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau : x \in U\}$. When there is no chance for confusion, we will simply write $A^\star$ for $A^\star(\mathcal{I}, \tau)$. In general, $X^\star$ is a proper subset of X. The hypothesis $X = X^\star$ is equivalent to the hypothesis $\tau \cap \mathcal{I} = \emptyset$. According to [3], we call the ideals which satisfy this condition *codense* ideals. A codense ideal also is called *τ-boundary* ideal in [10]. Note that $\mathrm{Cl}^\star(A) = A \cup A^\star$ defines a Kuratowski closure for a topology $\tau^\star(\mathcal{I})$, finer than τ. When there is no chance for confusion, we will simply write $\tau^\star$ for $\tau^\star(\mathcal{I})$. For a subset A of an ideal topological space $(X, \tau, \mathcal{I})$, the *δ-local function* of A with respect to $\mathcal{I}$ is defined in [5] as $A^{\delta\star}(\mathcal{I}, \tau) = \{x \in X : U \cap A \notin \mathcal{I} \text{ for every } U \in \tau_\delta(x)\}$, where $\tau_\delta(x) = \{U \in \tau_\delta : x \in U\}$. We will simply write $A^{\delta\star}$ for $A^{\delta\star}(\mathcal{I}, \tau)$. A Kuratowski closure operator $\delta Cl^\star$ for a topology $\tau^{\delta\star}$ finer than τ_δ, is defined by $\delta Cl^\star(A) = A \cup A^{\delta\star}$ [5]. A point $x \in X$ is called a *$\delta\theta$-$\mathcal{I}$-cluster point* of A if $\delta Cl^\star(U) \cap A \neq \emptyset$ for every open set U of X containing x (see [12]). The set of all $\delta\theta$-$\mathcal{I}$-cluster points of A is called the *$\delta\theta$-$\mathcal{I}$-closure* of A and is denoted by $\delta Cl^\star_\theta(A)$. A subset A of X is said to be *$\delta\theta$-$\mathcal{I}$-closed* if $\delta Cl^\star_\theta(A) = A$. The complement of a $\delta\theta$-$\mathcal{I}$-closed set is said to be *$\delta\theta$-$\mathcal{I}$-open*. A point $x \in X$ is called a *$\delta\theta$-$\mathcal{I}$-interior point* of a subset A if there exists an open set U such that $x \in U \subset \delta\mathrm{Cl}^\star(U) \subset A$. The set of all $\delta\theta$-$\mathcal{I}$-interior points of A is called the *$\delta\theta$-$\mathcal{I}$-interior* of A and is denoted by $\delta Int^\star_\theta(A)$.

Definition 2.1. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be weakly continuous [9] (resp. θ-continuous [4], strongly θ-continuous [11]), if for each $x \in X$ and each open set V in Y containing $f(x)$, there exists an open set U containing x such that $f(U) \subset Cl(V)$ (resp. $f(Cl(U)) \subset Cl(V)$, $f(Cl(U)) \subset V$).

Definition 2.2. A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is said to be weakly $\mathcal{J}$-continuous [1] (resp. θ-$\mathcal{I}$-continuous [15], strongly θ-$\mathcal{I}$-continuous [15]), if for each $x \in X$ and each open set V in Y containing $f(x)$, there exists an open set U containing x such that $f(U) \subset Cl^\star(V)$ (resp. $f(Cl^\star(U)) \subset Cl^\star(V)$, $f(Cl^\star(U)) \subset V$).

Remark 2.1. From Definitions 2.1 and 2.2, we have the following diagram:



Lemma 2.1. *If A is an open subset of a topological space (X, τ), then $Cl(A) = \delta Cl(A)$.*

Proof. It follows from [13, Lemma 1.4]. □

Lemma 2.2. *[12, Lemma 3.1] If A is any subset of an ideal topological space $(X, \tau, \mathcal{I})$, then the following properties hold:*

(1) $A^\star \subset A^{\delta\star}$,

(2) $Cl^\star(A) \subset \delta Cl^\star(A) \subset \delta Cl(A)$.

3. $\delta\theta$-$\mathcal{I}$-Continuous Functions

In this section, the $\delta\theta$-$\mathcal{I}$-continuous functions are defined and characterized. Also, we find the relationship with another types of continuous functions that are defined in terms of the operators $Cl^\star$ and $\delta Cl^\star$.

The following lemma extends the results given in [8, Theorem 17], using the operator $\delta Cl^\star$.

Lemma 3.1. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then, $\mathcal{I}$ is codense if and only if $\delta Cl(U) = Cl(U) = U^{\delta\star} = U^\star = \delta Cl^\star(U) = Cl^\star(U)$ for all open set U in X.*

Proof. Suppose that $\mathcal{I}$ is codense, then from [8, Theorem 17] and Lemma 2.2, for all open subset U in X, we obtain that $Cl(U) = Cl(U^\star) = U^\star = Cl^\star(U) \subset \delta Cl^\star(U) \subset \delta Cl(U)$ and also $Cl(U) = Cl(U^\star) = U^\star \subset U^{\delta\star} \subset \delta Cl^\star(U) \subset \delta Cl(U)$. Now, by Lemma 2.1, $\delta Cl(U) = Cl(U)$ for all open subset U in X. In consequence, $\delta Cl(U) = Cl(U) = U^{\delta\star} = U^\star = \delta Cl^\star(U) = Cl^\star(U)$ for all open set U in X. Conversely, if $\delta Cl(U) = Cl(U) = U^{\delta\star} = U^\star = \delta Cl^\star(U) = Cl^\star(U)$ for all open set U in X, again using [8, Theorem 17], conclude that $\mathcal{I}$ is codense. □

Definition 3.1. A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is said to be $\delta\theta$-$\mathcal{I}$-continuous (resp. weakly δ-$\mathcal{J}$-continuous, strongly $\delta\theta$-$\mathcal{I}$-continuous), if for each $x \in X$ and each open set V in Y containing $f(x)$, there exists an open set U containing x such that $f(\delta Cl^\star(U)) \subset \delta Cl^\star(V)$ (resp. $f(U) \subset \delta Cl^\star(V)$, $f(\delta Cl^\star(U)) \subset V$).

The following theorem give to us, the relations between weakly $\mathcal{J}$-continuous, weakly δ-$\mathcal{J}$-continuous, weakly continuous, $\delta\theta$-$\mathcal{I}$-continuous, strongly $\delta\theta$-$\mathcal{I}$-continuous and strongly θ-$\mathcal{I}$-continuous functions.

