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WHEN THE MAPPING CARRYING SUBMODULES TO THEIR RADICALS IS A LATTICE HOMOMORPHISM

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Abstract: Let R be a commutative ring and M be a unital R-module. In this paper, we investigate when the mapping $\eta_M : \mathcal{L}(M) \to \mathcal{R}(M)$, from the lattice of submodules of M to the lattice of radical submodules of M defined by $\eta_M(N) = \operatorname{rad} N$ is a lattice homomorphism. We show that if M is an R-module which satisfies the radical formula, then η_M is a lattice homomorphism if and only if $\operatorname{rad}(L \cap N) = \operatorname{rad} L \cap \operatorname{rad} N$, for all finitely generated submodules L and N of M.

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1. Introduction

Throughout this paper all rings will be commutative with identity and all modules will be unitary. Let R be a ring and M be an R-module. It is well-known that the set of all submodules of M forms a lattice with the following operations:

$$L \vee N = L + N \text{ and } L \wedge N = L \cap N,$$

for all submodules L and N of M. We shall denote this lattice by $\mathcal{L}(_RM)$ (or simply $\mathcal{L}(M)$) and if $M = R$ by $\mathcal{L}(R)$. As in [8], we consider the mapping $\lambda : \mathcal{L}(R) \to \mathcal{L}(_RM)$ given by $\lambda(I) = IM$, for all ideals I of R, and the mapping $\mu : \mathcal{L}(_RM) \to \mathcal{L}(R)$ given by $\mu(N) = (N : M)$, for all submodules N of M,

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where $(N : M) := \{r \in R \mid rM \subseteq N\}$. It is easily seen that $\lambda(I \vee J) = \lambda(I) \vee \lambda(J)$ and $\mu(N \wedge L) = \mu(N) \wedge \mu(L)$. An R-module M is called a *λ-module* (resp. *μ-module*), if λ (resp. μ) is a lattice homomorphism [8].

A submodule P of M is called a *prime submodule* (or *p-prime submodule*) if $P \neq M$ and, for $p = (P : M)$, whenever $rm \in P$ for $r \in R$ and $m \in M$, we have $r \in p$ or $m \in P$ (see, for example, [5]). The *radical* of a submodule N of M, denoted $\operatorname{rad} N$, is the intersection of all prime submodules of M containing N or, in case there are no such prime submodules, $\operatorname{rad} N$ is M; in particular $\operatorname{rad} M = M$. A submodule N of M is called a *radical submodule* if $\operatorname{rad} N = N$ [2]. For an ideal I of a ring R, we assume that $\sqrt{I}$ denotes the radical of I. An R-module M is said to be *multiplication* if, for every submodule N of M, there exists an ideal I of R such $N = IM$ [1].

Now let $\mathcal{R}({}_RM)$ (or simply $\mathcal{R}(M)$) be the set of radical submodules of an R-module M. In general $\mathcal{R}({}_RM)$ is not a sublattice of $\mathcal{L}({}_RM)$ [6, p. 36]. However $\mathcal{R}(M)$ forms a lattice with respect to the following operations:

$$L \vee N = \operatorname{rad}(L + N) \text{ and } L \wedge N = L \cap N.$$

In [6], the authors have examined the properties of the mapping $\rho : \mathcal{R}(R) \to \mathcal{R}(M)$ defined by $\rho(I) = \operatorname{rad}(\lambda(I)) = \operatorname{rad}(IM)$ for every ideal I of R and the mapping $\sigma : \mathcal{R}(M) \to \mathcal{R}(R)$ defined by $\sigma(N) = \mu(N) = (N : M)$ for every submodule N of M. In particular, they have considered when these mappings are lattice homomorphisms. In this paper, we are going to explore the relationship between the lattices $\mathcal{L}({}_RM)$ and $\mathcal{R}(M)$ via the mapping $\eta_M : \mathcal{L}(M) \to \mathcal{R}(M)$, defined by $\eta_M(N) = \operatorname{rad} N$. In particular, we consider this mapping when it is a homomorphism. Although $\eta_M(L \vee N) = \eta_M(L) \vee \eta_M(N)$ for all submodules L and N of M, the mapping η_M is not a homomorphism in general (Example 1). It is shown that η_M is a homomorphism if and only if $\operatorname{rad}(L \cap N) = \operatorname{rad} L \cap \operatorname{rad} N$ for all finitely generated submodules L and N of M (Lemma 6). Moreover, if M is a multiplication module (i.e., if λ is surjective), then η_M is a homomorphism (Theorem 10).

