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# ON CONVEX FUNCTIONS, *E*-CONVEX FUNCTIONS AND THEIR GENERALIZATIONS: APPLICATIONS TO NON-LINEAR OPTIMIZATION PROBLEMS

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**Abstract:** An important class of generalized convex sets and convex functions, called E-convex sets and E-convex functions, have been introduced and studied by Youness and other researchers. This class is proved to be useful in pure and applied mathematical fields. In this paper, some new characterizations of convex function, E-convex function, and their generalizations are discussed in terms of some level sets and different forms of epigraphs which are related to these functions. As an application of generalized convex functions in optimization problems, the optimality conditions of non-linear optimization problem using E-convex function (respectively, strictly E-convex function) and its generalizations such as E-quasiconvex function (respectively, strictly E-quasiconvex) function, and strictly quasi semi E-convex function are discussed.

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**Key Words:** convex function, quasi convex function, E-convex function, E-convex set, slack 2-convex set, semi E-convex function

## 1. Introduction and Preliminaries

Classical convex analysis is an important field of mathematics which plays a vital role in optimization and operation research. The main ingredients of convex

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analysis is related to convex sets and convex functions which are fundamentally employed in convex optimization problems [1], [20] and [21]. This field may include other types of functions with less restrictive convexity assumptions, such as quasiconvex and pseudoconvex functions (see Bazarra et.al [17]). The latter types of functions represent generalizations of convex functions. This area of the classical convex analysis has been generalized into other kinds of convexity by many researchers. For instance, the concept of convex functions has been extended to the class of $B$-vex and invex functions by [3] and [15], respectively (see also [14] and [25], for more recent papers). Another type of generalized convexity is $E$-convexity introduced first by Youness in 1999 [5]. Youness introduced $E$- convex sets, $E$-convex functions, and $E$-convex programming by relaxing the definitions of the ordinary convex set and convex function. The effect of an operator called $E : \mathbb{R}^n \to \mathbb{R}^n$ on the sets take the major place in defining this type of general convexity. Since then, the research on $E$- convexity is continued, improved, generalized, and extended in different directions. For more recent papers on $E$-convex sets (see [10] and [11]), and for $E$-convex functions and their generalizations see (e.g., [2], [4], [6], [7], [8], [12], [16], [18], [19], [22], [23], [24] and [26]). Note that some results appeared in Youness's first paper on $E$-convexity [5] are improved by other researchers. In specific, the proof of the relationship between $E$-convex function and it $E$-epigraph has some erroneous (see [4, Counterexample 2.1]). This motivates Duca and Lupsa [4] to introduce an alternative characterization of $E$-convex function f using different kinds of epigraphs, namely, $epi f$ and $epi_E f$. Chen [22] and [23], on the other hand, has introduced semi $E$-convex functions and characterizes this class of functions by using another type of epigraph called $epi^E f$. One of our purposes in this paper is to give new characterizations of semi-$E$-convex function, $E$-convex function, and convex function using the epigraph sets $epi f, epi_E f$, and $epi^E f$ and slack 2-convex (see Propositions 16- 26). We also give some new characterizations of convex function and its generalizations in terms of -level sets of $f$ (see Propositions 7, 8, 11, 12). These level sets are usually associated with the epigraphs mentioned earlier. We describe the optimality conditions of generalized non-linear optimization problem using generalized convex functions such as $E$-convex functions (respectively, strictly $E$-convex functions), $E$-quasiconvex functions (respectively, strictly $E$-quasiconvex functions), and strictly quasi semi $E$-convex functions. The organization of this paper is as follows. For the rest of this section, we recall some needed definitions and preliminary results. In Section 2, we provide new properties which relate convex function and its generalizations with the $\alpha$-level sets and the epigraphs of these functions. Section 3 is devoted to introduce some applications of generalized convexity in non-linear problems. Such optimization problems are considered as a generalization of convex problems.

Throughout this paper, and for simplicity in appearance, we often refer to the following assumption.

**Assumption A** Let $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$ be a real valued function, $S$ is a non-empty set, and $E : \mathbb{R}^n \to \mathbb{R}^n$ is a given mapping.

**Definition 1.** [5] A set $S \subseteq \mathbb{R}^n$ is said to be $E$-convex (i.e., $S$ is convex with respect to a mapping $E : \mathbb{R}^n \to \mathbb{R}^n$) if and only if it contains the E-convex combination of every two of its elements. Mathematically, $S$ is $E$-convex set if and only if $\forall s_1, s_2 \in S$ and for every $0 \leq \lambda \leq 1$, we have $\lambda E(s_1)+(1-\lambda)E(s_2) \in S$.