Theorem 3.1. *For a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$, the following properties hold.*

(1) *Every weakly $\mathcal{J}$-continuous function is weakly δ-$\mathcal{J}$-continuous.*

(2) *Every weakly δ-$\mathcal{J}$-continuous function is weakly continuous.*

(3) *Every $\delta\theta$-$\mathcal{I}$-continuous function is weakly δ-$\mathcal{J}$-continuous.*

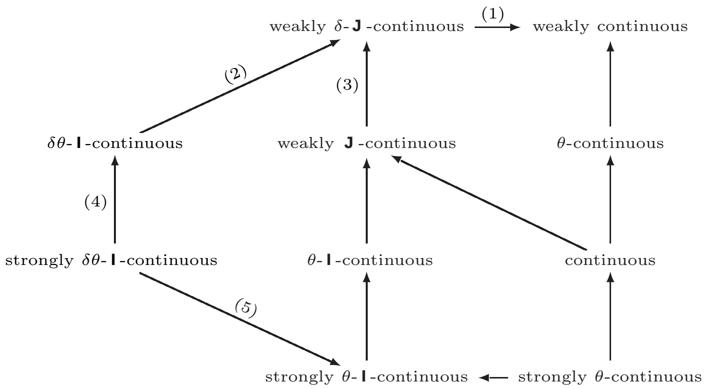
(4) *Every strongly $\delta\theta$-$\mathcal{I}$-continuous function is $\delta\theta$-$\mathcal{I}$-continuous.*

(5) *Every strongly $\delta\theta$-$\mathcal{I}$-continuous function is strongly θ-$\mathcal{I}$-continuous.*

Proof. (1) Let f be a weakly $\mathcal{J}$-continuous function. Then for all $x \in X$ and each open set V of Y containing $f(x)$, there exists an open set U containing x such that $f(U) \subset Cl^\star(V)$. From Lemma 2.2, $Cl^\star(V) \subset \delta Cl^\star(V)$ for all subset V of Y. Thus, $f(U) \subset Cl^\star(V) \subset \delta Cl^\star(V)$. Therefore, $f(U) \subset \delta Cl^\star(V)$ and so f is weakly δ-$\mathcal{J}$-continuous.

The proof of (2)-(5) are similar to (1). □

Remark 3.1. The following diagram shows the relationship between the functions in the diagram in Remark 2.1 and the functions defined in Definition 3.1:



The following examples show that in general none of the implications (1) – (5) is reversible.

Example 3.1. A weakly continuous function need not be weakly δ-$\mathcal{J}$-continuous. Let $X = \{a, b\}$, $\tau = \{X, \emptyset\}$ and $Y = \{1, 2, 3, 4\}$, $\sigma = \{Y, \emptyset, \{1\}, \{2, 3\}, \{1, 2, 3\}\}$ with $\mathcal{J} = \{\emptyset, \{1\}\}$. Define a function $f : (X, \tau) \to (Y, \sigma, \mathcal{J})$ as $f = \{(a, 1), (b, 4)\}$. Observe that is easy to see that f is weakly continuous.

On the other hand, $V_1 = \{1\}$ is an open set in Y containing $f(a) = 1$, but $f(U) = f(X) = f(\{a, b\}) = \{1, 4\} \not\subset \{1\} = \delta Cl^\star(V_1)$. Therefore, f is not weakly δ-$\mathcal{J}$-continuous.

Example 3.2. A weakly δ-$\mathcal{J}$-continuous function need not be $\delta\theta$-$\mathcal{I}$-continuous. Let $X = \{1, 2, 3, 4\}$, $\tau = \{X, \emptyset, \{1, 2, 3\}, \{3\}, \{3, 4\}\}$ with $\mathcal{I} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ and $Y = \{a, b, c, d\}$, $\sigma = \{Y, \emptyset, \{a, b\}, \{b\}, \{d\}, \{b, d\}, \{a, b, d\}, \{b, c, d\}\}$ with $\mathcal{J} = \{\emptyset, \{c\}\}$. Define a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ as $f = \{(1, a), (2, b), (3, c), (4, d)\}$. Note that:

(1) The only open sets in Y containing $f(1) = a$ are $V_1 = \{a, b\}$, $V_2 = \{a, b, d\}$ and $V_3 = Y$. In addition, $U_1 = \{1, 2, 3\}$ is an open set in X containing 1 such that $f(U_1) = f(\{1, 2, 3\}) = \{a, b, c\} = Cl^\star(\{a, b\}) = Cl^\star(V_1)$, $f(U_1) = \{a, b, c\} \subset Y = Cl^\star(\{a, b, d\}) = Cl^\star(V_2)$ and $f(U_1) = \{a, b, c\} \subset Y = Cl^\star(Y) = Cl^\star(V_3)$.

(2) The only open sets in Y containing $f(2) = b$ are $V_1 = \{a, b\}$, $V_2 = \{a, b, d\}$, $V_3 = Y$, $V_4 = \{b\}$, $V_5 = \{b, d\}$ and $V_6 = \{b, c, d\}$. In addition, $U_1 = \{1, 2, 3\}$ is an open set in X containing 2 such that $f(U_1) = \{a, b, c\} = Cl^\star(V_1)$, $f(U_1) = \{a, b, c\} \subset Y = Cl^\star(V_2)$, $f(U_1) = \{a, b, c\} \subset Y = Cl^\star(V_3)$, $f(U_1) = \{a, b, c\} = Cl^\star(\{b\}) = Cl^\star(V_4)$, $f(U_1) = \{a, b, c\} \subset Y = Cl^\star(\{b, d\}) = Cl^\star(V_5)$ and $f(U_1) = \{a, b, c\} \subset Y = Cl^\star(\{b, c, d\}) = Cl^\star(V_6)$.

(3) The only open sets in Y containing $f(3) = c$ are $V_3 = Y$ and $V_6 = \{b, c, d\}$. In addition, $U_1 = \{1, 2, 3\}$ is an open set in X containing 3 such that $f(U_1) = \{a, b, c\} \subset Y = Cl^\star(V_3)$ and $f(U_1) = \{a, b, c\} \subset Y = Cl^\star(V_6)$.

(4) The only open sets in Y containing $f(4) = d$ are $V_2 = \{a, b, d\}$, $V_3 = Y$, $V_5 = \{b, d\}$, $V_6 = \{b, c, d\}$ and $V_7 = \{d\}$. In addition, $U_2 = \{3, 4\}$ is an open set in X containing 4 such that $f(U_2) = f(\{3, 4\}) = \{c, d\} \subset Y = Cl^\star(V_2)$, $f(U_2) = \{c, d\} \subset Y = Cl^\star(V_3)$, $f(U_2) = \{c, d\} \subset Y = Cl^\star(V_5)$ and $f(U_2) = \{c, d\} \subset Y = Cl^\star(V_6)$.

By (1)-(4) f is weakly $\mathcal{J}$-continuous and by Theorem 3.1, it is weakly δ-$\mathcal{J}$-continuous. Now, we show that f is not $\delta\theta$-$\mathcal{I}$-continuous. Indeed, the only open sets in X containing $f(a) = 1$ are $U_1 = X$ and $U_2 = \{1, 2, 3\}$. In addition, V_1 is an open set in Y such that $f(\delta Cl^\star(U_1)) = f(\delta Cl^\star(X)) = f(X) = Y \not\subset \{a, b, c\} = \delta Cl^\star(V_1)$ and $f(\delta Cl^\star(U_2)) = f(\delta Cl^\star(\{1, 2, 3\})) = f(X) = Y \not\subset \{a, b, c\} = \delta Cl^\star(V_1)$. Therefore, f is not $\delta\theta$-$\mathcal{I}$-continuous.

Example 3.3. A weakly δ-$\mathcal{J}$-continuous function need not be weakly $\mathcal{J}$-continuous. Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{b\}, \{b, c\}\}$ and $Y = \{1, 2, 3, 4\}$, $\sigma = \{Y, \emptyset, \{1, 2, 3\}, \{2\}, \{2, 4\}\}$ with $\mathcal{J} = \{\emptyset, \{2\}, \{4\}, \{2, 4\}\}$. We define a function $f : (X, \tau) \to (Y, \sigma, \mathcal{J})$ as $f = \{(a, 1), (b, 2), (c, 4)\}$. Note that:

(1) The only open sets in Y containing $f(a) = 1$ are $V_1 = \{1, 2, 3\}$ and $V_2 = Y$. In addition, $U_1 = X$ is an open set in X containing a such that $f(U_1) = f(X) = \{1, 2, 4\} \subset Y = \delta Cl^\star(\{1, 2, 3\}) = \delta Cl^\star(V_1)$ and $f(U_1) = \{1, 2, 4\} \subset Y = \delta Cl^\star(Y) = \delta Cl^\star(V_2)$.

