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OSCILLATION CRITERIA FOR SECOND ORDER NONLINEAR NEUTRAL DELAY DYNAMIC EQUATIONS ON TIME SCALES

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Abstract: The purpose of this paper is to obtain new sufficient conditions for the oscillation of solutions of a class of second-order neutral delay dynamic equations. The obtained results improve and extend some known results in the literature. Three examples are given to illustrate the main results.

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1. Introduction

The study of dynamic equations on time scales has received a lot of attentions, since it was introduced by Hilger's landmark paper [4]. A rapidly expanding body of literature has sought to unify, extend, and generalize ideas from discrete ,continuous, and quantum calculus to arbitrary time scale calculus. A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$.The cases when $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$ represent the classical theories of differential and difference equations respectively.

For $t \in \mathbb{T}$ we define The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ and The

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backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s \geq t\}, \qquad \rho(t) := \sup\{s \in \mathbb{T} : s \leq t\}.$$

A point $t \in \mathbb{T}$ is said to be right-scattered if $\sigma(t) > t$, right-dense if $\sigma(t) = t$, left-scattered if $t > \rho(t)$, left-dense if $\rho(t) = t$, and dense point if $\rho(t) = t = \sigma(t)$. The graininess function μ for the time scale $\mathbb{T}$ is defined by $\mu(t) = \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$, the notation $f^{\sigma}(t)$ denotes $f(\sigma(t))$. A function $g : \mathbb{T} \to \mathbb{R}$ is said to be *rd*-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of all such *rd*-continuous functions is denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$. The set of functions $f : \mathbb{T} \to \mathbb{R}$ which are differentiable and whose derivative is an *rd*-continuous function is denoted by $C^1_{rd}(\mathbb{T}, \mathbb{R})$.

Definition 1. The function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive function if

$$1 + \mu(t)p(t) \neq 0, \qquad for\ all\ t \in \mathbb{T}^k.$$

The set of all regressive functions denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})$

Definition 2. For $p \in \mathcal{R}$, the exponential function is defined by

$$e_p(t,s) = \exp\left\{\int_s^t \xi_\mu(u)p(u)\Delta u\right\} \qquad t, s \in \mathbb{T}$$

where the cylinder transformation is defined by

$$\xi_\mu(u) = \begin{cases} \log \frac{1+\mu u}{\mu} & if \mu \neq 0 \\ u & if \mu = 0 \end{cases}.$$

In the recent years there has been a great interest concerning the oscillation and nonoscillation of solutions of dynamic equations on time scales, see [7, 8, 10, 11, 14].

El-Sheikh, et al [3] discussed the oscillatory behavior of the second order dynamic equation

$$(r(t)\psi(x(t))x^{\Delta}(t))^{\Delta} + p(t)x^{\Delta}(t) + f(t, x(\tau(t))) = 0,$$

where r, ψ and p are real-valued positive rd-continuous functions defined on $\mathbb{T}$.

L.Erbe, et al [7] investigated the oscillation of the second order nonlinear functional dynamic equation

$$\left(a(t)(x^{\Delta}(t))^{\gamma}\right)^{\Delta} + \sum_{i=0}^{n} p_i(t)\phi_{\alpha_i}(x(g_i(t))) = 0,$$

where γ is a quotient of odd positive integers in the both cases $\int_{t_0}^{\infty} a^{-\frac{1}{\gamma}}(s)\Delta s = \infty$ and $\int_{t_0}^{\infty} a^{-\frac{1}{\gamma}}(s)\Delta s < \infty$.

The aim of this paper is to extend and improve some results of [2] for the class of second order nonlinear neutral delay differential equation

$$\left[r(t)\psi(x(t))(z'(t))^{\alpha}\right]' + \sum_{i=1}^{n} q_i(t) f_i(x(\delta_i(t))) = 0,$$

To the best of our knowledge, the obtained results of this paper are new.

Consider the second-order nonlinear neutral delay dynamic equation of the form

$$\left[r(t)\psi(x(t))(z^{\Delta}(t))^{\alpha}\right]^{\Delta} + \sum_{i=1}^{n} q_i(t) f_i(x(\delta_i(t))) = 0, \tag{1}$$

where $z(t) = x(t) + p(t)x(\tau(t))$, and α is a quotient of odd positive integers. Assume that the following conditions hold

(I_1) $r \in C_{rd}^1([t_0,\infty)_{\mathbb{T}}, (0,\infty))$, $p \in C_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R})$ with; $0 \le p(t) < 1$;

(I_2) $\tau \in C_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{T}), \tau(t) \le t, lim_{t\to\infty}\tau(t) = \infty$;

(I_3) $q_i(t) \in C_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R})$, $\delta_i \in C_{rd}^1([t_0,\infty)_{\mathbb{T}}, \mathbb{T}), \delta_i(t) \le t, lim_{t\to\infty}, \delta_i(t) = \infty, i = 1,2,3,...,n$;

