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STABILITY OF FUNCTIONAL EQUATION IN NON-ARCHIMEDEAN ORTHOGONALITY SPACES

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Abstract: In this paper, the authors established the Hyers-Ulam stability of a new type of additive functional equation

$$f(3x+y)+f(x+3y)=4f(x)+4f(y), \quad \forall x,y \text{ with } x\perp y, \tag{0.1}$$

in non-Archimedean orthogonal spaces by fixed point method. Here $\perp$ is the orthogonality in the sense of Rätz.

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1. Introduction

In 1897, Hensel [14] introduced a normed space which does not have the Archimedean property. It turned out that non-Archimedean spaces have many nice applications (see [5, 19, 20, 26]).

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A valuation is a function $|\cdot|$ from a field K into $[0,\infty)$ such that 0 is the unique element having the 0 valuation, $|rs| = |r|\cdot|s|$ and the triangle inequality holds, i.e.,

$$|r+s| \le |r| + |s|, \quad \forall r,s \in K.$$

A field K is called a valued field if K carries a valuation. Throughout this paper, we assume that the base field is a valued field, hence call it simply a field. The usual absolute values of R and C are examples of valuations.

Let us consider a valuation which satisfies a stronger condition than the triangle inequality. If the triangle inequality is replaced by

$$|r+s| \le max\{|r|,|s|\}, \quad \forall r,s \in K,$$

then the function $|\cdot|$ is called a non-Archimedean valuation, and the field is called a non-Archimedean field. Clearly $|1| = |-1| = 1$ and $|n| \le 1$ for all $n \in N$. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except for 0 into 1 and $|0| = 0$.

Definition 1.1. [25] Let X be a vector space over a field K with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\| : X \to [0,\infty)$ is said to be a non-Archimedean norm if it satisfies the following conditions:

(i) $\|x\| = 0$ if and only if $x = 0$;

(ii) $\|rx\| = |r|\,\|x\| \quad (r \in K, x \in X)$;

(iii) the strong triangle inequality

$$\|x+y\| \le max\{\|x\|,\|y\|\} \quad \forall x,y \in X$$

holds. Then $(X, \|\cdot\|)$ is called a non-Archimedean normed space.

Definition 1.2. (i) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X. Then the sequence $\{x_n\}$ is called Cauchy if for a given $\epsilon > 0$ there is a positive integer N such that

$$\|x_n - x_m\| \le \epsilon$$

for all $n, m \ge N$.

(ii) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X. Then the sequence $\{x_n\}$ is called convergent if for a given $\epsilon > 0$ there are a positive integer N and an $x \in X$ such that

$$\|x_n - x_m\| \le \epsilon$$

for all $n \geq N$. Then we call $x \in X$ a limit of the sequence $\{x_n\}$, and denote by $\lim_{n\to\infty} \{x_n\} = x$.

(iii) If every Cauchy sequence in X converges, then the non-Archimedean normed space X is called a non-Archimedean Banach space.

Assume that X is a real inner product space and $f : X \to R$ is a solution of the orthogonal Cauchy functional equation $f(x + y) = f(x) + f(y), \langle x, y\rangle = 0$. By the Pythagorean theorem $f(x) = \|x\|^2$ is a solution of the conditional equation. Of course, this function does not satisfy the additivity equation everywhere. Thus orthogonal Cauchy equation is not equivalent to the classic Cauchy equation on the whole inner product space.

The stability problem of functional equations originated from a question of S.M. Ulam [38] in 1940, concerning the stability of group homomorphisms.

Let $(G_1, \cdot)$ be a group and let $(G_2, *)$ be a metric group with the metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$, such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x\cdot y), h(x)(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$.

The case of approximately additive functions was solved by D.H. Hyers [15] under the assumption that G_1 and G_2 are Banach spaces. In 1951 and in 1978, a generalized version of the theorem of Hyers for approximately linear mappings was given by T. Aoki [1] and Th.M. Rassias [35]. In 1982, J.M. Rassias [33, 34] provided a generalizations of the Hyers stability theorem which allows the Cauchy difference to be bounded.

There are several orthogonality notations on a real normed space are available. But here, we present the orthogonality concept introduced by J.Rätz[36]. This is given in the following definition.