(2) The only open sets in Y containing $f(b) = 2$ are $V_1 = \{1, 2, 3\}$, $V_2 = Y$, $V_3 = \{2\}$ and $V_4 = \{2, 4\}$. In addition, $U_2 = \{b\}$ is an open set

in X containing b such that $f(U_2) = f(\{b\}) = \{2\} \subset Y = \delta Cl^\star(V_1)$, $f(U_2) = \{2\} \subset Y = \delta Cl^\star(V_2)$, $f(U_2) = \{2\} \subset \delta Cl^\star(\{2\}) = \delta Cl^\star(V_3)$ and $f(U_2) = \{2\} \subset \{2,4\} = \delta Cl^\star(\{2,4\}) = \delta Cl^\star(V_4)$.

(3) The only open sets in Y containing $f(c) = 4$ are $V_2 = Y$ and $V_4 = \{2,4\}$. In addition, $U_3 = \{b,c\}$ is an open set in X containing c such that $f(U_3) = f(\{b,c\}) = \{2,4\} \subset Y = \delta Cl^\star(V_2)$ and $f(U_3) = \{2,4\} = \delta Cl^\star(V_4)$.

From (1)-(3), we conclude that f is weakly δ-$\mathcal{J}$-continuous. On the other hand, since the only open set in X containing a is $U_1 = X$ and also $V_1 = \{1,2,3\}$ is an open set in Y containing $f(a) = 1$, we have that $f(U_1) = \{1,2,4\} \not\subset \{1,2,3\} = Cl^\star(\{1,2,3\}) = Cl^\star(V_1)$. Therefore, f is not weakly $\mathcal{J}$-continuous.

Example 3.4. A $\delta\theta$-$\mathcal{I}$-continuous function need not be strongly $\delta\theta$-$\mathcal{I}$-continuous. Let $X = \{a,b,c,d,e\}$, $\tau = \{X, \emptyset, \{a,b\}, \{a,b,e\}, \{e\}\}$ with $\mathcal{I} = \{\emptyset, \{e\}\}$ and $Y = \{1,2,3,4,5\}$, $\sigma = \{Y, \emptyset, \{1,2\}, \{1,2,5\}, \{5\}\}$ with $\mathcal{J} = \{\emptyset, \{5\}\}$. Define a function $f : (X,\tau,\mathcal{I}) \to (Y,\sigma,\mathcal{J})$ as $f = \{(a,1),(b,2),(c,3),(d,4),(e,5)\}$. Note that:

(1) The only open sets in Y containing $f(a) = 1$ are $V_1 = \{1,2\}$, $V_2 = \{1,2,5\}$ and $V_3 = Y$. In addition, $U_1 = X$, $U_2 = \{a,b\}$ and $U_3 = \{a,b,e\}$ are open sets in X containing a such that $f(\delta Cl^\star(U_1)) = f(\delta Cl^\star(X)) = f(X) = Y = \delta Cl^\star(Y) = \delta Cl^\star(V_3)$, $f(\delta Cl^\star(U_2)) = f(\delta Cl^\star(\{a,b\})) = f(\{a,b,c,d\}) = \{1,2,3,4\} = \delta Cl^\star(\{1,2\}) = \delta Cl^\star(V_1)$ and $f(\delta Cl^\star(U_3)) = f(\delta Cl^\star(\{a,b,e\})) = f(X) = Y = \delta Cl^\star(\{1,2,5\}) = \delta Cl^\star(V_2)$.

(2) Using the sets U_1, U_2, U_3, V_1, V_2 and V_3 and making the same calculations that in (1), the result is obtained for the point $f(b) = 2$.

(3) The only open set in Y containing the points $f(c) = 3$ and $f(d) = 4$ is $V_3 = Y$. In addition, $U_1 = X$ is an open set in X containing the points c and d such that $f(\delta Cl^\star(U_1)) = f(\delta Cl^\star(X)) = f(X) = Y = \delta Cl^\star(V_3)$.

(4) The only open sets in Y containing the point $f(e) = 5$ are $V_4 = \{5\}$ and $V_3 = Y$. In addition, $U_1 = X$ and $U_4 = \{e\}$ are open sets in X containing e such that $f(\delta Cl^\star(U_1)) = f(\delta Cl^\star(X)) = f(X) = Y = \delta Cl^\star(V_3)$ and $f(\delta Cl^\star(U_4)) = f(\delta Cl^\star(\{e\})) = f(\{e\}) = \{5\} = \delta Cl^\star(\{5\}) = \delta Cl^\star(V_4)$.

By (1)-(4), f is $\delta\theta$-$\mathcal{I}$-continuous. On the other hand, the only open sets in X containing a are $U_1 = X$, $U_2 = \{a,b\}$ and $U_3 = \{a,b,e\}$. In addition, $V_1 = \{1,2\}$ is an open set in Y containing $f(a) = 1$ such that $f(\delta Cl^\star(U_1)) = f(\delta Cl^\star(X)) = f(X) = Y \not\subset \{1,2\} = V_1$, $f(\delta Cl^\star(U_2)) = f(\delta Cl^\star(\{a,b\})) =$

$f(\{a,b,c,d\}) = \{1,2,3,4\} \not\subset \{1,2\} = V_1$ and $f(\delta Cl^\star(U_3)) = f(\delta Cl^\star(\{a,b,e\})) = f(X) = Y \not\subset \{1,2\} = V_1$. Therefore, f is not strongly $\delta\theta$-$\mathcal{I}$-continuous.

Example 3.5. A strongly θ-$\mathcal{I}$-continuous function need not be strongly $\delta\theta$-$\mathcal{I}$-continuous. Let $X = \{a,b,c,d\}$, $\tau = \{\emptyset, X, \{b\}, \{a,b\}, \{b,c\}, \{a,b,c\}, \{a,b,d\}\}$ with $\mathcal{I} = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$ and $Y = \{1,2,3,4\}$, $\sigma = \{\emptyset, Y, \{2\}, \{1,2\}, \{2,3\}, \{1,2,3\}, \{1,2,4\}\}$. Define a function $f : (X,\tau,\mathcal{I}) \to (Y,\sigma)$ as $f = \{(a,1),(b,2),(c,3),(d,4)\}$. Using an argument similar as in the above examples, we conclude that f is strongly θ-$\mathcal{I}$-continuous. On the other hand, the only open sets in X containing d are $U_4 = \{a,b,d\}$ and $U_5 = X$. In addition, $V_4 = \{1,2,4\}$ is an open set in Y containing $f(d) = 4$ such that $f(\delta Cl^\star(U_4)) = f(\delta Cl^\star(\{a,b,d\})) = f(X) = Y \not\subset \{1,2,4\} = V_4$ and $f(\delta Cl^\star(U_5)) = f(\delta Cl^\star(X)) = f(X) = Y \not\subset \{1,2,4\} = V_4$. Therefore, f is not strongly $\delta\theta$-$\mathcal{I}$-continuous.