(I_4) $\psi, f \in C_{rd}^1([t_0,\infty)_{\mathbb{T}}, \mathbb{R}), \psi(x) > 0$, and $\frac{f_i(x_i)}{|x_i|^{\alpha-1}|x_i|} \ge \eta_i > 0$ for $x_i(t) \ne 0$, η_i are constant ,$i = 1,2,3,...,n$ and $\psi(x) \le L^{-1}$,where L constant ;

By a solution of Eq.(1), we mean a nontrivial real-valued function $x(t)$ which has the property $[x(t) + p(t)x(\tau(t))] \in C_{rd}^1([t_0,\infty)_{\mathbb{T}})$ and satisfy

$$[r(t)\psi(x(t))[x(t) + p(t)x(\tau(t))]^{\Delta}]^{\alpha} \in C_{rd}^1([t_0,\infty)_{\mathbb{T}}).$$

A solution $x(t)$ of Eq.(1) is said to be oscillatory if it is neither eventually positive nor eventually negative. otherwise; it is called nonoscillatory . The equation itself is said to be oscillatory if all its solutions are oscillatory.

We study the oscillatory behavior for (1) in the two possible cases

$$\lim_{t\to\infty} \int_{t_0}^{t} r^{-\frac{1}{\alpha}}(s)\Delta s = \infty.$$

and

$$\lim_{t\to\infty} \int_{t_0}^{t} r^{-\frac{1}{\alpha}}(s)\Delta s < \infty.$$

2. Main Results

For simplicity, we define the following notations to used through this paper. $R(t) = \int_{t_0}^{t} r^{-\frac{1}{\alpha}}(s)\Delta s$, $\vartheta(t) = \int_{t_0}^{\infty} r^{-\frac{1}{\alpha}}(s)\Delta s$.

First, we consider the case

$$\lim_{t\to\infty}\int_{t_0}^{t} r^{-\frac{1}{\alpha}}(s)\Delta s = \infty. \tag{2}$$

The following preliminaries will be needed

Theorem 3. [1] *Suppose that the first nonhomogeneous linear equation*

$$y^{\Delta} = p(t)y + f(t)$$

is regressive. Let $t_0 \in \mathbb{T}$ and $x_0 \in \mathbb{R}$. The unique solution of the inial value problem

$$x^{\Delta} = -p(t)x^{\sigma} + f(t), \quad x(t_0) = x_0,$$

is given by

$$x(t) = e_{\ominus p}(t,t_0)x_0 + \int_{t_0}^{t} e_{\ominus p}(t,\tau)f(\tau)\Delta\tau$$

Lemma 4. *Let the conditions (I_1)-(I_4) and (2) hold. If $x(t)$ is an eventually positive solution of (1), then there exists $t_1 \in [t_0,\infty)_{\mathbb{T}}$ sufficiently large such that*

$$z(t) > 0, \quad z^{\Delta}(t) > 0, \quad z^{\Delta\Delta}(t) \le 0, \qquad t \ge t_1.$$

Lemma 5. [6] *Suppose that the condition (I_3) holds. If $z(t) \in C^2_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R})$ satisfy*

$$z(t) > 0, \quad z^{\Delta}(t) > 0, \quad z^{\Delta\Delta}(t) \le 0, \qquad t \ge t_0.$$

Then for each $0 < m_i < 1$, there exists a $T_2 \ge t_0$, such that

$$z(\delta_i(t)) \ge m_i z(t)\frac{\delta_i(t)}{t}, \qquad i = 1,2,3,...,n. \tag{3}$$

Theorem 6. *Assume that $\alpha \ge 1$, (2) and(I_1)-(I_4) hold. Furthermore, assume that there exists a function $\rho \in C_{rd}([t_0,\infty),\mathbb{R}^+)_{\mathbb{T}}$ such that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\rho(s)Q(s) - \frac{[\rho^{\Delta}(s)]^2 r(s)}{4\alpha\left(\frac{s}{2}\right)^{\alpha-1}\rho(s)L}\right]\Delta s = \infty \tag{4}$$

where $Q(t) = \sum_{i=1}^{n} q_i(t)\eta_i\frac{(m_i\delta_i(t))^{\alpha}}{t^{\alpha}}[1-p(\delta_i(t))]^{\alpha}$, $\rho^{\Delta}_{+}(t) = \max\{0,\rho^{\Delta}(t)\}$.

Then Eq(1) is oscillatory.