2. The Mapping η_M

Recall that $\eta_M : \mathcal{L}(M) \to \mathcal{R}(M)$ defined by $\eta_M(N) = \operatorname{rad} N$ is a mapping which is not necessarily a lattice homomorphism as the following example shows.

Example 1. Consider $M = \mathbb{Z} \oplus \mathbb{Z}$ as a $\mathbb{Z}$-module and let $N = (2, 3)\mathbb{Z}$ and $L = (4, 0)\mathbb{Z} + (0, 1)\mathbb{Z}$ be two submodules of M. Then we can see that

$\eta_M(N \wedge L) = \mathrm{rad}(N \cap L) = (4,6)\mathbb{Z} \subsetneq (2,3)\mathbb{Z} = \mathrm{rad}\, N \cap \mathrm{rad}\, L = \eta_M(N) \wedge \eta_M(L)$. For more details see [7, Example 2.3].

We shall say that M is an *η-module* in case the mapping η_M is a homomorphism. It is clear that every ring R is an η-module over R. The following example shows that every vector space V over a field F is also an η-module.

Example 2. Let V be a vector space over the field F. It is easily seen that every proper subspace W of V is a prime submodule of V since $(W : V) = 0$. Let W_1 and W_2 be proper subspaces of V. Then $W_1 \cap W_2 \subseteq W_1 \subsetneq V$. Thus $W_1 \cap W_2$ is prime and hence $\mathrm{rad}(W_1 \cap W_2) = W_1 \cap W_2 = \mathrm{rad}\, W_1 \cap \mathrm{rad}\, W_2$. Thus V is an η-module.

Note the following result.

Lemma 3. *Let R be a ring and M be an R-module. Then M is an η-module if and only if $\mathrm{rad}(L \cap N) = \mathrm{rad}\, L \cap \mathrm{rad}\, N$ for all submodules L and N of M.*

Proof. Let L and N be any submodules of M . Hence we have

$$\begin{aligned}\eta_M(L \vee N) &= \mathrm{rad}(L + N) = \mathrm{rad}(\mathrm{rad}\, L + \mathrm{rad}\, N) = \mathrm{rad}\, L \vee \mathrm{rad}\, N \\ &= \eta_M(L) \vee \eta_M(N).\end{aligned}$$

for all submodules L and N of M. Thus M is an η-module if and only if $\eta_M(L \wedge N) = \eta_M(L) \wedge \eta_M(N)$ if and only if $\mathrm{rad}(L \cap N) = \mathrm{rad}\, L \cap \mathrm{rad}\, N$. □

Proposition 4. *Let M be an R-module such that $(L : M) + (N : M) = R$ for all incomparable submodules L and N of M. Then M is an η-module.*

Proof. Let L and N be two submodules of M. By Lemma 3 we need to show that $\mathrm{rad}(L \cap N) = \mathrm{rad}\, L \cap \mathrm{rad}\, N$. If N and L are comparable, then there is nothing to prove. Therefore we let N and L to be incomparable and hence $(L : M) + (N : M) = R$, by assumption. Let P be a prime submodule of M containing $L \cap N$. Then $(P : M)$ is a prime ideal of R and $(L : M) \cap (N : M) = (L \cap N : M) \subseteq (P : M)$. Thus $(L : M) \subseteq (P : M)$ or $(N : M) \subseteq (P : M)$. Let $(L : M) \subseteq (P : M)$ and hence $(N : M) \nsubseteq (P : M)$. Let $r \in (N : M) \setminus (P : M)$ and $l \in L$. Then $rl \in L \cap N \subseteq P$ which implies that $l \in P$, since P is prime. Thus $L \subseteq P$. So, we have $L \subseteq P$ which implies that $\mathrm{rad}\, L \cap \mathrm{rad}\, N \subseteq \mathrm{rad}(L \cap N)$ and hence we have the desired equality. □