Theorem 3.2. *For a function $f : (X,\tau,\mathcal{I}) \to (Y,\sigma,\mathcal{J})$, the following implications:* (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4) *hold. Moreover, the implication* (4) $\Rightarrow$ (1) *holds if $\mathcal{J}$ is codense.*

(1) *f is $\delta\theta$-$\mathcal{I}$-continuous.*

(2) $f^{-1}(V) \subset \delta Int^\star_\theta(f^{-1}(\delta Cl^\star(V)))$ *for every open set V of Y.*

(3) $\delta Cl^\star_\theta(f^{-1}(V)) \subset f^{-1}(Cl(V))$ *for every open set V of Y.*

(4) *For each $x \in X$ and each open set V of Y containing $f(x)$, there exists an open set U of X containing x such that $f(\delta Cl^\star(U)) \subset Cl(V)$.*

Proof. (1)$\Rightarrow$(2). Suppose that V is any open set of Y and $x \in f^{-1}(V)$, then $f(x) \in V$. By hypothesis f is $\delta\theta$-$\mathcal{I}$-continuous, thus there exists an open set U in X containing x such that $f(\delta Cl^\star(U) \subset \delta Cl^\star(V))$. Therefore, $x \in U \subset \delta Cl^\star(U) \subset f^{-1}(\delta Cl^\star(V))$. This show that $x \in \delta Int^\star_\theta(f^{-1}(\delta Cl^\star(V)))$. Consequently, $f^{-1}(V) \subset \delta Int^\star_\theta(f^{-1}(\delta Cl^\star(V)))$.

(2)$\Rightarrow$(1). Let $x \in X$ and $V \in \sigma$ containing $f(x)$. Then $x \in f^{-1}(V)$ and by (2), $x \in \delta Int^\star_\theta(f^{-1}(\delta Cl^\star(V)))$. Thus, there exists an open set U in X such that $x \in U \subset \delta Cl^\star(U) \subset f^{-1}(\delta Cl^\star(V))$ and consequently $f(\delta Cl^\star(U)) \subset \delta Cl^\star(V)$. Therefore, f is $\delta\theta$-$\mathcal{I}$-continuous.

(2)$\Rightarrow$(3). Suppose that V is an open set in Y and $x \notin f^{-1}(Cl(V))$. Then $f(x) \notin Cl(V)$ and so there exists an open set W in Y containing $f(x)$ such that $W \cap V = \emptyset$. Thus, $\delta Cl^\star(W) \cap V \subset \delta Cl(W) \cap V \subset Cl(W) \cap V = \emptyset$, that is, $\delta Cl^\star(W) \cap V = \emptyset$ and $f^{-1}(\delta Cl^\star(W) \cap V) = \emptyset$, which means that $f^{-1}(\delta Cl^\star(W)) \cap f^{-1}(V) = \emptyset$. Since $x \in f^{-1}(W)$, by (2) we have $x \in \delta Int^\star_\theta(f^{-1}(\delta Cl^\star(W)))$, so there exists an open set U containing x such that $x \in U \subset \delta Cl^\star(U) \subset f^{-1}(\delta Cl^\star(W))$, this is $\delta Cl^\star(U) \subset f^{-1}(\delta Cl^\star(W))$ and so $\delta Cl^\star(U) \cap f^{-1}(V) = \emptyset$. This shows that $x \notin \delta Cl^\star_\theta(f^{-1}(V))$. Therefore, $\delta Cl^\star_\theta(f^{-1}(V)) \subset f^{-1}(Cl(V))$.

(3)⇒(4). Suppose that $x \in X$ and V is any open set in Y containing $f(x)$. Then $V \cap (Y - Cl(V)) = \emptyset$ and $f(x) \notin Cl(Y - Cl(V))$. Thus, $x \notin f^{-1}(Cl(Y - Cl(V)))$ and by (3), $x \notin \delta Cl^\star_\theta(f^{-1}(Y - Cl(V)))$, hence there exists an open set U in X containing x such that $\delta Cl^\star(U) \cap f^{-1}(Y - Cl(V)) = \emptyset$, which implies that $f(\delta Cl^\star(U)) \cap (Y - Cl(V)) = \emptyset$. Therefore, $f(\delta Cl^\star(U)) \subset Cl(V)$.

(4)⇒(3). Let V be any open set in Y and suppose that $x \notin f^{-1}(Cl(V))$. Then $f(x) \notin Cl(V)$ and so there exists an open set W in Y containing $f(x)$ such that $W \cap V = \emptyset$. By (4), there exists an open set U in X containing x such that $f(\delta Cl^\star(U)) \subset Cl(W)$. Since V is an open set in Y, $Cl(W) \cap V = \emptyset$ and $f(\delta Cl^\star(U)) \cap V \subset Cl(W) \cap V = \emptyset$, which implies that $\delta Cl^\star(U) \cap f^{-1}(V) = \emptyset$. This show that $x \notin \delta Cl^\star_\theta(f^{-1}(V))$. Therefore, $\delta Cl^\star_\theta(f^{-1}(V)) \subset f^{-1}(Cl(V))$.

(4)⇒(1). Suppose that $\mathcal{J}$ is codense. Then by Lemma 3.1, $Cl(V) = \delta Cl^\star(V)$ for all open subset V in Y. By (4), for each $x \in X$ and each open set V in Y containing $f(x)$, there exists an open set U in X containing x such that $f(\delta Cl^\star(U)) \subset Cl(V) = \delta Cl^\star(V)$ and consequently $f(\delta Cl^\star(U)) \subset \delta Cl^\star(V)$. Therefore, f is $\delta\theta$-$\mathcal{I}$-continuous. □

Remark 3.2. The function given in Example 3.1, satisfy the condition (4) of the Theorem 3.2, but it is not $\delta\theta$-$\mathcal{I}$-continuous.

Now, we characterize the $\delta\theta$-$\mathcal{I}$-continuous functions in terms of their direct and inverse images.

Theorem 3.3. *For a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$, the following properties are equivalent:*

(1) *f is $\delta\theta$-$\mathcal{I}$-continuous.*

(2) *$\delta Cl^\star_\theta(f^{-1}(B)) \subset f^{-1}(\delta Cl^\star_\theta(B))$ for all subset B of Y.*

(3) *$f(\delta Cl^\star_\theta(A)) \subset \delta Cl^\star_\theta(f(A))$ for all subset A of X.*

Proof. (1)⇒(2). Let B be any subset of Y and suppose that $x \notin f^{-1}(\delta Cl^\star_\theta(B))$. Then $f(x) \notin \delta Cl^\star_\theta(B)$ and so there exists an open set V in Y containing $f(x)$ such that $\delta Cl^\star(V) \cap B = \emptyset$. Since f is $\delta\theta$-$\mathcal{I}$-continuous, there exists an open set U in X containing x such that $f(\delta Cl^\star(U)) \subset \delta Cl^\star(V)$. Consequently, $f(\delta Cl^\star(U)) \cap B \subset \delta Cl^\star(V) \cap B = \emptyset$ and $f(\delta Cl^\star(U)) \cap B = \emptyset$, which implies that $\delta Cl^\star(U) \cap f^{-1}(B) = \emptyset$. Therefore, $x \notin \delta Cl^\star_\theta(f^{-1}(B))$ and so $\delta Cl^\star_\theta(f^{-1}(B)) \subset f^{-1}(\delta Cl^\star_\theta(B))$ for all subset B of Y.