Definition 1.3. [36] A vector space X is called an orthogonality vector space if there is a relation $x\perp y$ on X such that

(i) $x\perp 0, 0\perp x$ for all $x \in X$;

(ii) if $x\perp y$ and $x, y \neq 0$, then x, y are linearly independent;

(iii) $x\perp y$, $ax\perp by$ for all $a, b \in R$;

(iv) if P is a two-dimensional subspace of X; then: (a) for every $x \in P$ there exists $0 \neq y \in P$ such that $x\perp y$;
(b) there exists vectors $x, y \neq 0$ such that $x\perp y$ and $x + y\perp x - y$.

Any vector space can be made into an orthogonality vector space if we define $x\perp 0, 0\perp x$ for all x and for non zero vector x, y define $x\perp y$ iff x, y are linearly independent. The relation $\perp$ is called symmetic if $x \perp y$ implies that $y \perp x$ for all $x, y \in X$. The pair $(X, \perp)$ is called anorthogonality space. It becomes orthogonality normed space when the orthogonality space is equipped with a norm.

The orthogonal Cauchy functional equation

$$f(x+y) = f(x) + f(y), x \perp y$$

in which $\perp$ is an abstract orthogonally was first investigated by S.Gudder and D. Strawther [13].

The orthogonally quadratic equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y), x \perp y$$

was first investigated by F. Vajzovic [39] when X is a Hilbert space, Y is the scalar field, f is continuous and $\perp$ means the Hilbert space orthogonality. Later, Drljevic [6], Fochi [11], Moslehian [25, 24] and Szab [37] generalized this result.

Let X be a set. A function $d : X \to X \to [0, \infty]$ is called a generalized metric on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, z) = d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We recall a fundamental result in fixed point theory.

Theorem 1.4. [2, 7] *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty, \quad \forall n \geq n_0$;

(2) the sequence $\{J^n x\}$ converges to a fixed point $y*$ of J;

(3) $y*$ is the unique fixed point of J in the set $Y = \{y \in X / d(J^{n_0} x, y) < \infty\}$;

(4) $d(y, y*) \leq \frac{1}{1-\alpha} d(y, Jy)$ for all $y \in Y$.

In 1996, Isac and Rassias [16] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors [3, 4, 8, 9, 10, 18, 22, 27, 28, 29, 30, 31, 32].

In the present paper, the authors introduced a new type of additive functional equation (0.1) and proved the Hyers-Ulam stability of the additive functional equation (0.1) in non-Archimedean orthogonality spaces.

Throughout this paper, assume that $(X, \perp)$ is a non-Archimedean orthogonality space and that $(Y, \|\cdot\|_Y)$ is a real non-Archimedean Banach space. Assume that $|2| \neq 1$.

2. Stability of the Orthogonally Additive Functional Equation

In this section, we deal with the stability problem for the orthogonally additive functional equation

$$Df(x,y) := f(3x+y) + f(x+3y) - 4f(x) - 4f(y) = 0 \tag{2.1}$$

for all $x, y \in X$ with $x \perp y$ in non-Archimedean Banach spaces.

Theorem 2.1. *Let $\phi : X^2 \to [0,\infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(x,y) \leq |4|\, \alpha \varphi\left(\frac{x}{4}, \frac{y}{4}\right) \tag{2.2}$$

for all $x, y \in X$ with $x \perp y$. Let $f : X \to Y$ be an odd mapping satisfying

$$\|Df(x,y)\|_Y \leq \varphi(x,y) \tag{2.3}$$

for all $x, y \in X$ with $x \perp y$. Then there exists a unique orthogonally additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\|_Y \leq \frac{1}{|8| - |8|\,\alpha} \varphi(x,x) \tag{2.4}$$

for all $x \in X$.

Proof. Putting $y = x$ in (2.3), we get

$$\|2f(4x) - 8f(x)\|_Y \leq \varphi(x,x) \tag{2.5}$$

for all $x \in X$, since $x \perp 0$. So

$$\left\| f(x) - \frac{1}{4} f(4x) \right\|_Y \leq \frac{1}{|8|} \varphi(x,x) \tag{2.6}$$

for all $x \in X$.

Consider the set

$$S := \{h : X \to Y\}$$

and introduce the generalized metric on S:

$$d(g,h) = inf\left\{\mu \in R_+ : \|g(x) - h(x)\|_Y \leq \mu\varphi(x,x), \quad \forall x \in X\right\},$$

where, as usual, inf $\varphi = +\infty$. It is easy to show that (S, d) is complete (see [31]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{4} g(4x)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \epsilon$. Then

$$\|g(x) - h(x)\|_Y \leq \varphi(x,x)$$

for all $x \in X$. Hence

$$\|Jg(x) - Jh(x)\|_Y = \left\| \frac{1}{4} g(4x) - \frac{1}{4} h(4x) \right\|_Y \leq \alpha\varphi(x,x)$$

for all $x \in X$. So $d(g,h) = \epsilon$ implies that $d(Jg, Jh) \leq \alpha\epsilon$. This means that

$$d(Jg, Jh) \leq \alpha d(g,h)$$

for all $g, h \in S$.