Remark 5. Let R be a ring and M be an R-module. Define the relation $\sim$ on $\mathcal{L}(M)$ by

$$N \sim L \Leftrightarrow \mathrm{rad}\, N = \mathrm{rad}\, L,$$

for every submodules N and L of M. It is clear that $\sim$ is an equivalence relation in $\mathcal{L}(M)$. We denote the set of all equivalence classes by $\frac{\mathcal{L}(M)}{\sim}$. Clearly $\overline{\eta} : \frac{\mathcal{L}(M)}{\sim} \to \mathcal{R}(M)$ defined by $\overline{\eta}(\sqrt{N}) = \operatorname{rad} N$is bijective and hence by [8, Lemma 2.1], is an isomorphism.

Let M be an R-module and N a proper submodule of M. Let $E_M(N) = \{rx : r \in R \text{ and } x \in M \text{ such that } r^n x \in N \text{ for some } n \in \mathbb{N}\}$. The *envelop submodule* of N in M is defined to be the submodule of M generated by $E_M(N)$, that is $RE_M(N)$. An R-module M is said to *satisfy the radical formula* if $\operatorname{rad} N = RE_M(N)$, for each submodule N of M.

Lemma 6. *Let M be an R-module which satisfies the radical formula. Then the following statements are equivalent:*

(i) *M is an η-module.*

(ii) $\operatorname{rad}(N_1 \cap N_2 \cap \ldots \cap N_n) = \operatorname{rad} N_1 \cap \operatorname{rad} N_2 \cap \ldots \cap \operatorname{rad} N_n$, *for every positive integer n and submodules $N_i (1 \le i \le n)$ of M.*

(iii) $\operatorname{rad}(L \cap N) = \operatorname{rad} L \cap \operatorname{rad} N$, *for all finitely generated submodules L and N of M.*

Proof. (i) $\Rightarrow$ (ii) By Lemma 3 and induction on n.

(ii) $\Rightarrow$ (iii) Clear.

(iii) $\Rightarrow$ (i) Let L and N be any submodules of M. Clearly $\operatorname{rad}(L \cap N) \subseteq \operatorname{rad} L \cap \operatorname{rad} N$. Let $m \in \operatorname{rad} L \cap \operatorname{rad} N$. Since M satisfy the radical formula, $m \in RE_M(L) \cap RE_M(N)$. Hence $m = \sum\limits_{i=1}^{s} r_i x_i$ for some $r_i \in R$ and $x_i \in M (1 \leqslant i \leqslant s)$ where $x_i = a_i u_i$ and $a_i^{n_i} u_i \in L$, for some $a_i \in R$, $u_i \in M$ and positive integers $n_i (1 \leqslant i \leqslant s)$. Also $m = \sum_{j=1}^{t} s_j y_j$ for some $s_j \in R$ and $y_j \in M (1 \leqslant j \leqslant t)$ where $y_j = b_j v_j$ and $b_j^{m_j} v_j \in N$, for some $b_j \in R$, $v_j \in M$ and positive integers $m_j (1 \leqslant j \leqslant t)$. Now let $L_1 = Ra_1^{n_1} u_1 + Ra_2^{n2} u_2 + \ldots + Ra_s^{n_s} u_s \subseteq L$ and $N_1 = Rb_1^{m_1} v_1 + Rb_2^{m_2} v_2 + \ldots + Rb_t^{m_t} v_t \subseteq N$. Clearly L_1 and N_1 are finitely generated submodules of M and $m \in RE_M(L_1) \cap RE_M(N_1) = \operatorname{rad}(L_1) \cap \operatorname{rad}(N_1)$. Thus $m \in \operatorname{rad}(L_1 \cap N_1)$, by hypothesis. But $\operatorname{rad}(L_1 \cap N_1) \subseteq \operatorname{rad}(L \cap N)$ and we have $m \in \operatorname{rad}(L \cap N)$. $\square$

Let R be a ring and M be an R-module. M is said to be an *arithmetical module* if the following equivalent conditions are satisfied for all submodules K, L and N of M:

(i) $(K + L) \cap N = (K \cap N) + (L \cap N)$,

(ii) $(K \cap L) + N = (K + N) \cap (L + N)$.