(2)⇒(1). Let $x \in X$ and V be any open set of Y containing $f(x)$. Then we have $\delta Cl^\star(V) \cap (Y - \delta Cl^\star(V)) = \emptyset$ and so $f(x) \notin \delta Cl^\star_\theta(Y - \delta Cl^\star(V))$. Therefore, $x \notin f^{-1}(\delta Cl^\star_\theta(Y - \delta Cl^\star(V)))$ and by (2), we have $x \notin \delta Cl^\star_\theta(f^{-1}(Y - \delta Cl^\star(V)))$. This show that there exists an open set U in X containing x such that $\delta Cl^\star(U) \cap f^{-1}(Y - \delta Cl^\star(V)) = \emptyset$, so $\delta Cl^\star(U)) \subset X - f^{-1}(Y - \delta Cl^\star(V)) = X - (X - f^{-1}(\delta Cl^\star(V))) = f^{-1}(\delta Cl^\star(V))$, which implies that $f(\delta Cl^\star(U)) \subset \delta Cl^\star(V)$. Consequently, f is $\delta\theta$-$\mathcal{I}$-continuous.

(2)⇒(3). For each subset A of X,

$$\delta Cl^\star_\theta(A) \subset \delta Cl^\star_\theta(f^{-1}(f(A))) \subset f^{-1}(\delta Cl^\star_\theta(f(A)))$$

and hence $f(\delta Cl^\star_\theta(A)) \subset \delta Cl^\star_\theta(f(A))$.

(3)⇒(2). Let B be any subset of Y. Then we have

$$f(\delta Cl^\star_\theta(f^{-1}(B)) \subset \delta Cl^\star_\theta(f(f^{-1}(B))) \subset \delta Cl^\star_\theta(B)$$

and hence $\delta Cl^\star_\theta(f^{-1}(B)) \subset f^{-1}(\delta Cl^\star_\theta(B))$. □

4. Some Properties of δθ-$\mathcal{I}$-Continuous Functions

In this section, we looking for some topological conditions in order to find the relationship between the $\delta\theta$-$\mathcal{I}$-continuous, strongly $\delta\theta$-$\mathcal{I}$-continuous and weakly δ-$\mathcal{J}$-continuous functions.

Definition 4.1. An ideal topological space $(X, \tau, \mathcal{I})$ is said to be $\delta\star$-Urysohn if for each pair of distinct points x and y in X, there exist two open subsets U and V of X containing x and y, respectively, such that $\delta Cl^\star(U) \cap \delta Cl^\star(V) = \emptyset$.

Example 4.1. A $\delta\star$-Urysohn space. Let $X = \{a, b\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}\}$ with $\mathcal{I} = \{\emptyset, \{a\}\}$. Note that: $a \in \{a\}$, $b \in \{b\}$, $\delta Cl^\star(\{a\}) = \{a\}$, $\delta Cl^\star(\{b\}) = \{b\}$ and $\delta Cl^\star(\{a\}) \cap \delta Cl^\star(\{b\}) = \emptyset$. Therefore, $(X, \tau, \mathcal{I})$ is a $\delta\star$-Urysohn space.

Proposition 4.1. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space.*

(1) *If $(X, \tau, \mathcal{I})$ is an Urysohn space, then it is $\delta\star$-Urysohn.*

(2) *If $(X, \tau, \mathcal{I})$ is a $\delta\star$-Urysohn space, then it is Hausdorff.*

Proof. (1) Let $(X, \tau, \mathcal{I})$ be an Urysohn space. Then for each pair of distinct points $x, y \in X$, there exist two open sets U and V such that $x \in U$, $y \in V$ and $Cl(U) \cap Cl(V) = \emptyset$. By Lemma 2.1 and Lemma 2.2, we have $\delta Cl^\star(U) \cap \delta Cl^\star(V) \subset \delta Cl(U) \cap \delta Cl(V) = Cl(U) \cap Cl(V) = \emptyset$. Consequently, $\delta Cl^\star(U) \cap \delta Cl^\star(V) = \emptyset$. Therefore, $(X, \tau, \mathcal{I})$ is a $\delta\star$-Urysohn space.

(2) The proof is similar to (1). □

Theorem 4.1. *If $f, g : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ are $\delta\theta$-$\mathcal{I}$-continuous functions and $(Y, \sigma, \mathcal{J})$ is a $\delta\star$-Urysohn space, then the set $A = \{x \in X : f(x) = g(x)\}$ is a $\delta\theta$-$\mathcal{I}$-closed subset of $(X, \tau, \mathcal{I})$.*

Proof. We prove that $X - A$ is a $\delta\theta$-$\mathcal{I}$-open set. Let $x \in X - A$. Then $f(x) \neq g(x)$ and as by hypothesis $(Y, \sigma, \mathcal{J})$ is a $\delta\star$-Urysohn space, there exist two open sets V_1 and V_2 such that $f(x) \in V_1$, $g(x) \in V_2$ and $\delta Cl^\star(V_1) \cap \delta Cl^\star(V_2) = \emptyset$. Since f and g are $\delta\theta$-$\mathcal{I}$-continuous functions, there exist two open sets U_1, U_2 in X containing x such that $\delta Cl^\star(U_1) \subset f^{-1}(f(\delta Cl^\star(U_1))) \subset f^{-1}(\delta Cl^\star(V_1))$ and $\delta Cl^\star(U_2) \subset g^{-1}(g(\delta Cl^\star(U_2))) \subset g^{-1}(\delta Cl^\star(V_2))$. Let $U = U_1 \cap U_2$. Then, U is an open set in X such that $x \in U \subset \delta Cl^\star(U) \subset \delta Cl^\star(U_1) \cap \delta Cl^\star(U_2) \subset f^{-1}(\delta Cl^\star(V_1)) \cap g^{-1}(\delta Cl^\star(V_2)) \subset X - A$. This shows that $X - A$ is a $\delta\theta$-$\mathcal{I}$-open set and hence, A is a $\delta\theta$-$\mathcal{I}$-closed set. □

Follows from Remark 2.1, that the notions of θ-continuous, θ-$\mathcal{I}$-continuous and $\delta\theta$-$\mathcal{I}$-continuous functions are independent. But, if consider some additional conditions on the ideals, we obtain that all of them are equivalent.

Proposition 4.2. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be a function. If $\mathcal{I}$ and $\mathcal{J}$ are codense, then the following properties are equivalent:*

(1) *f is $\delta\theta$-$\mathcal{I}$-continuous.*

(2) *f is θ-$\mathcal{I}$-continuous.*

(3) *f is θ-continuous.*

Proof. Follows from Definitions 2.1, 2.2, 3.1 and Lemma 3.1. □

Similarly, to the proposition 4.2, we have the following two results for weakly δ-$\mathcal{J}$-continuous and strongly $\delta\theta$-$\mathcal{I}$-continuous functions, respectively.

Proposition 4.3. *Let $f : (X, \tau) \to (Y, \sigma, \mathcal{J})$ be a function. If $\mathcal{J}$ is codense, then the following properties are equivalent:*

(1) *f is weakly δ-$\mathcal{J}$-continuous.*

(2) *f is weakly $\mathcal{J}$-continuous.*

(3) *f is weakly continuous.*

Proposition 4.4. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a function. If $\mathcal{I}$ is codense, then the following properties are equivalent:*

(1) *f is strongly $\delta\theta$-$\mathcal{I}$-continuous.*

(2) *f is strongly θ-$\mathcal{I}$-continuous.*

(3) *f is strongly θ-continuous.*

The weakly δ-$\mathcal{J}$-continuous functions can be characterized in terms of inverse images of open sets, as we shown in the following result.

