GAUSS-WINKLER TYPE INEQUALITY FOR SUGENO INTEGRALS

Dug Hun Hong

Department of Mathematics
Myongji University
Yongin Kyunggido, 449-728, SOUTH KOREA

Abstract: This paper proposes a Gauss-Winkler type inequality for Sugeno integrals. Indeed, we find the optimal constant $H$ for which the following Gauss-Winkler type inequality for fuzzy integrals

$$\left((S) \int_0^1 x^2 f(x) d\mu\right)^2 \leq H \left((S) \int_0^1 f(x) d\mu\right) \left((S) \int_0^1 x^4 f(x) d\mu\right)$$

holds where $f : [0, 1] \rightarrow [0, \infty)$ is a nondecreasing function and $\mu$ is the Lebesgue measure on $\mathbb{R}$. Some examples are provided to illustrate the validity of the proposed inequality.

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1. Introduction and Preliminaries

A number of studies have examined the Sugeno integral since its introduction in 1974 [16], it has been exhaustively investigated by many authors. Ralescu and Adams [12] generalized a range of fuzzy measures and gave several equivalent definitions of fuzzy integrals. Wang and Klir [17] provided an overview of fuzzy measure theory.


In this paper, we propose a Gauss-Winkler type inequality for Sugeno integrals and find an optimal constant for which Gauss-Winkler type inequality for Sugeno integrals holds for nondecreasing functions. Some examples are provided to illustrate the validity of the proposed inequality.

**Definition 1.** Let $\Sigma$ be a $\sigma$-algebra of subsets of $\mathbb{R}$ and let $\mu : \Sigma \rightarrow [0, \infty]$ be a non-negative, extended real-valued set function. We say that $\mu$ is a fuzzy measure if and only if

(a) $\mu(\emptyset) = 0$.

(b) $E, F \in \Sigma$ and $E \subseteq F$ imply $\mu(E) \leq \mu(F)$ (monotonicity).

(c) $\{E_p\} \subseteq \Sigma$ and $E_1 \subseteq E_2 \subseteq \cdots$ imply $\lim_{p \to \infty} \mu(E_p) = \mu\left(\bigcup_{p=1}^{\infty} E_p\right)$ (continuity form below).

(d) $\{E_p\} \subseteq \Sigma$, $E_1 \supseteq E_2 \supseteq \cdots$, and $\mu(E_1) < \infty$ imply $\lim_{p \to \infty} \mu(E_p) = \mu\left(\bigcap_{p=1}^{\infty} E_p\right)$ (continuity form above).

If $f$ is a non-negative real-valued function defined on $\mathbb{R}$, then we denote by $F_\alpha = \{x \in X | f(x) \geq \alpha\} = \{f \geq \alpha\}$ the $\alpha$-level of $f$, for $\alpha > 0$, and $F_0 = \{x \in X | f(x) > 0\} = \text{supp}(f)$ is the support of $f$.

We note that

$\alpha \leq \beta \Rightarrow \{f \geq \beta\} \subseteq \{f \geq \alpha\}$

If $\mu$ is a fuzzy measure on $A \subset \mathbb{R}$, then we define the following:

$\mathcal{F}^\mu(A) = \{f : A \to [0, \infty) | f \text{ is } \mu\text{-measurable}\}$.

**Definition 2.** Let $\mu$ be a fuzzy measure on $(\mathbb{R}, \Sigma)$. If $f \in \mathcal{F}^\mu(\mathbb{R})$ and $A \in \Sigma$, then the Sugeno integral(or the fuzzy integral) of $f$ on $A$, with respect to the fuzzy measure $\mu$, is defined as

$$(S) \int_A f \, d\mu = \sup_{\alpha \in [0, \infty)} [\alpha \wedge \mu(A \cap F_\alpha)]$$. 
In particular, if $A = X$ then

$$\left( S \right) \int_{\mathbb{R}} f \, d\mu = (S) \int f \, d\mu = \sup_{\alpha \in [0, \infty)} [\alpha \wedge \mu(F_\alpha)]$$

The following properties of the Sugeno integral are well known and can be found in [17]:

**Proposition 1.** (see [17]) If $\mu$ is a fuzzy measure on $\mathbb{R}$ and $f, g \in \mathcal{F}_\mu(\mathbb{R})$, then

(i) $(S) \int_A f \, d\mu \leq \mu(A)$;

(ii) $(S) \int_A K \, d\mu = K \wedge \mu(A)$ for any constant $K \in [0, \infty)$;

(iii) $(S) \int_A f \, d\mu \leq (S) \int_A g \, d\mu$, if $f \leq g$ on $A$;

(iv) $\mu(A \cap \{f \geq \alpha\}) \geq \alpha \Rightarrow (S) \int_A f \, d\mu \geq \alpha$;

(v) $\mu(A \cap \{f \geq \alpha\}) \leq \alpha \Rightarrow (S) \int_A f \, d\mu \leq \alpha$;

(vi) $(S) \int_A f \, d\mu < \alpha \Leftrightarrow$ there exists $\gamma < \alpha$ such that $\mu(A \cap \{f \geq \gamma\}) < \alpha$;

(vii) $(S) \int_A f \, d\mu > \alpha \Leftrightarrow$ there exists $\gamma > \alpha$ such that $\mu(A \cap \{f \geq \gamma\}) > \alpha$.