Proposition 7. *Let R be a zero-dimensional ring or a one-dimensional domain and let M be a non-faithful arithmetical R-module. Then η_M is a homomorphism.*

Proof. Let Max(R) denote the set of maximal ideals of R. Let N be a submodule of M and P be a prime submodule of M containing N. Note that if R is a zero-dimensional ring, then $(P : M)$ is clearly a maximal ideal of R, and if R is a one-dimensional domain, then $(P : M)$ is a maximal ideal of R since M is not faithful. Now, since

$$(P : M) \subseteq ((P : M)M : M) \subseteq (N + (P : M)M : M) \subseteq (P : M).$$

we have $(N + (P : M)M : M) = (P : M)$ and so $N + (P : M)M$ is a prime submodule of M. Moreover, it is easily seen that $N + (P : M)M$ is minimal among $(P : M)$-prime submodules of M containing N. It shows that $\operatorname{rad}(N) = \bigcap_{p \in \operatorname{Max(R)}} (N + pM)$. Now, since M is arithmetical, we have:

$$\begin{aligned}
\operatorname{rad}(L \cap N) &= \bigcap_{p \in \operatorname{Max(R)}} (L \cap N + pM) \\
&= \bigcap_{p \in \operatorname{Max(R)}} ((L + pM) \cap (N + pM)) \\
&= (\bigcap_{p \in \operatorname{Max(R)}} (L + pM)) \cap (\bigcap_{p \in \operatorname{Max(R)}} (N + pM)) \\
&= \operatorname{rad} L \cap \operatorname{rad} N.
\end{aligned}$$

□

Proposition 8. *Let M be an R-module such that $\operatorname{rad} N = \sqrt{N : M}M$ for any submodule N of M, and $(I \cap J)M = IM \cap JM$ for any radical ideals I and J of R. Then M is an η-module.*

Proof. Let L and N be any submodules of M . Then

$$\begin{aligned}
\operatorname{rad}(L \cap N) &= \sqrt{(L \cap N : M)}M = \sqrt{(L : M) \cap (N : M)}M \\
&= (\sqrt{L : M} \cap \sqrt{N : M})M = \sqrt{L : M}M \cap \sqrt{N : M}M \\
&= \operatorname{rad} L \cap \operatorname{rad} N.
\end{aligned}$$

By Lemma 3, M is an η-module. □

Proposition 9. *Let M be a finitely generated λ-module over a ring R and $(N : M)M = N$ for all radical submodules N of M. Then the R-module M is an η-module.*

Proof. Using [7, Theorem 2.9], we have

$$\begin{aligned}\operatorname{rad}(L \cap N) &= (\operatorname{rad}(L \cap N) : M)M = (\operatorname{rad} L \cap \operatorname{rad} N : M)M \\ &= ((\operatorname{rad} L : M) \cap (\operatorname{rad} N : M))M \\ &= (\operatorname{rad} L : M)M \cap (\operatorname{rad} N : M)M \\ &= \operatorname{rad} L \cap \operatorname{rad} N.\end{aligned}$$

Thus M is an η-module by Lemma 3. □

Theorem 10. *Let R be a ring and M be an R-module such that* $\operatorname{rad} N = \operatorname{rad}((N : M)M)$ *for all submodules N of M. Then M is an η-module. In particular, if M is a multiplication module, then M is an η-module.*

Proof. Let ρ and μ be the mappings as before, and consider the following diagram:

$$\begin{array}{ccc} \mathcal{L}(M) & \xrightarrow{\eta_M} & \mathcal{R}(M) \\ {\scriptstyle\mu}\downarrow & & \uparrow{\scriptstyle\rho} \\ \mathcal{L}(R) & \xrightarrow{\eta_R} & \mathcal{R}(R) \end{array}$$

By hypothesis $\rho\eta_R\mu(N) = \eta_M(N)$ for all submodules N of M. Since, by [6, p. 37], ρ is always a homomorphism, and since ρ and η_R are clearly homomorphisms, we have $\rho\eta_R\mu(N \wedge L) = \rho\eta_R\mu(N) \wedge \rho\eta_R\mu(L)$ and hence $\eta_M(N \wedge L) = \eta_M(N) \wedge \eta_M(L)$. Now Lemma 3 implies M is an η-module. Note that if M is a multiplication module $N = (N : M)M$ for all submodule N of M. Thus it follows the "in particular" part. □

Remark 11. The converse of Theorem 10 is not true necessarily. For example, let V be a vector space over a field F. By Example 2, V is an η-module. However for a non-zero proper subspace W of V we have $\rho\eta\mu(W) = \rho\eta(0) = \rho(0) = \operatorname{rad} 0 = 0 \neq W = \eta(W)$.