Lemma 4.1. *Let $f : (X, \tau) \to (Y, \sigma, \mathcal{J})$ be a function. Then, f is weakly δ-$\mathcal{J}$-continuous if and only if for each open set V in Y, $f^{-1}(V) \subset Int(f^{-1}(\delta Cl^{\star}(V)))$.*

Proof. Let V be any open set in Y and suppose that $x \in f^{-1}(V)$. Then $f(x) \in V$ and since f is weakly δ-$\mathcal{J}$-continuous, there exists an open set U in X containing x such that $f(U) \subset \delta Cl^{\star}(V)$, which implies that $x \in U \subset f^{-1}(f(U)) \subset f^{-1}(\delta Cl^{\star}(V))$. Therefore, $x \in Int(f^{-1}(\delta Cl^{\star}(V)))$. Conversely, let $x \in X$ and V be an open set in Y such that $f(x) \in V$. Then by hypothesis $x \in f^{-1}(V) \subset Int(f^{-1}(\delta Cl^{\star}(V)))$ and so there exists an open set U in X such that $x \in U \subset f^{-1}(\delta Cl^{\star}(V))$, which implies that $f(U) \subset f(f^{-1}(\delta Cl^{\star}(V)) \subset \delta Cl^{\star}(V)$ and hence $f(U) \subset \delta Cl^{\star}(V)$. Therefore, f is weakly δ-$\mathcal{J}$-continuous. □

Definition 4.2. An ideal topological space $(X, \tau, \mathcal{I})$ is said to be $\delta\star$-extremally disconnected, if $\delta Cl^{\star}(A)$ is an open subset of X for every open subset A of X.

Remark 4.1. The ideal topological space $(X, \tau, \mathcal{I})$ considered in the Example 4.1 is $\delta\star$-extremally disconnected.

Theorem 4.2. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be a function, where $(Y, \sigma, \mathcal{J})$ is a $\delta\star$-extremally disconnected space and $\delta Cl^{\star}(f^{-1}(W)) \subset f^{-1}(\delta Cl^{\star}(W))$ for every open set W in Y. Then, f is $\delta\theta$-$\mathcal{I}$-continuous if and only if it is weakly δ-$\mathcal{J}$-continuous.*

Proof. By Theorem 3.1, every $\delta\theta$-$\mathcal{I}$-continuous function is weakly δ-$\mathcal{J}$-continuous. Conversely, let $x \in X$ and V be an open set in Y such that $f(x) \in V$. Since f is weakly δ-$\mathcal{J}$-continuous, by Lemma 4.1 $x \in f^{-1}(V) \subset$

$Int(f^{-1}(\delta Cl^{\star}(V)))$. Now, let $U = Int(f^{-1}(\delta Cl^{\star}(V)))$. Since $(Y, \sigma, \mathcal{J})$ is a $\delta\star$-extremally disconnected space and $\delta Cl^{\star}(f^{-1}(W)) \subset f^{-1}(\delta Cl^{\star}(W))$ for every open set W in Y, then

$$\begin{aligned} f(\delta Cl^{\star}(U)) &= f(\delta Cl^{\star}(Int(f^{-1}(\delta Cl^{\star}(V))))) \\ &\subset f(\delta Cl^{\star}(f^{-1}(\delta Cl^{\star}(V)))) \\ &\subset f(f^{-1}(\delta Cl^{\star}(\delta Cl^{\star}(V)))) \\ &\subset \delta Cl^{\star}(\delta Cl^{\star}(V)) \\ &\subset \delta Cl^{\star}(V). \end{aligned}$$

Therefore, $f(\delta Cl^{\star}(U)) \subset \delta Cl^{\star}(V)$ and so f is $\delta\theta$-$\mathcal{I}$-continuous. □

Definition 4.3. An ideal topological space $(X, \tau, \mathcal{I})$ is said to be $\delta\star$-regular if for each closed subset F of X and each point $x \notin F$, there exist $V \in \tau$ and $U \in \tau^{\delta\star}$ such that $x \in V$, $F \subset U$ and $U \cap V = \emptyset$.

Example 4.2. A $\delta\star$-regular space. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{a, b\}, \{a, d\}, \{a, b, d\}\}$ with $\mathcal{I} = \{\emptyset, \{a\}, \{b\}, \{d\}, \{a, b\}, \{a, d\}, \{b, d\}, \{a, b, d\}\}$. Note that the τ-closed sets are $\emptyset$, X, $\{b, c, d\}$, $\{c, d\}$, $\{c\}$ and $\{b, c\}$, also $\tau_\delta = \{\emptyset, X\}$. In addition, $\tau^{\delta\star} = \{\emptyset, X, \{c\}, \{a, c\}, \{c, d\}, \{b, c\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}\}$ and so we have:

(1) For $F_1 = \{b, c, d\}$ and $a \in X - F_1$, there exist $V_1 = \{a\} \in \tau$ and $U_1 = \{b, c, d\} \in \tau^{\delta\star}$ such that $a \in V_1$, $F_1 \subset U_1$ and $U_1 \cap V_2 = \emptyset$.

(2) For $F_2 = \{c, d\}$ and $a \in X - F_2$, there exist $V_1 = \{a\} \in \tau$ and $U_2 = \{c, d\} \in \tau^{\delta\star}$ such that $a \in V_1$, $F_2 \subset U_2$ and $U_2 \cap V_1 = \emptyset$.

(3) For $F_2 = \{c, d\}$ and $b \in X - F_2$, there exist $V_2 = \{a, b\} \in \tau$ and $U_2 = \{c, d\} \in \tau^{\delta\star}$ such that $b \in V$, $F_2 \subset U_2$ and $U_2 \cap V_2 = \emptyset$.

(4) For $F_3 = \{b, c\}$ and $a \in X - F_3$, there exist $V_1 = \{a\} \in \tau$ and $U_3 = \{b, c\} \in \tau^{\delta\star}$ such that $a \in V_1$, $F_3 \subset U_3$ and $U_3 \cap V_1 = \emptyset$.

(5) For $F_3 = \{b, c\}$ and $d \in X - F_3$, there exist $V_3 = \{a, d\} \in \tau$ and $U_3 = \{b, c\} \in \tau^{\delta\star}$ such that $d \in V_3$, $F_3 \subset U_3$ and $U_3 \cap V_3 = \emptyset$.

(6) For $F_4 = \{c\}$ and $a \in X - F_4$, there exist $V_1 = \{a\} \in \tau$ and $U_2 = \{c, d\} \in \tau^{\delta\star}$ such that $a \in V_1$, $F_4 \subset U_2$ and $U_2 \cap V_1 = \emptyset$.

(7) For $F_4 = \{c\}$ and $b \in X - F_4$, there exist $V_2 = \{a, b\} \in \tau$ and $U_4 = \{c\} \in \tau^{\delta\star}$ such that $b \in V_2$, $F_4 \subset U_4$ and $U_4 \cap V_2 = \emptyset$.

(8) For $F_4 = \{c\}$ and $d \in X - F_4$, there exist $V_3 = \{a, d\} \in \tau$ and $U_4 = \{c\} \in \tau^{\delta\star}$ such that $a \in V_3$, $F_4 \subset U_4$ and $U_4 \cap V_3 = \emptyset$.