Proof. Assume that $x(t)$ has a nonoscillatory solution of (1) with $x(t) > 0$ on $[t_0, \infty)_{\mathbb{T}}$, then there exists $T_0 \geq t_0)$ such that

$$x(t) > 0, \quad x(\tau(t)) > 0, \quad x(\delta_i(t)) > 0, \;\; i = 1, 2, 3, ..., n, \quad on \;\; [T_0, \infty),$$

Since $z^{\Delta}(t) > 0$, then from (1) and (I_4), we get

$$\left[r(t)\psi(x(t))(z^{\Delta}(t))^{\alpha}\right]^{\Delta} + \sum_{i=1}^{n} q_i(t)\eta_i(x(\delta_i(t)))^{\alpha} \leq 0. \tag{5}$$

Since $x(t) = z(t) - p(t)x(\tau(t))$, then

$$\begin{aligned} x(\delta_i(t)) &= z(\delta_i(t)) - p(\delta_i(t))x(\tau(\delta_i(t))) \\ &\geq z(\delta_i(t))[1 - p(\delta_i(t))]. \end{aligned} \tag{6}$$

Substituting from (6) into (5), we get

$$\left[r(t)\psi(x(t))(z^{\Delta}(t))^{\alpha}\right]^{\Delta} + \sum_{i=1}^{n} q_i(t)\eta_i z^{\alpha}(\delta_i(t))[1 - p(\delta_i(t))]^{\alpha} \leq 0. \tag{7}$$

Applying Lemma 5, then (2.8) takes the form

$$\left[r(t)\psi(x(t))(z^{\Delta}(t))^{\alpha}\right]^{\Delta} + z^{\alpha}(t)\sum_{i=1}^{n} q_i(t)\eta_i \frac{(m_i\delta_i(t))^{\alpha}}{t^{\alpha}}[1 - p(\delta_i(t))]^{\alpha} \leq 0. \tag{8}$$

Now define

$$\omega(t) = \rho(t)\frac{r(t)\psi(x(t))(z^{\Delta}(t))^{\alpha}}{z^{\alpha}(t)}. \tag{9}$$

It is clear that $\omega(t) > 0$, and

$$\omega^{\Delta}(t) = \left[r(t)\psi(x(t))(z^{\Delta}(t))^{\alpha}\right]^{\Delta}\left[\frac{\rho(t)}{z^{\alpha}(t)}\right] + \left[r(t)\psi(x(t))(z^{\Delta}(t))^{\alpha}\right]^{\sigma}\left[\frac{\rho(t)}{z^{\alpha}(t)}\right]^{\Delta}$$

i.e.

$$\omega^{\Delta}(t) \leq -\rho(t)Q(t) + \frac{\rho^{\Delta}(t)}{\rho(\sigma(t))}\omega(\sigma(t)) - \rho(t)\frac{\left[r(t)\psi(x(t))(z^{\Delta}(t))^{\alpha}\right]^{\sigma}[z^{\alpha}(t)]^{\Delta}}{z^{\alpha}(t)z^{\alpha}(\sigma(t))} \tag{10}$$

By the Pötzsche chain rule , we have

$$[z^{\alpha}(t)]^{\Delta} = \left\{\alpha\int_0^1 \left[z(t) + \mu h z^{\Delta}(t)\right]^{\alpha-1} dh\right\} z^{\Delta}(t)$$

$$\begin{aligned} &= \left\{\alpha \int_0^1 [(1-h)z(t)+hz(\sigma(t))]^{\alpha-1}\, dh\right\} z^{\Delta}(t) \\ &\geq \alpha z^{\alpha-1}(t) z^{\Delta}(t), \end{aligned}$$

Substituting into (10), we obtain

$$\omega^{\Delta}(t) \leq -\rho(t)Q(t) + \frac{\rho^{\Delta}(t)}{\rho(\sigma(t))}\omega(\sigma(t)) - \alpha\rho(t)\frac{\left[r(t)\psi(x(t))(z^{\Delta}(t))^{\alpha}\right]^{\sigma} z^{\alpha-1}(t) z^{\Delta}(t)}{z^{\alpha}(t) z^{\alpha}(\sigma(t))}. \quad (11)$$

Now by Lemma 4, there exists $t_2 > 2t_1$ such that

$$z(t) = z(t_1) + \int_{t_1}^{t} z^{\Delta}(s)\Delta s > \int_{t_1}^{t} z^{\Delta}(s)\Delta s \geq z^{\Delta}(t)(t-t_1) \geq \frac{t}{2} z^{\Delta}(t) \quad (12)$$

and so

$$\begin{aligned} \alpha z^{\alpha-1}(t) z^{\Delta}(t) &\geq \alpha \left(\frac{t}{2}\right)^{\alpha-1} (z^{\Delta}(t))^{\alpha} \\ &\geq \alpha \left(\frac{t}{2}\right)^{\alpha-1} \frac{r(t)\psi(x(t))}{r(t)\psi(x(t))}(z^{\Delta}(t))^{\alpha}. \quad (13) \end{aligned}$$

Now since $\left[r(t)\psi(x(t))(z^{\Delta}(t))^{\alpha}\right]^{\Delta} \leq 0$, we have

$$\left[r(t)\psi(x(t))(z^{\Delta}(t))^{\alpha}\right]^{\sigma} \leq \left[r(t)\psi(x(t))(z^{\Delta}(t))^{\alpha}\right]. \quad (14)$$