**Proposition 2.** *[5] If $S$ and $E$ are defined as in assumption A such that $S$ is an $E$-convex, then $E(S) \subseteq S$.*

**Definition 3.** [9], [13] Let $S_1$ and $S_2$ be two subsets of $\mathbb{R}^n$. Then $S_1$ is said to be slack 2-convex with respect to $S_2$ (for short, $S_1$ is s. 2-convex w.r.t. $S_2$) if, for every for every $s_1, s_2 \in S_1 \cap S_2$ and every $0 \leq \lambda \leq 1$ such that $(1-\lambda)s_1 + \lambda s_2 \in S_2$, we get $(1-\lambda)s_1 + \lambda s_2 \in S_1$.

**Definition 4.** Let $f, S$, and $E$ are defined as in assumption A. Then $f$ is referred to as

(i) Convex function on $S$ if and only if $S$ is a convex set and for each $s_1, s_2 \in S$, and each $0 \leq \lambda \leq 1$, we have

$$f(\lambda s_1 + (1-\lambda)s_2) \leq \lambda f(s_1) + (1-\lambda)f(s_2). \quad [20]$$

(ii) $E$-convex function on $S$ if and only if $S$ is an $E$-convex set and for each $s_1, s_2 \in S$, and each $0 \leq \lambda \leq 1$, we have

$$f(\lambda E(s_1) + (1-\lambda)E(s_2)) \leq \lambda f(E(s_1)) + (1-\lambda)f(E(s_2)).$$

On the other hand, $f$ is strictly $E$-convex if for each $s_1, s_2 \in S$, $s_1 \neq s_2$, and each $0 \leq \lambda \leq 1$, we have

$$f(\lambda E(s_1) + (1-\lambda)E(s_2)) < \lambda f(E(s_1)) + (1-\lambda)f(E(s_2)) \quad [5].$$

(iii) Semi-$E$-convex on $S$ if and only if $S$ is $E$-convex set and for each $s_1, s_2 \in S$, $0 \leq \lambda \leq 1$, we have

$$f(\lambda E(s_1) + (1-\lambda)E(s_2)) \leq \lambda f(s_1) + (1-\lambda)f(s_2) \quad [22].$$

(iv) Quasi semi $E$-convex function if and only if $S$ is an $E$-convex set and for each $s_1, s_2 \in S$ and $0 \leq \lambda \leq 1$, we have

$$f(\lambda E(s_1) + (1-\lambda)E(s_2)) \leq \max f(s_1), f(s_2),$$

and$f$ is strictly quasi semi $E$-convex function if and only if $S$ is a $E$-convex set and for each $s_1, s_2 \in S$ with $s_1 \neq s_2$, and $0 < \lambda < 1$, we have

$$f(\lambda E(s_1) + (1-\lambda)E(s_2)) < \max\{f(s_1), f(s_2)\} \quad [22].$$

(v) Quasiconvex if and only if $S$ is a convex set, and for each $s_1, s_2 \in S$, $0 < \lambda < 1$, we have

$$f[\lambda s_1 + (1-\lambda)s_2] \leq \max\{f(s_1), f(s_2)\} \quad [17, \text{Definition } 3.5.1].$$

(vi) $E$- quasiconvex if and only if for each $s_1, s_2 \in S$, and for each $0 \leq \lambda \leq 1$,

$$f(\lambda E(s_1) + (1-\lambda)E(s_2)) \leq \max\{f(E(s_1), f(E(s_2)\};$$

and $f$ is strictly $E$-quasiconvex if and only if for each $s_1, s_2 \in S$, $E(s_1) \neq E(s_2)$, and each $\lambda \in [0,1]$, we have

$$f(\lambda E(s_1) + (1-\lambda)E(s_2)) < \max\{f(E(s_1), f(E(s_2)\} \quad [6], [26].$$

when studying convex functions in the classical sense, the set of points located on or above the graph of $f$, which is called the epigraph of $f$, is useful for characterizing convex functions. However, in generalized convexity (when the functions are $E$-convex, semi $E$-convex, quasi semi $E$-convex, etc), we deal with three other different notions of epigraphs [4], [5] and [23]. These epigraphs are associated with the operator $E$. We list below the ordinary epigraph and its generalized versions.

**Definition 5.** Let $f, S$, and $E$ are defined as in assumption A. Then, the ordinary epigraph is defined as

$$epi f = \{(s, \alpha) \in S \times \mathbb{R} : f(s) \leq \alpha\} \quad [20],$$

while the epigraphs associated with the operator $E$ are classified as

$$E - e(f) = \{(s, \alpha) \in S \times \mathbb{R} : f(E(s)) \leq \alpha\} \quad [5];$$

$$epi_E f = \{(E(s), \alpha) \in E(S) \times \mathbb{R} : f(E(s)) \leq \alpha\} \quad [4];$$

and

$$epi^E f = \{(E(s), \alpha) \in E(S) \times \mathbb{R} : f(s) \leq \alpha\} \quad [23].$$

Associated with each epigraph defined above, an $\alpha$-level set is defined, respectively, as follows.