In the following corollary, by R_S (resp. M_S) we mean that the ring (resp. module) of fractions with respect to a multiplicatively closed subset S.

Corollary 12. *Let M be a multiplication η-module over a ring R and S be a multiplicatively closed subset of R. Then the R_S-module M_S is an η-module.*

Proof. Since M is a multiplication R-module, it is evident that M_S is a multiplication R_S-module. Now, it is done by Theorem 10. □

Theorem 13. *Let M and M' be R-modules and $\varphi : M \to M'$ be an epimorphism of R-modules such that $Ker\varphi \subseteq N$, for all non-zero submodules N of M. Then M is an η-module if and only if M' is an η-module.*

Proof. ($\Rightarrow$) Let M be an η-module and let L' and N' be submodules of M'. Then $L' = \varphi(L)$ and $N' = \varphi(N)$ for some submodules L and N of M. Since $Ker\varphi \subseteq N$, clearly $\varphi(L \cap N) = \varphi(L) \cap \varphi(N)$, for all submodules L and N of M. Thus using [3, Corollary 1.3], we have $\mathrm{rad}(L' \cap N') = \mathrm{rad}(\varphi(L) \cap \varphi(N)) = \mathrm{rad}(\varphi(L \cap N)) = \varphi(\mathrm{rad}(L \cap N)) = \varphi(\mathrm{rad}\, L \cap \mathrm{rad}\, N) = \varphi(\mathrm{rad}\, L) \cap \varphi(\mathrm{rad}\, N) = \mathrm{rad}(\varphi(L)) \cap \mathrm{rad}(\varphi(N)) = \mathrm{rad}\, L' \cap \mathrm{rad}\, N'$. Therefore M' is an η-module.

($\Leftarrow$) Conversly, assume that M' is an η-module. Let L and N be submodules of M. Then

$$\begin{aligned}\varphi(\mathrm{rad}\, L \cap \mathrm{rad}\, N) &= \varphi(\mathrm{rad}\, L) \cap \varphi(\mathrm{rad}\, N)\\ &= \mathrm{rad}\, \varphi(L) \cap \mathrm{rad}\, \varphi(N) = \mathrm{rad}(\varphi(L) \cap \varphi(N))\\ &= \mathrm{rad}\, \varphi(L \cap N) = \varphi(\mathrm{rad}(L \cap N)).\end{aligned}$$

Hence we have $\mathrm{rad}\, L \cap \mathrm{rad}\, N = \mathrm{rad}(L \cap N)$. Thus, by Lemma 3, M is an η-module. □

Proposition 14. *Let R be any ring. Then every direct summand of an η-module is an η-module.*

Proof. Let K be a direct summand of M. So, there exists an R-module N such that $M = K \oplus N$ and hence $K = \frac{M}{N}$. Let L and L' be submodules of K. Thus $L = \frac{H}{N}$ and $L' = \frac{H'}{N}$ for some submodules H, H' of M. Now, by considering the epimorphism $M \longrightarrow \frac{M}{N}$ and using [3, Corollary 1.3] togather with assumption, we have

$$\begin{aligned}\mathrm{rad}(L \cap L') &= \mathrm{rad}(\frac{H}{N} \cap \frac{H'}{N}) = \mathrm{rad}(\frac{H \cap H'}{N}) = \frac{\mathrm{rad}(H \cap H')}{N}\\ &= \frac{\mathrm{rad}\, H \cap \mathrm{rad}\, H'}{N} = \frac{\mathrm{rad}\, H}{N} \cap \frac{\mathrm{rad}\, H'}{N} = \mathrm{rad}\, L \cap \mathrm{rad}\, L'.\end{aligned}$$

□

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