It follows from (2.6) that $d(f, Jf) \leq \frac{1}{8}$.

By Theorem 1.4, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, i.e.,

$$A(4x) = 4A(x) \tag{2.7}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(h,g) < \infty\}.$$

This implies that A is a unique mapping satisfying (2.7) such that there exists a $\mu \in (0, \infty)$ satisfying

$$\|f(x) - A(x)\|_Y \le \mu\varphi(x, x)$$

for all $x \in X$.

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{1}{4^n} f(4^n x) = A(x)$$

for all $x \in X$.

(3) $d(f, A) \le \frac{1}{1-\alpha} d(f, Jf)$, which implies the inequality

$$d(f, A) \le \frac{1}{|8| - |8|\,\alpha}.$$

This implies that the inequality (2.4) holds.

It follows from (2.2) and (2.3) that

$$\begin{aligned}\|DA(x, y)\|_Y &= \lim_{n\to\infty} \frac{1}{|4|^n} \|Df(4^n x, 4^n y)\|_Y \\ &\le \lim_{n\to\infty} \frac{1}{|4|^n} \varphi(4^n x, 4^n y) \le \lim_{n\to\infty} \frac{|4|^n\,\alpha^n}{|4|^n} \varphi(x, y) = 0\end{aligned}$$

for all $x, y \in X$ with $x \perp y$. So

$$DA(x, y) = 0$$

for all $x, y \in X$ with $x \perp y$. Since f is odd, A is odd. Hence $A : X \to Y$ is an orthogonally additive mapping, i.e.,

$$A(3x + y) + A(x + 3y) = 4A(x) + 4A(y)$$

for all $x, y \in X$ with $x \perp y$. Thus $A : X \to Y$ is a unique orthogonally additive mapping satisfying (2.4), as desired. □

From now on, in corollaries, assume that $(X, \perp)$ is a non - Archimedean orthogonality normed space.

Corollary 2.2. *Let θ be a positive real number and p a real number with $0 < p < 1$. Let $f : X \to Y$ be an odd mapping satisfying*

$$\|Df(x,y)\|_Y \leq \theta\left(\|x\|^p + \|y\|^p\right) \tag{2.8}$$

for all $x, y \in X$ with $x \perp y$. Then there exists a unique orthogonally additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\|_Y \leq \frac{|4|^{p\theta}}{|8|\left(|4|^p - |4|\right)}\|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.1 by taking $\phi(x,y) = \theta\left(\|x\|^p + \|y\|^p\right)$ for all $x, y \in X$ with $x \perp y$. Then we can choose $\alpha = 4^{1-p}$ and we get the desired result. □

Theorem 2.3. *Let $f : X \to Y$ be an odd mapping satisfying* (2.3) *for which there exists a function $\phi : X^2 \to [0, \infty)$ such that*

$$\varphi(x,y) \leq \frac{\alpha}{|4|}\varphi\left(4x, 4x\right)$$

for all $x, y \in X$ with $x \perp y$. Then there exists a unique orthogonally additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\|_Y \leq \frac{\alpha}{|8| - |8|\,\alpha}\varphi(x,x) \tag{2.9}$$

for all $x \in X$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1. Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{4}\right)$$

for all $x \in X$.

It follows from (2.5) that $d(f, Jf) \leq \frac{\alpha}{|8|}$. So

$$d(f, A) \leq \frac{\alpha}{|8| - |8|\,\alpha}$$

. Thus we obtain the inequality (2.9).

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.4. *Let θ be a positive real number and p a real number with $p > 1$. Let $f : X \to Y$ be an odd mapping satisfying* (2.8)*. Then there exists a unique orthogonally additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\|_Y \leq \frac{|4|^{p\theta}}{|8|\left(|4| - |4|^p\right)} \|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.3 by taking

$$\phi(x, y) = \theta\left(\|x\|^p + \|y\|^p\right),$$

for all $x, y \in X$ with $x \perp y$. Then we can choose $\alpha = 4^{p-1}$ and we get the desired result. □

3. Conclusion

In this paper, we proved the Hyers-Ulam stability of the additive functional equation (0.1) in non-Archimedean orthogonality spaces.

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