Thus

$$\alpha z^{\alpha-1}(t) z^{\Delta}(t) \geq \alpha \left(\frac{t}{2}\right)^{\alpha-1} \frac{\left[r(t)\psi(x(t))(z^{\Delta}(t))^{\alpha}\right]^{\sigma}}{r(t)\psi(x(t))} \quad (15)$$

Therefore (11) takes the form

$$\begin{aligned} \omega^{\Delta}(t) &\leq -\rho(t)Q(t) + \frac{\rho^{\Delta}(t)}{\rho(\sigma(t))}\omega(\sigma(t)) - \alpha\rho(t)\left(\frac{t}{2}\right)^{\alpha-1} \frac{\left[\left[r(t)\psi(x(t))(z^{\Delta}(t))^{\alpha}\right]^{\sigma}\right]^2}{r(t)\psi(x(t)) z^{2\alpha}(\sigma(t))}, \\ &\leq -\rho(t)Q(t) + \frac{\rho^{\Delta}(t)}{\rho(\sigma(t))}\omega(\sigma(t)) - \frac{\alpha\left(\frac{t}{2}\right)^{\alpha-1}\rho(t)L}{[\rho(\sigma(t))]^2 r(t)}\omega(\sigma(t))^2, \end{aligned}$$

i.e.

$$\omega^{\Delta}(t) \leq -\rho(t)Q(t) - \left[\frac{\alpha\left(\frac{t}{2}\right)^{\alpha-1}\rho(t)L}{[\rho(\sigma(t))]^2 r(t)}\omega(\sigma(t))^2 - \frac{\rho^{\Delta}(t)}{\rho(\sigma(t))}\omega(\sigma(t))\right].$$

Thus

$$\omega^{\Delta}(t) \leq -\left[\rho(t)Q(t) - \frac{[\rho^{\Delta}(t)]^2 r(t)}{4\alpha\left(\frac{t}{2}\right)^{\alpha-1}\rho(t)L}\right] \tag{16}$$

Integrating (16)) from T_0 to t, and takes lim sup

$$\limsup_{t\to\infty}\int_{T_0}^{t}\left[\rho(s)Q(s) - \frac{[\rho^{\Delta}(s)]^2 r(s)}{4\alpha\left(\frac{s}{2}\right)^{\alpha-1}\rho(s)L}\right]\Delta s \leq \omega(T_0) - \omega(t) \leq \omega(T_0). \tag{17}$$

This contradicts (4).This completes the proof. □

Theorem 7. *Let (2) holds. If there exists a function $\rho \in C_{rd}([t_0,\infty),\mathbb{R}^+)_{\mathbb{T}}$ such that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\rho(s)Q(s) - \frac{1}{(1+\alpha)^{1+\alpha}}\frac{(\rho^{\Delta}(s))^{\alpha+1} r(\sigma(s))}{L\rho^{\alpha}(\sigma(s))\rho^{\alpha}(s)}\right]\Delta s = \infty, \tag{18}$$

then Eq.(1) is oscillatory.

Proof. Assume that $x(t)$ has a nonoscillatory solution of (1). Proceeding as in the proof of Theorem 6, since by the Pötzsche chain rule , we have

$$\begin{aligned}
[z^{\alpha}(t)]^{\Delta} &= \left\{\alpha\int_0^1\left[z(t)+\mu h z^{\Delta}(t)\right]^{\alpha-1} dh\right\} z^{\Delta}(t)\\
&= \left\{\alpha\int_0^1 [(1-h)z(t)+hz(\sigma(t))]^{\alpha-1} dh\right\} z^{\Delta}(t)\\
&\geq \alpha\begin{cases} z^{\alpha-1}(t)z^{\Delta}(t) & \alpha \geq 1\\ z^{\alpha-1}(\sigma(t))z^{\Delta}(t) & 0<\alpha\leq 1\end{cases},
\end{aligned}$$

Since by Lemma4, $z^{\Delta}(t) > 0$, then $z(t) \leq z(\sigma(t))$ for $t \in [T_0,\infty)_{\mathbb{T}}$. Hence for $\alpha > 0$

$$[z^{\alpha}(t)]^{\Delta} \geq \alpha z^{\alpha-1}(t)z^{\Delta}(t).$$

This with (10), leads to

$$\omega^{\Delta}(t) \leq -\rho(t)Q(t) + \frac{\rho^{\Delta}(t)}{\rho(\sigma(t))}\omega(\sigma(t)) - \frac{\alpha\rho(t)L^{\frac{1}{\alpha}}}{\rho^{\frac{1}{\alpha}}(\sigma(t))r^{\frac{1}{\alpha}}(\sigma(t))}\omega^{\frac{\alpha+1}{\alpha}}(\sigma(t)). \tag{19}$$