**Note 1.** Let $F(\alpha) = \mu(A \cap \{f \geq \alpha\})$, then by Proposition 1, (v), (vi),

$$F(\alpha) = \alpha \Rightarrow (S) \int_0^1 f(x) \, d\mu = \alpha.$$  

**Theorem 1.** (see [10]) Let $f : [0, \infty) \to [0, \infty)$ be continuous and non-increasing or non-decreasing functions and $\mu$ be the Lebesgue measure on $\mathbb{R}$. Let

$$(S) \int_0^a f(x) \, d\mu = p.$$  

If $0 < p < a$, then $f(p) = p, f(a - p) = p$, respectively.
2. Gauss-Winkler Type Inequality for Sugeno Integrals

The classical Gauss-Winkler inequality provides the following inequality [1, p 94]:
\[
\left( \int_0^1 x^2 f(x) d\mu \right)^2 \leq \frac{5}{9} \left( \int_0^1 f(x) d\mu \right) \left( \int_0^1 x^4 f(x) d\mu \right)
\]  
(1)

where \( f : [0, 1] \rightarrow [0, \infty) \) is nondecreasing.

However, this inequality is not valid for the Sugeno integral as is shown in the following example.

**Example 1.** Let \( f(x) = x \) for \( x \in [0, 1/2] \) and \( 1/2 \) for \( x \in (1/2, 1] \). Then, a straightforward calculus shows that

\[
(S) \int_0^1 f(x) d\mu = \frac{1}{2}.
\]

By Theorem 1, by solving the equation \( x = \frac{1}{2}(1-x)^2 \), we obtain

\[
(S) \int_0^1 x^2 f(x) d\mu = 2 - \sqrt{3} \approx 0.2679,
\]

and by solving the equation \( x = \frac{1}{2}(1-x)^4 \), we obtain

\[
(S) \int_0^1 x^4 f(x) d\mu \approx 0.2024.
\]

Consequently

\[
0.0718 \approx \left( (S) \int_0^1 x^2 f(x) d\mu \right)^2 > 0.0562 = \left( \frac{5}{9} \right) \left( \frac{1}{2} \right) (0.2024)
\]

\[
\approx \frac{5}{9} \left( (S) \int_0^1 f(x) d\mu \right) \left( (S) \int_0^1 x^4 f(x) d\mu \right).
\]

Therefore, inequalities of (1) does not hold for Sugeno integrals.

The following result shows a Gauss-Winkler type inequality for Sugeno integrals.

**Theorem 2.** (Fuzzy Gauss-Winkler Inequality) Let \( f : [0, 1] \rightarrow [0, \infty) \) be a nondecreasing function and that \( \mu \) the Lebesgue measure on \( \mathbb{R} \). Then

\[
\left( (S) \int_0^1 x^2 f(x) d\mu \right)^2 \leq (2 - \alpha) \left( (S) \int_0^1 f(x) d\mu \right) \left( (S) \int_0^1 x^4 f(x) d\mu \right)
\]  
(2)
where $0 < \alpha \leq 1$ satisfies the following equation

$$16\alpha(1 - \alpha)^2 - \alpha(2 - \alpha)^3 = 0.$$  

**Proof.** Let

$$H = \sup \left\{ \frac{\left( (S) \int_{0}^{1} x f(x) d\mu \right)^2}{(S) \int_{0}^{1} f(x) d\mu \left( (S) \int_{0}^{1} x f(x) d\mu \right)^2} \mid f : \text{nondecreasing on}[0, 1] \right\}.$$  

and let

$$H_\alpha = \sup \left\{ \frac{\left( (S) \int_{0}^{1} x f(x) d\mu \right)^2}{(S) \int_{0}^{1} f(x) d\mu \left( (S) \int_{0}^{1} x f(x) d\mu \right)^2} \mid f : \text{nondecreasing}, \right. \quad \left. (S) \int_{0}^{1} x f(x) d\mu = \alpha \right\}.$$  

Then

$$H = \sup_{0 < \alpha < 1} H_\alpha.$$  

We consider $H_\alpha$. Let

$$f_0(x) = \begin{cases} 0, & \text{if } x \in [0, 1 - \alpha), \\ \frac{\alpha}{(1 - \alpha)^2}, & \text{if } x \in (1 - \alpha, 1] \end{cases}$$  

Then it is easy to check that

$$f_0 = \inf \left\{ f : \text{nondecreasing on}[0, 1] \mid (S) \int_{0}^{1} x f(x) d\mu = \alpha \right\}.$$  

Hence we have

$$H_\alpha = \frac{\alpha^2}{\left( (S) \int_{0}^{1} f_0(x) d\mu \right) \left( (S) \int_{0}^{1} x f_0(x) d\mu \right)}.$$  