By (1)-(8) , $(X, \tau, \mathcal{I})$ is a $\delta\star$-regular space.

Lemma 4.2. *A ideal topological space $(X, \tau, \mathcal{I})$ is $\delta\star$-regular if and only if for each $x \in X$ and each open set U containing x, there exists an open set V such that $x \in V \subset \delta Cl^\star(V) \subset U$.*

Proof. Let $x \in X$, $U \in \tau$ containing x and $F = X - U$. Then F is a closed set not containing x. Since $(X, \tau, \mathcal{I})$ is $\delta\star$-regular, there exist an open set V and a $\tau^{\delta\star}$-open set W such that $x \in V$, $F \subset W$ and $V \cap W = \emptyset$. Thus, $X - W$ is $\tau^{\delta\star}$-closed and $\delta Cl^\star(X - W) = X - W \subset X - F = U$. On the other hand, as $V \cap W = \emptyset$ it follows that $V \subset X - W = \delta Cl^\star(X - W)$ and so $x \in V \subset \delta Cl^\star(V) \subset \delta Cl^\star(\delta Cl^\star(X - W)) = \delta Cl^\star(X - W) \subset U$. Therefore, $x \in V \subset \delta Cl^\star(V) \subset U$. Conversely, let F be a closed set and $x \notin F$. Then $U = X - F$ is an open set containing x. Using hypothesis, there exists an open set V containing x such that $x \in V \subset \delta Cl^\star(V) \subset U = X - F$. Let $W = X - \delta Cl^\star(V)$, then $F \subset X - \delta Cl^\star(V) = W$, $W \in \tau^{\delta\star}$ and $V \cap W \subset \delta Cl^\star(V) \cap (X - \delta Cl^\star(V)) = \emptyset$, which implies that $x \in V$, $F \subset W$ and $V \cap W = \emptyset$. This shows that $(X, \tau, \mathcal{I})$ is $\delta\star$-regular. □

Theorem 4.3. *Let $(X, \tau, \mathcal{I})$ be a $\delta\star$-regular space. A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is $\delta\theta$-$\mathcal{I}$-continuous if and only if it is weakly δ-$\mathcal{J}$-continuous.*

Proof. By Theorem 3.1, every $\delta\theta$-$\mathcal{I}$-continuous function is weakly δ-$\mathcal{J}$-continuous. Conversely, let $x \in X$ and V be an open set in Y containing $f(x)$. Since f is weakly δ-$\mathcal{J}$-continuous, there exists an open set U in X containing x such that $f(U) \subset \delta Cl^\star(V)$. Now, since $(X, \tau, \mathcal{I})$ is a $\delta\star$-regular space, by Lemma 4.2 there exists an open set W in X such that $x \in W \subset \delta Cl^\star(W) \subset U$. Thus, $f(\delta Cl^\star(W)) \subset f(U) \subset \delta Cl^\star(V)$, which implies that $f(\delta Cl^\star(W)) \subset \delta Cl^\star(V)$. Therefore, f is $\delta\theta$-$\mathcal{I}$-continuous. □

The notions of pre-$\mathcal{I}$-open sets and pre-$\mathcal{I}$-continuous functions were introduced by J. Dontchev [2]. A subset A of $(X, \tau, \mathcal{I})$ is said to be pre-$\mathcal{I}$-open if $A \subset Int(Cl^\star(A))$. On the other hand, a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be pre-$\mathcal{I}$-continuous if the image inverse of any open set in Y, is a pre-$\mathcal{I}$-open set in X. The following result shows that under some conditions every pre-$\mathcal{I}$-continuous function is $\delta\theta$-$\mathcal{I}$-continuous.

Theorem 4.4. *If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is a pre-$\mathcal{I}$-continuous function and $\delta Cl^\star(f^{-1}(V)) \subset f^{-1}(\delta Cl^\star(V))$ for each open set V in Y, then f is $\delta\theta$-$\mathcal{I}$-continuous.*

Proof. Let $x \in X$ and V be any open set in Y containing $f(x)$. Since f is pre-$\mathcal{I}$-continuous, $f^{-1}(V)$ is a pre-$\mathcal{I}$-open set in X and hence $f^{-1}(V) \subset Int(Cl^\star(f^{-1}(V))) \subset Int(\delta Cl^\star(f^{-1}(V)))$, which implies that $x \in f^{-1}(V) \subset Int(\delta Cl^\star(f^{-1}(V)))$. Thus, there exists an open set U in X such that $x \in U \subset \delta Cl^\star(f^{-1}(V))$. Using the assumption that $\delta Cl^\star(f^{-1}(V)) \subset f^{-1}(\delta Cl^\star(V))$, we obtain $\delta Cl^\star(U) \subset f^{-1}(\delta Cl^\star(V))$ and consequently $f(\delta Cl^\star(U)) \subset \delta Cl^\star(V)$. Therefore, f is $\delta\theta$-$\mathcal{I}$-continuous. □

5. Preservation of Topological Notions

In this section, using the notions of δ-local function and $\delta\theta$-$\mathcal{I}$-open set, we introduce some topological notions to study their behavior under direct and inverse images of a $\delta\theta$-$\mathcal{I}$-continuous function.

Theorem 5.1. *If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is an $\delta\theta$-$\mathcal{I}$-continuous injective function and Y is a $\delta\star$-Urysohn space, then X is a $\delta\star$-Urysohn space.*

Proof. Suppose that $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is a $\delta\theta$-$\mathcal{I}$-continuous injective function and Y is a $\delta\star$-Urysohn space. Let $x_1, x_2 \in X$ such that $x_1 \neq x_2$. Then $f(x_1) \neq f(x_2)$ and since Y is $\delta\star$-Urysohn, there exist open sets V_1 and V_2 in Y such that $f(x_1) \in V_1$, $f(x_2) \in V_2$ and $\delta Cl^\star(V_1) \cap \delta Cl^\star(V_2) = \emptyset$. Now, using the $\delta\theta$-$\mathcal{I}$-continuity of f, there exist open sets U_1 and U_2 in X such that $x_1 \in U_1$, $x_2 \in U_2$, $f(\delta Cl^\star(U_1)) \subset \delta Cl^\star(V_1)$ and $f(\delta Cl^\star(U_2)) \subset \delta Cl^\star(V_2)$. It follows that,

$$\begin{aligned} \delta Cl^\star(U_1) \cap \delta Cl^\star(U_2) &\subset f^{-1}(f(\delta Cl^\star(U_1))) \cap f^{-1}(f(\delta Cl^\star(U_2))) \\ &\subset f^{-1}(\delta Cl^\star(V_1)) \cap f^{-1}(\delta Cl^\star(V_2)) \\ &= f^{-1}(\delta Cl^\star(V_1) \cap \delta Cl^\star(V_2)) \\ &= f^{-1}(\emptyset) = \emptyset \end{aligned}$$

and so $\delta Cl^\star(U_1) \cap \delta Cl^\star(U_2) = \emptyset$. Therefore, X is a $\delta\star$-Urysohn space. □

Definition 5.1. A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is said to be $\delta\theta$-$\mathcal{I}$-irresolute if for each $\delta\theta$-$\mathcal{J}$-open set U in Y, $f^{-1}(U)$ is a $\delta\theta$-$\mathcal{I}$-open set in X.