Applying the inequality

$$B\omega - A\omega^{\frac{\alpha+1}{\alpha}} \leq \frac{\alpha^{\alpha}}{(1+\alpha)^{1+\alpha}}\frac{B^{1+\alpha}}{A^{\alpha}} \tag{20}$$

where $B = \frac{\rho^\Delta(t)}{\rho(\sigma(t))} > 0$ and $A = \frac{\alpha\rho(t)L^{\frac{1}{\alpha}}}{\rho^{\frac{1}{\alpha}}(\sigma(t))r^{\frac{1}{\alpha}}(\sigma(t))} > 0$. This with (19), leads to

$$\omega^\Delta(t) \leq -\rho(t)Q(t) + \frac{1}{(1+\alpha)^{1+\alpha}} \frac{(\rho^\Delta(t))^{\alpha+1} r(\sigma(t))}{L\rho^\alpha(\sigma(t))\rho^\alpha(t)}. \tag{21}$$

Integrating from t_2 to t, we get

$$\int_{t_2}^t \left[\rho(s)Q(s) - \frac{1}{(1+\alpha)^{1+\alpha}} \frac{(\rho^\Delta(s))^{\alpha+1} r(\sigma(s))}{L\rho^\alpha(\sigma(s))\rho^\alpha(s)}\right] \Delta s \leq \omega(t_2). \tag{22}$$

Taking $\limsup_{t\to\infty}$, we get a contradiction with (18). This completes the proof. □

Theorem 8. *Let (2) holds. If there exists $h \in C^1_{rd}([t_0,\infty),\mathbb{R}^+)_{\mathbb{T}}$ such that*

$$\limsup_{t\to\infty} \int_{t_0}^t \left[Q(t) - \alpha r^{-\frac{1}{\alpha}}(s) L^{\frac{1}{\alpha}} h^{\alpha+1}(s)\right] e_p(s,t_0)\Delta s = \infty, \tag{23}$$

where

$$e_p(s,t_0) = \exp\left\{(\alpha+1)L^{\frac{1}{\alpha}} \int_{t_0}^s \xi_\mu(u) r^{-\frac{1}{\alpha}}(u) h(u)\Delta u\right\}$$

with

$$\xi_\mu(u) = \begin{cases} \log\frac{1+\mu u}{\mu} & \text{if } \mu \neq 0 \\ u & \text{if } \mu = 0 \end{cases},$$

then Eq.(1) is oscillatory.

Proof. Without loss of generality, assume that (1)possesses a nonosilatory solution $x(t) > 0$ such that

$$x(t) > 0, \quad x(\tau(t)) > 0, \quad x(\delta_i(t)) > 0, \quad i = 1,2,3,...,n, \quad \text{on } [T_0,\infty).$$

Proceeding as in the proof of Theorem6, and define

$$\nu(t) = \frac{r(t)\psi(x(t))(z^\Delta(t))^\alpha}{z^\alpha(t)}. \tag{24}$$

Obviously, $\nu(t) > 0$, and

$$\begin{aligned} \nu^\Delta(t) &= \frac{[r(t)\psi(x(t))(z^\Delta(t))^\alpha]^\Delta}{z^\alpha(t)} + [r(t)\psi(x(t))(z^\Delta(t))^\alpha]^\sigma \left[\frac{1}{z^\alpha(t)}\right]^\Delta \\ &\leq -Q(t) - \left[r(t)\psi(x(t))(z^\Delta(t))^\alpha\right]^\sigma \frac{[z^\alpha(t)]^\Delta}{z^\alpha(t) z^\alpha(\sigma(t))}, \end{aligned}$$

i.e.

$$\nu^{\Delta}(t) \leq -Q(t) - \alpha\nu(\sigma(t))\frac{z^{\Delta}(t)}{z(t)}.$$

Since $\left[r(t)\psi(x(t))(z^{\Delta}(t))^{\alpha}\right]^{\Delta} \leq 0$, and $z^{\Delta}(t) > 0$, we have

$$\begin{aligned} \nu^{\Delta}(t) &\leq -Q(t) - \alpha\nu(\sigma(t))\frac{r^{\frac{1}{\alpha}}(t)\psi^{\frac{1}{\alpha}}(x(t))}{r^{\frac{1}{\alpha}}(t)\psi^{\frac{1}{\alpha}}(x(t))}\frac{z^{\Delta}(t)}{z(t)} \\ &\leq -Q(t) - \alpha\nu(\sigma(t))\frac{[r^{\frac{1}{\alpha}}(t)\psi^{\frac{1}{\alpha}}(x(t))]^{\sigma}}{r^{\frac{1}{\alpha}}(t)\psi^{\frac{1}{\alpha}}(x(t))}\frac{z^{\Delta}(\sigma(t))}{z(\sigma(t))}. \end{aligned} \tag{25}$$

