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REFLEXIVITY ON SPECIAL FUNCTION SPACES

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Abstract: In this paper we prove directly the reflexivity of any powers of the multiplication operator acting on Banach spaces of formal Laurent series, and in our proof, we do not use any lemma or theorems that was used in the same resent results.

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1. Introduction

Let $\{\beta(n)\}_{n=-\infty}^{\infty}$ be a sequence of positive numbers satisfying $\beta(0) = 1$. If $1 < p < \infty$, the space $L^p(\beta)$ consists of all *formal Laurent series* $f(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n)z^n$ such that the norm

$$\|f\| = \|f\|_\beta = (\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^p \beta(n)^p)^{\frac{1}{p}}$$

is finite. These are reflexive Banach spaces with the norm $\|\cdot\|_\beta$. Let $\hat{f}_k(n) = \delta_k(n)$. So $f_k(z) = z^k$ and then $\{f_k\}_{k\in\mathbf{Z}}$ is a basis for $L^p(\beta)$ such that $\|f_k\| = \beta(k)$. We denote the set of multipliers of $L^p(\beta)$ by $L^p_\infty(\beta)$ and the linear operator of multiplication by φ on $L^p(\beta)$ by M_φ.

The functional of point evaluation at $\lambda \in \mathbf{C}$, $e(\lambda) : L^p(\beta) \longrightarrow \mathbf{C}$ is defined

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by $e(\lambda)(f) = f(\lambda)$.

By the same method used in [3] we can see that $L^p(\beta)^* = L^q(\beta^{\frac{p}{q}})$, where $\frac{1}{p} + \frac{1}{q} = 1$. Also if $f(z) = \sum_n \hat{f}(n)z^n \in L^p(\beta)$ and $g(z) = \sum_n \hat{g}(n)z^n \in L^q(\beta^{\frac{p}{q}})$, then clearly

$$< f, g >= \sum_n \hat{f}(n)\overline{\hat{g}(n)}\beta(n)^p$$

where the notation $< f, g >$ stands for $g(f)$. For some sources on formal series, we refer the reader to [1–5].

Recall that if E is a separable Banach space and $A \in B(E)$, then $Lat(A)$ is by definition the *lattice of all invariant subspaces* of A, and $AlgLat(A)$ is the algebra of all operators B in $B(E)$ such that $Lat(A) \subset Lat(B)$. An operator A in $B(E)$ is said to be *reflexive* if $AlgLat(A) = W(A)$, where $W(A)$ is the smallest subalgebra of $B(E)$ that contains A and the identity I and is closed in the weak operator topology.

2. Main Result

In [1], under sufficient conditions, by using the well-known Farrel-Rubel-Shields Theorem, it has been proved that the multiplication operator, M_z, is reflexive on the Hilbert space $L^2(\beta)$. Also, in [4], under the condition of polynomially boundedness, and also by using the Farrel-Rubel-Shields Theorem, it has been proved that M_z is reflexive on Banach spaces $L^p(\beta)$, $1 < p < \infty$. In [5], reflexivity of the powers of M_z acting on Banach spaces $L^p(\beta)$ has been proved and Lemma 2.2 in [5], played an essential role in the proof. In this manuscript, we prove the reflexivity of any positive powers of the multiplication operator M_z acting on $L^p_\infty(\beta)$, without using the above mentioned conditions and the Farrel-Rubel-Shields Theorem.

We will use the following notations:

$$\begin{aligned} r_{01} =& \overline{\lim}\beta(-n)^{\frac{-1}{n}}, \\ r_{11} =& \underline{\lim}\beta(n)^{\frac{1}{n}}, \\ \Omega_{01} =& \{z \in \mathbf{C} : |z| > r_{01}\}, \\ \Omega_{11} =& \{z \in \mathbf{C} : |z| < r_{11}\}, \\ \Omega_1 =& \Omega_{01} \cap \Omega_{11}. \end{aligned}$$

It is possible for a Banach space of analytic functions to be a Banach algebra at the same time. From now on, we suppose that $L^p(\beta) = L^p_\infty(\beta)$ and $\Omega_1 \neq \emptyset$. In the following by $H(\Omega)$ and $H^\infty(\Omega)$ we will mean respectively the set of analytic functions on a plane domain Ω and the set of bounded analytic functions on Ω.