Theorem 5.2. *Every $\delta\theta$-$\mathcal{I}$-continuous function is $\delta\theta$-$\mathcal{I}$-irresolute.*

Proof. Suppose that $f:(X,\tau,\mathcal{I})\to(Y,\sigma,\mathcal{J})$ is a $\delta\theta$-$\mathcal{I}$-continuous function. Let V be any $\delta\theta$-$\mathcal{J}$-open set in Y and $x\in f^{-1}(V)$. Then $f(x)\in V$ and so there exists an open set U in Y such that $f(x)\in U\subset\delta Cl^{\star}(U)\subset V$. Since f is $\delta\theta$-$\mathcal{I}$-continuous, there exists an open set W if X such that $x\in W$ and $f(\delta Cl^{\star}(W))\subset\delta Cl^{\star}(U)\subset V$. Thus, $x\in W\subset\delta Cl^{\star}(W)\subset f^{-1}(f(\delta Cl^{\star}(W)))\subset f^{-1}(V)$, which implies that $f^{-1}(V)$ is a $\delta\theta$-$\mathcal{I}$-open set in X and hence, f is $\delta\theta$-$\mathcal{I}$-irresolute. □

Example 5.1. A $\delta\theta$-$\mathcal{I}$-irresolute function need not be $\delta\theta$-$\mathcal{I}$-continuous. Let f be the function given in Example 3.2, f is not $\delta\theta$-$\mathcal{I}$-continuous. Now, note that $\emptyset$ and Y are the only $\delta\theta$-$\mathcal{J}$-open sets in Y and $f^{-1}(\emptyset)=\emptyset$, $f^{-1}(Y)=X$. Since $\emptyset$ and X are $\delta\theta$-$\mathcal{I}$-open sets in X, we conclude that f is $\delta\theta$-$\mathcal{I}$-irresolute.

Definition 5.2. An ideal topological space $(X,\tau,\mathcal{I})$ is said to be $\delta\theta$-$\mathcal{I}$-compact if every cover of X by $\delta\theta$-$\mathcal{I}$-open sets have a finite subcover.

Theorem 5.3. *If $f:(X,\tau,\mathcal{I})\to(Y,\sigma,\mathcal{J})$ is a $\delta\theta$-$\mathcal{I}$-irresolute surjective function and X is a $\delta\theta$-$\mathcal{I}$-compact space, then Y is a $\delta\theta$-$\mathcal{J}$-compact space.*

Proof. Let $\mathcal{V}=\{U_\alpha:\alpha\in\Lambda\}$ be a cover of Y by $\delta\theta$-$\mathcal{J}$-open sets. Since f is $\delta\theta$-$\mathcal{I}$-irresolute, the collection $\mathcal{U}=\{f^{-1}(U_\alpha):U_\alpha\in\mathcal{V}\}$ is a cover of X by $\delta\theta$-$\mathcal{I}$-open sets and as X is $\delta\theta$-$\mathcal{I}$-compact, there exists a finite subcollection $\{f^{-1}(U_{\alpha_i}):i=1,...,n\}$ of $\mathcal{U}$ that cover X. Now, using that f is onto, $\{U_{\alpha_i}:i=1,...,n\}$ is a finite subcollection of $\mathcal{V}$ which cover Y. Therefore, Y is $\delta\theta$-$\mathcal{J}$-compact. □

Corollary 5.1. *If $f:(X,\tau,\mathcal{I})\to(Y,\sigma,\mathcal{J})$ is a $\delta\theta$-$\mathcal{I}$-continuous surjective function and X is a $\delta\theta$-$\mathcal{I}$-compact space, then Y is a $\delta\theta$-$\mathcal{J}$-compact space.*

Definition 5.3. An ideal topological space $(X,\tau,\mathcal{I})$ is said to be $\delta\star$-Lindelöf if every open cover $\{U_\alpha:\alpha\in\Lambda\}$ of X, there exists a countable subcollection $\{U_{\alpha_n}:n\in\mathbb{N}\}$ of $\{U_\alpha:\alpha\in\Lambda\}$ such that $X=\bigcup_{n\in\mathbb{N}}\delta Cl^{\star}(U_{\alpha_n})$.

Theorem 5.4. *If $f:(X,\tau,\mathcal{I})\to(Y,\sigma,\mathcal{J})$ is a $\delta\theta$-$\mathcal{I}$-continuous surjective function and X is a $\delta\star$-Lindelöf space, then Y is a $\delta\star$-Lindelöf space.*

Proof. Let $\{V_\alpha:\alpha\in\Lambda\}$ be an open cover of Y. For each $x\in X$, there exists $\alpha(x)\in\Lambda$ such that $f(x)\in V_{\alpha(x)}$. Since f is $\delta\theta$-$\mathcal{I}$-continuous, there exists an open set $U_{\alpha(x)}$ of X containing x such that $f(\delta Cl^{\star}(U_{\alpha(x)}))\subset\delta Cl^{\star}(V_{\alpha(x)})$. Note that $\{U_{\alpha(x)}:x\in X\}$ is an open cover of the $\delta\star$-Lindelöf space X. So there exists a countable subset $\{U_{\alpha(x_n)} : n \in \mathbb{N}\}$ such that $X = \bigcup_{n\in\mathbb{N}} \delta Cl^\star(U_{\alpha(x_n)})$. Consequently,

$$Y = f(X) = f(\bigcup_{n\in\mathbb{N}} \delta Cl^\star(U_{\alpha(x_n)})) \subset \bigcup_{n\in\mathbb{N}} f(\delta Cl^\star(U_{\alpha(x_n)})) \subset \bigcup_{n\in\mathbb{N}} \delta Cl^\star(V_{\alpha(x_n)}).$$

Therefore, Y is $\delta\star$-Lindelöf. □

Definition 5.4. An ideal topological space $(X, \tau, \mathcal{I})$ is said to be $\delta\theta$-$\mathcal{I}$-connected if X can not be written as the union of two disjoint nonempty $\delta\theta$-$\mathcal{I}$-open sets.

Example 5.2. A $\delta\theta$-$\mathcal{I}$-connected space. Let $X = \{a, b\}$, $\tau = \{\emptyset, X\}$ with $\mathcal{I} = \{\emptyset, \{a\}\}$. Obviously, there is not separation of X by $\delta\theta$-$\mathcal{I}$-open sets. Therefore, X is a $\delta\theta$-$\mathcal{I}$-connected space.

Theorem 5.5. *If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is a $\delta\theta$-$\mathcal{I}$-irresolute surjective function and X is a $\delta\theta$-$\mathcal{I}$-connected space, then Y is $\delta\theta$-$\mathcal{J}$-connected space.*

Proof. Suppose that Y is not a $\delta\theta$-$\mathcal{J}$-connected space. Then there exist nonempty $\delta\theta$-$\mathcal{J}$-open sets U and V in Y such that $U \cap V = \emptyset$ and $U \cup V = Y$. Therefore, we have $f^{-1}(U) \cap f^{-1}(V) = \emptyset$ and $f^{-1}(U) \cup f^{-1}(V) = X$. In addition, $f^{-1}(U)$ and $f^{-1}(V)$ are nonempty $\delta\theta$-$\mathcal{I}$-open sets in X. Therefore, X is not a $\delta\theta$-$\mathcal{I}$-connected space. □

Corollary 5.2. *If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is a $\delta\theta$-$\mathcal{I}$-continuous surjective function and X is a $\delta\theta$-$\mathcal{I}$-connected space, then Y is a $\delta\theta$-$\mathcal{J}$-connected space.*

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