Thus from (I_4), (25) takes the form

$$\begin{aligned} \nu^{\Delta}(t) \leq &- Q(t) - \alpha r^{-\frac{1}{\alpha}}(t)L^{\frac{1}{\alpha}}\nu^{\frac{\alpha+1}{\alpha}}(\sigma(t)) \\ \leq &- \left[Q(t) - \alpha r^{-\frac{1}{\alpha}}(t)L^{\frac{1}{\alpha}}h^{\alpha+1}(t)\right] - r^{-\frac{1}{\alpha}}(t)L^{\frac{1}{\alpha}}\left[\alpha\nu^{\frac{\alpha+1}{\alpha}}(\sigma(t)) + h^{\alpha+1}(t)\right] \end{aligned}$$

Since by Young's inequality, $\alpha\nu^{\frac{\alpha+1}{\alpha}}(\sigma(t)) + h^{\alpha+1}(t) \geq (\alpha+1)h(t)\nu(t)$, then

$$\nu^{\Delta}(t) \leq -\left[Q(t) - \alpha r^{-\frac{1}{\alpha}}(t)L^{\frac{1}{\alpha}}h^{\alpha+1}(t)\right] - (\alpha+1)r^{-\frac{1}{\alpha}}(t)L^{\frac{1}{\alpha}}h(t)\nu(\sigma(t)) \tag{26}$$

Now by Theorem3, (26) takes the form

$$[e_{\ominus p}(t,t_0)\nu(t)]^{\Delta} \leq -\left[Q(t) - \alpha r^{-\frac{1}{\alpha}}(t)L^{\frac{1}{\alpha}}h^{\alpha+1}(t)\right]e_{\ominus p}(t,t_0) \tag{27}$$

Integrating (27) from T to t

$$\begin{aligned} 0 &< e_{\ominus p}(t,t_0)\nu(t) \\ &\leq e_{\ominus p}(T,t_0)\nu(T) - \int_T^t \left[Q(s) - \alpha r^{-\frac{1}{\alpha}}(s)L^{\frac{1}{\alpha}}h^{\alpha+1}(s)\right]e_{\ominus p}(s,t_0)\Delta s. \end{aligned} \tag{28}$$

Taking $\limsup_{t\to\infty}$, we get a contradiction with (23).This completes the proof. □

Now, we consider the case

$$\lim_{t\to\infty} R(t) = \int_{t_0}^{t} r^{-\frac{1}{\alpha}}(s)\Delta s < \infty. \tag{29}$$

Theorem 9. *Assume that (I_1)-(I_4) and (29) hold. If*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\vartheta(s)Q(s)-\frac{1}{(1+\alpha)^{1+\alpha}}\frac{r^{-\frac{\alpha+1}{\alpha}}(s)r(\sigma(s))}{L\vartheta^{\alpha}(\sigma(s))\vartheta^{\alpha}(s)}\right]\Delta s=\infty, \tag{30}$$

and

$$\limsup_{t\to\infty}\int^{t}\left[Q(s)\vartheta^{\alpha}(s)-\left(\frac{\alpha}{\alpha+1}\right)^{\alpha}\frac{r(\sigma(s))}{Lr^{\frac{\alpha+1}{\alpha}}(s)\vartheta(s)}\right]\Delta s=\infty, \tag{31}$$

then Eq.(1) is oscillatory.

Proof. Assume that $x(t)$ has a nonoscillatory solution of (1) with $x(t)>0$ on $[t_0,\infty)_{\mathbb{T}}$, then there exists $T_0\geq t_0$) such that

$$x(t)>0,\quad x(\tau(t))>0,\quad x(\delta_i(t))>0,\ \ i=1,2,3,...,n,\quad on\ \ [T_0,\infty).$$

It is clear from (7) that $[r(t)\psi(x(t))(z^{\Delta}(t))^{\alpha}]$ is decreasing on $[t_1,\infty)_{\mathbb{T}}$, this means that $z^{\Delta}(t)>0$ or $z^{\Delta}(t)<0$.

Case 1. $z^{\Delta}(t)>0$ Proceeding as the same as the proof of Theorem 7 by choosing $\rho(t)=\vartheta(t)$, we get a contradiction by (30).

Case 2. $z^{\Delta}(t)<0$ for $t\geq T_0$. Define

$$u(t)=\frac{r(t)\psi(x(t))(z^{\Delta}(t))^{\alpha}}{z^{\alpha}(t)}. \tag{32}$$

It is clear that $u(t)<0$. Noting that $[r(t)\psi(x(t))(z^{\Delta}(t))^{\alpha}]$ is nonincreasing, we have for $s\geq t\geq T_1$

$$r^{\frac{1}{\alpha}}(s)\psi^{\frac{1}{\alpha}}(x(s))z^{\Delta}(s)\geq r^{\frac{1}{\alpha}}(t)\psi^{\frac{1}{\alpha}}(x(t))z^{\Delta}(t),$$

i.e.