Because $\mu\{f_0 \geq \alpha\} = \alpha$, noting that $\frac{\alpha}{(1 - \alpha)^2} \geq \alpha$, we have by Note 1

$$(S) \int_{0}^{1} f_0(x) d\mu = \alpha.$$
Now, let
\[(S) \int_0^1 x^4 f_0(x) d\mu = x_0.\]
Since \(f_0\) is continuous and increasing on \((1 - \alpha, 1]\) and right limit of \(f_0\) at \(1 - \alpha\) is less than \(\alpha\), then by Theorem 1,

\[x_0 = (1 - x_0)^4 f_0((1 - x_0)) = (1 - x_0)^4 \frac{\alpha}{(1 - \alpha)^2}\]

and hence
\[H_\alpha = \frac{(1 - \alpha)^2}{(1 - x_0)^4} = \frac{\alpha}{x_0}.\]

To find \(H\) we now consider the optimization problem:

\[H = \sup_{0 < \alpha < 1} H_\alpha\]

\[= \text{Maximize} \frac{\alpha}{x}\]

subject to \(x = (1 - x)^4 \frac{\alpha}{(1 - \alpha)^2}, \ 0 < \alpha < 1\).

Let
\[f(x, \alpha) = \frac{\alpha}{x}, \ g(x, \alpha) = \frac{\alpha}{(1 - \alpha)^2} - \frac{x}{(1 - x)^4}.\]

Then we have
\[\nabla f(x, \alpha) = \left( -\frac{\alpha}{x^2}, \frac{1}{x} \right)\]
\[\nabla g(x, \alpha) = \left( -\frac{1 + 3x}{(1 - x)^5}, \frac{1 + \alpha}{(1 - \alpha)^2} \right).\]

By using the Lagrange multiplier method, we have
\[f(x, \alpha) = \lambda \nabla g(x, \alpha), \ g(x, \alpha) = 0\]

which imply that
\[-\frac{\alpha}{x^2} = \lambda \frac{1 + 3x}{(1 - x)^5}, \ \frac{1}{x} = \lambda \frac{1 + \alpha}{(1 - \alpha)^2}.\]

By solving these equations, we obtain the solution
\[x = \frac{\alpha}{2 - \alpha}.\]
where $\alpha$ satisfies the following equation.

$$g(x, \alpha) = \frac{\alpha}{(1 - \alpha)^2} - \frac{x}{(1 - x)^4} = \frac{16\alpha(1 - \alpha)^2 - \alpha(2 - \alpha)^3}{16(1 - \alpha)^4} = 0$$

and the optimal value is

$$H = 2 - \alpha.$$

**Note 2.** If we solve the equation

$$16\alpha(1 - \alpha)^2 - \alpha(2 - \alpha)^3 = 0, \ 0 < \alpha < 1$$

with the aid of computer work, then we obtain

$$\alpha^* \approx 0.5745, \ x^* \approx 0.4030.$$ 

Hence we can conclude that $f(x, \alpha)$ assumes maximum at $(x^*, \alpha^*)$ and the optimal value is

$$H = \frac{\alpha^*}{x^*} = 2 - \alpha^* \approx 1.4255.$$

**Example 2.** In Example 1, if we substitute 1.4255 to $\frac{5}{7}$, then the inequality (1) holds since

$$0.0718 \approx \left(\int_0^1 x^2 f(x) d\mu\right)^2 \leq (1.4255) \left(\int_0^1 f(x) d\mu\right) \left(\int_0^1 x^4 f(x) d\mu\right) \approx 0.1443.$$

The following example shows that the constant $H \approx 1.4255$ in Theorem 2 is optimal for Inequality (2).

**Example 3.** Let $f(x) = 0$ for $x \in [0, 0.4255]$ and 3.1731 for $x \in (0.4255, 1]$. Then, a straightforward calculus shows that

$$(S) \int_0^1 f(x) d\mu = 0.5745.$$ 

By Theorem 1, by solving the equation $x = 0.5745(1 - x)^2$, we obtain

$$(S) \int_0^1 x^2 f(x) d\mu \approx 0.5745,$$
and by solving the equation \( x = 0.5746(1 - x)^4 \), we obtain

\[
(S) \int_0^1 x^4 f(x) d\mu \approx 0.4030.
\]

Consequently

\[
\left( (S) \int_0^1 x^2 f(x) d\mu \right)^2 \approx 0.3300,
\]

\[
1.4255 \left( (S) \int_0^1 f(x) d\mu \right) \left( (S) \int_0^1 x^4 f(x) d\mu \right)
\]

\[
\approx (1.4255)(0.5746)(0.4030) = 0.3300.
\]

Therefore, the constant \( H \approx 1.4255 \) in Theorem 2 is optimal.

The case for non-increasing function is similar.

**Theorem 3.** (Fuzzy Gauss-Winkler Inequality) Let \( f : [0, 1] \rightarrow [0, \infty) \) be a non-increasing function and that \( \mu \) the Lebesgue measure on \( \mathbb{R} \). Then

\[
\left( (S) \int_0^1 (1 - x)^2 f(x) d\mu \right)^2
\]

\[
\leq 1.4255 \left( (S) \int_0^1 f(x) d\mu \right) \left( (S) \int_0^1 (1 - x)^4 f(x) d\mu \right).
\]

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**References**