Theorem 2.1. *Let M_z be bounded on $L^p(\beta)$. For all $k \geq 1$, the operator M_{z^k} is reflexive.*

Proof. First note that each point of Ω_1 is a bounded point evaluation ([3]) and indeed $L^p(\beta) \subset H(\Omega_1)$. Let $A \in AlgLat(M_{z^k})$. Since $Lat(M_z) \subset Lat(M_{z^k})$, thus $Lat(M_z) \subset Lat(A)$. This implies that $A \in AlgLat(M_z)$. Then by an argument similar to the proof of Lemma 1 in [1] or by a similar method used in the proof of Theorem 2.4 in [5], we can see that $A = M_\varphi$ for some $\varphi \in L^p_\infty(\beta)$, hence $\varphi \in H^\infty(\Omega_1)$. Set

$$L_0 = L^p(\beta) \cap H(\Omega_{11}).$$

Since $L^p(\beta)$ contains the constants, $L_0 \neq \{0\}$. Clearly every function in L_0 is analytic in Ω_{11}. Now we show that L_0 is a closed subspace of $L^p(\beta)$ that is invariant under M_z. To see this, let $\{g_n\}$ be a sequence in L_0 such that g_n converges to f in $L^p(\beta)$. By applying the Cauchy Integral Theorem we can write $f = f_0 + f_1$ where $f_0 \in H(\Omega_{11})$ and $f_1 \in H_o(\Omega_{01}$, where $H_o(\Omega_{01}$ denotes the space of all functions in $H(\Omega_{01})$ that vanishes at ∞. Clearly $g_n - f_0$ converges uniformly to f_1 on compact subsets of Ω_1 and so $z^i(g_n - f_0)$ converges uniformly to $z^i f_1$ on compact subsets of Ω_1 for $i = 0, 1, 2, \cdots$. Since $z^i(g_n - f_0) \in H(\Omega_{11})$, we have

$$\int_{\gamma_0} \xi^i (g_n(\xi) - f_0(\xi)) d\xi = 0, \quad i = 0, 1, 2, \ldots, \ n = 1, 2, \ldots,$$

where γ_0 is the circle $\{z : \ |z| = r_{11} - \epsilon_0\}$ in Ω_1 where ϵ_0 has been chosen such that $r_{01} < r_{11} - \epsilon_0$. Choose the circle γ_0' sufficiently close to γ_0 with smaller radius so that γ_0 lies in $\text{ext}(\gamma_0')$. We can write

$$f_1(z) = \sum_{n=-\infty}^{-1} a_n z^n, \quad z \in ext(\gamma_0'),$$
$$a_n = \frac{1}{2\pi i} \int_{\gamma_0} f_1(\xi)/\xi^{n+1}\, d\xi, \quad n < 0.$$

But

$$\int_{\gamma_0} \xi^k f_1(\xi) = 0, \quad k = 0, 1, 2, \cdots .$$

From this it follows that $f_1(z) = 0$, $z \in ext(\gamma_0')$. Hence $f_1 \equiv 0$. Therefore $f = f_0$ is analytic in Ω_{11} and so L_0 is closed. Clearly L_0 is invariant under M_z, and contains the constants. Since $AL_0 \subset L_0$ and $1 \in L_0$, we see that $A1 = \phi \in L_0$. But $L_0 \subset H(\Omega_{11})$ and $\varphi \in H^\infty(\Omega_1)$. Thus $\varphi \in H^\infty(\Omega_{11})$ and so we can suppose that φ has a representation of the form $\varphi(z) = \sum_{n=0}^{\infty} \hat{\varphi}(n)z^n$.

Now let $\mathcal{M}_k$ be the closed linear span of the set $\{f_{nk} : n \geq 0\}$ (recall that $f_i(z) = z^i$ for all i). We have

$$M_{z^k} f_{nk} = f_{(n+1)k} \in \mathcal{M}_k$$

for all $n \geq 0$. Thus $\mathcal{M}_k \in Lat(M_{z^k})$, and so $\mathcal{M}_k \in Lat(M_\varphi)$. Since $1 \in \mathcal{M}_k$, thus $M_\varphi 1 = \varphi \in \mathcal{M}_k$. Hence $\hat{\varphi}(i) = 0$ for all $i \neq nk$, $n \geq 0$. Put

$$\varphi_N = \sum_{i=0}^{N} \hat{\varphi}(ik) f_{ik}$$

and note that $\varphi_N \in L^p_\infty(\beta)$ for all $N \geq 0$. Clearly each φ_N is a polynomial in z^k, i.e., $\varphi_N(z) = q_N(z^k)$ for some polynomial q_N. We have

$$\|\varphi - \varphi_N\|^p = \sum_{i=N+1}^{\infty} |\hat{\varphi}(ik)|^p \beta(ik)^p$$

converges to 0 as $N \to \infty$. Therefore,

$$M_{\varphi_N} = \varphi_N(M_z) = q_N(M_{z^k}) \to A = M_\varphi$$

in the weak operator topology, and hence $A \in W(M_{z^k})$. Thus M_{z^k} is reflexive and so the proof is complete. □

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