$$z^{\Delta}(s)\geq L^{\frac{1}{\alpha}}r^{\frac{1}{\alpha}}(t)\psi^{\frac{1}{\alpha}}(x(t))z^{\Delta}(t)\frac{1}{r^{\frac{1}{\alpha}}(s)}$$

Integration from t to l, we obtain

$$z(l)\leq z(t)+L^{\frac{1}{\alpha}}r^{\frac{1}{\alpha}}(t)\psi^{\frac{1}{\alpha}}(x(t))z^{\Delta}(t)\int_t^l\frac{1}{r^{\frac{1}{\alpha}}(s)}$$

Since $z^{\Delta}(t)<0$, letting $l\to\infty$, we get

$$0\leq z(t)+L^{\frac{1}{\alpha}}r^{\frac{1}{\alpha}}(t)\psi^{\frac{1}{\alpha}}(x(t))z^{\Delta}(t)\vartheta(t)$$

i.e.

$$\frac{r^{\frac{1}{\alpha}}(t)\psi^{\frac{1}{\alpha}}(x(t))z^{\Delta}(t)}{z(t)}\vartheta(t) \geq -L^{-\frac{1}{\alpha}}$$

Since $z^{\Delta}(t) < 0$ and $[r(t)\psi(x(t))(z^{\Delta}(t))^{\alpha}]$ is nonincreasing, then

$$\frac{[r^{\frac{1}{\alpha}}(t)\psi^{\frac{1}{\alpha}}(x(t))z^{\Delta}(t)]^{\sigma}}{z(\sigma(t))}\vartheta(t) \geq -L^{-\frac{1}{\alpha}}$$

i.e

$$-L^{-1} \leq \vartheta^{\alpha}(t)u(\sigma(t)) \leq 0, \quad t \geq T_1. \tag{33}$$

On the other hand, from (32), we have

$$u^{\Delta}(t) + Q(t) + \frac{\alpha L^{\frac{1}{\alpha}}}{r^{\frac{1}{\alpha}}(\sigma(t))}u^{\frac{\alpha+1}{\alpha}}(\sigma(t)) \leq 0, \quad t \geq T_1 \tag{34}$$

Multiplying (34) by $\vartheta^{\alpha}(t)$, we get

$$\vartheta^{\alpha}(t)u^{\Delta}(t) + Q(t)\vartheta^{\alpha}(t) + \alpha L^{\frac{1}{\alpha}}r^{-\frac{1}{\alpha}}(\sigma(t))u^{\frac{\alpha+1}{\alpha}}(\sigma(t))\vartheta^{\alpha}(t) \leq 0. \tag{35}$$

Integrating from T_1 to t , we get

$$\begin{aligned}&\vartheta^{\alpha}(t)u(t) - \vartheta^{\alpha}(T_1)u(T_1) + \alpha\int_{T_1}^{t} r^{-\frac{1}{\alpha}}(s)\vartheta^{\alpha-1}(s)u(\sigma(s))\Delta s +\\ &\int_{T_1}^{t} Q(s)\vartheta^{\alpha}(s)\Delta s + \alpha\int_{T_1}^{t} L^{\frac{1}{\alpha}}r^{-\frac{1}{\alpha}}(\sigma(s))u^{\frac{\alpha+1}{\alpha}}(\sigma(s))\vartheta^{\alpha}(s)\Delta s \leq 0.\end{aligned} \tag{36}$$

Applying the inequality (20), with $A = \frac{\alpha\vartheta^{\alpha}(s)L^{\frac{1}{\alpha}}}{r^{\frac{1}{\alpha}}(\sigma(s))}$ and $B = \frac{\alpha\vartheta^{\alpha-1}(s)}{r^{\frac{1}{\alpha}}(s)}$, we can get

$$\alpha r^{-\frac{1}{\alpha}}(t)\vartheta^{\alpha-1}(\sigma(t))u(\sigma(t)) \leq \frac{\alpha L^{\frac{1}{\alpha}}\vartheta^{\alpha}(t)}{r^{\frac{1}{\alpha}}(\sigma(t))}u^{\frac{\alpha+1}{\alpha}}(\sigma(t)) + \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{r(\sigma(t)}{Lr^{\frac{\alpha+1}{\alpha}}(t)\vartheta(t)} \tag{37}$$

From (36) and (37), we get

$$\vartheta^{\alpha}(t)u(t) + \int_{T_1}^{t}\left[Q(s)\vartheta^{\alpha}(s) - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha}\frac{r(\sigma(s))}{Lr^{\frac{\alpha+1}{\alpha}}(s)\vartheta(s)}\right]\Delta s \leq \vartheta^{\alpha}(T_1)u(T_1) \tag{38}$$

By (31), $\vartheta^{\alpha}(t)\nu(t) \to -\infty$ *as* $t \to \infty$, which contradicts the fact $-L^{-1} \leq \vartheta^{\alpha}(t)u(\sigma(t)) \leq 0$. This completes the proof. □

Example 1. *Consider the second order nonlinear differential equation*

$$\left[x(t)+\frac{1}{2}x(t-\pi)\right]''+t^2x(t-2\pi)+\frac{t}{3}x(t-3\pi)=0, \qquad t\in\mathbb{T}:=\mathbb{R}. \tag{39}$$

Here $r(t)=1$, $\psi(x)=1$, $p(t)=\frac{1}{2}$, $\alpha=1$, $q_1(t)=t^2$, $q_2(t)=\frac{t}{3}$, $\delta_1(t)=t-2\pi$, and $\delta_2(t)=t-3\pi$. i.e. $\lim_{t\to\infty}\int_{t_0}^{t}r^{-\frac{1}{\alpha}}(s)\Delta s=\infty$.

Now we can fix $L=1$, $\eta_1=\eta_2=1$, $m_1=m_2=\frac{1}{2}$, and $\rho(t)=t$.

Therefore,

$$Q(t)=\sum_{i=1}^{n}q_i(t)\eta_i\frac{(m_i\delta_i(t))^\alpha}{t^\alpha}[1-p(\delta_i(t))]^\alpha=\frac{3t(t-2\pi)+2(t-3\pi)}{12}, \tag{40}$$

and

$$\int_{t_0}^{t}\left[\rho(s)Q(s)-\frac{[\rho^\Delta(s)]^2r(s)}{4\alpha\left(\frac{s}{2}\right)^{\alpha-1}\rho(s)L}\right]\Delta s\to\infty \quad as \quad t\to\infty$$

Hence, all conditions of Theorem6 are satisfied, and Eq(39) is oscillatory.

Although the following example is discussed by [16], however, our technique is different

Example 2. *Consider the second order nonlinear differential equation*

$$\left[\frac{1}{1+x^2(t)}|z'(t)|^{\alpha-1}z'(t)\right]'+e^{\alpha\lambda t}|x(\lambda t)|^{\alpha-1}x(\lambda t)=0, \qquad t\geq 1, \tag{41}$$

where $\alpha>0$, $0<\lambda<1$, $r(t)=1$, $p(t)=1-e^{-t}$, $q(t)=e^{\alpha\lambda t}$, $\psi(x)=\frac{1}{1+x^2(t)}$, $\tau(t)=t-1$, and $\delta(t)=\lambda t$.

It is clear that $\eta=1$, $L=1$, and $\int_{t_0}^{t}r^{-\frac{1}{\alpha}}(s)\Delta s=\infty$.

Therefore,

$$Q(t)=\sum_{i=1}^{n}q_i(t)\eta_i\frac{(m_i\delta_i(t))^\alpha}{t^\alpha}[1-p(\delta_i(t))]^\alpha=\lambda m, \tag{42}$$

and for $\rho(t)=1$, we have

$$\int_{1}^{t}\left[\rho(s)Q(s)-\frac{[\rho^\Delta(s)]^2r(s)}{4\alpha\left(\frac{s}{2}\right)^{\alpha-1}\rho(s)L}\right]\Delta s=\lambda mt\to\infty \quad as \quad t\to\infty. \tag{43}$$

Hence, by Theorem6, then Eq.(41) is oscillatory.

Example 3. *Consider the second order neutral delay differential equation*

$$\left[\frac{t^2}{1+x^2}(x(t)+\frac{1}{t}x(\frac{t}{3}))'\right]' + tx(\frac{t}{2}). \quad t \geq 1 \tag{44}$$

In this equation $r(t) = t^2$, $\psi(x) = \frac{1}{1+x^2}$, $p(t) = \frac{1}{t}$, $\tau(t) = \frac{t}{3}$, $q(t) = t$, and $\delta(t) = \frac{t}{2}$. We can fix $L = 1$ and $\eta = 1$.

It is clear that $R(t) < \infty$ and $\vartheta(t) = 1$. Therefore,

$$Q(t) = \sum_{i=1}^{n} q_i(t)\eta_i \frac{(m_i\delta_i(t))^\alpha}{t^\alpha}[1 - p(\delta_i(t))]^\alpha = mt^2(t-2), \tag{45}$$

According to Theorem 6 we have

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\vartheta(s)Q(s) - \frac{1}{(1+\alpha)^{1+\alpha}} \frac{r^{-\frac{\alpha+1}{\alpha}}(s)r(\sigma(s))}{L\vartheta^\alpha(\sigma(s))\vartheta^\alpha(s)}\right]\Delta s = \infty, \tag{46}$$

and

$$\limsup_{t\to\infty} \int^{t} \left[Q(s)\vartheta^\alpha(s) - \left(\frac{\alpha}{\alpha+1}\right)^\alpha \frac{r(\sigma(s))}{Lr^{\frac{\alpha+1}{\alpha}}(s)\vartheta(s)}\right]\Delta s = \infty, \tag{47}$$

This mean that Eq.(44) is oscillatory.

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