**Definition 6.** [18] [20] and [26] Let $f, S$, and $E$ are defined as in assumption A and $\alpha \in \mathbb{R}$. Then

(i) $S_\alpha[f] = \{s \in S : f(s) \leq \alpha\}$.

(ii) $E - S_\alpha[f] = \{s \in S : f(E(s)) \leq \alpha\}$.

(iii) $S_{\alpha,E}[f] = \{E(s) \in E(S) : f(E(s)) \leq \alpha\}$.

(vi) $S_\alpha^E[f] = \{E(s) \in E(S) : f(s) \leq \alpha\}$.

## 2. Some Characterizations of Convex Function, E-Convex Function and Their Generalizations

In this section, we provide some characterizations of convex functions, quasiconvex functions, and quasi-semi $E$-convex functions using the $\alpha$-level sets $S_\alpha^E[f]$ and $E - S_\alpha[f]$ of a function $f$. Note that, some characterizations of $E$-convex function and its generalizations (semi $E$-convex functions, quasi-semi $E$-convex functions, and $E$-quasiconvex functions) are given using $S_\alpha[f]$ and $S_{\alpha,E}[f]$ (see [22, Proposition 4, Proposition 6] and [18, Theorem 3.10, Theorems 3.12-3.14]). We also introduce new characterizations of semi-$E$-convex function, $E$-convex function, and convex function using the epigraph sets $epi f, epi_E f$ and $epi^E f$.

The next two propositions give sufficient conditions for $S_\alpha^E[f]$ to be a convex set and a s. 2-convex w.r.t. $E(S)$, respectively.

**Proposition 7.** *Let $f, S$, and $E$ are defined as in assumption A such that $f$ is convex on the convex set $S, E$ is a linear mapping, and $E(S)$ is a convex set. Then $S_\alpha^E[f]$ is a convex set, for all $\alpha \in \mathbb{R}$.*

*Proof.* Let $\alpha \in \mathbb{R}$ and $E(s_1), E(s_2) \in S_\alpha^E[f]$, then $E(s_1), E(s_2) \in E(S)$ and $f(s_1) \leq \alpha, f(s_2) \leq \alpha$. Since $E(S)$ is a convex set, then

$$\lambda E(s_1) + (1 - \lambda)E(s_2) \in E(S), \tag{1}$$

for each $0 \leq \lambda \leq 1$. Using (1) and the linearity of $E$,

$$\lambda E(s_1) + (1 - \lambda)E(s_2) = E(\lambda s_1 + (1 - \lambda)s_2) \in E(S). \tag{2}$$

This means that $\lambda s_1 + (1 - \lambda)s_2 \in S$. From the convexity of $f$ we have

$$f(\lambda s_1 + (1 - \lambda)s_2) \leq \lambda f(s_1) + (1 - \lambda)f(s_2) \leq \alpha. \tag{3}$$

By (2) and (3), we get $\lambda E(s_1) + (1 - \lambda)E(s_2) \in S_\alpha^E[f]$. □

**Proposition 8.** *Let $f, S$, and $E$ are defined as in assumption A. If $f$ is a convex function on the convex set $S, E$ is a linear mapping. Then $S_\alpha^E[f]$ is a s.2-convex w.r.t. $E(s)$, for all $\alpha \in \mathbb{R}$.*

*Proof.* Let $\alpha \in \mathbb{R}$. Assume that $E(s_1), E(s_2) \in S_\alpha^E[f] \cap E(S)$ such that for each $0 \leq \lambda \leq 1$, we have $\lambda E(s_1) + (1-\lambda)E(s_2) \in E(S)$. Since $E(s_1), E(s_2) \in S_\alpha^E[f]$, then $s_1, s_2 \in S$, and $f(s_1) \leq \alpha, f(s_2) \leq \alpha$. By the linearity of $E$,

$$\lambda E(s_1) + (1-\lambda)E(s_2) = E(\lambda s_1 + (1-\lambda)s_2) \in E(S). \tag{4}$$

This means $\lambda s_1 + (1-\lambda)s_2 \in S$. Since $f$ is a convex function,then

$$f(\lambda s_1 + (1-\lambda)s_2) \leq \lambda f(s_1) + (1-\lambda)f(s_2) \leq \alpha. \tag{5}$$

From (4) and (5), $\lambda E(s_1) + (1-\lambda)E(s_2) \in S_\alpha^E[f]$, which implies, $S_\alpha^E[f]$ is a s. 2-convex w.r.t. $E(S)$. □

**Remark 9.** If the set $S_\alpha^E[f]$ is convex or s. 2-convex w.r.t. $E(S)$, it is not necessary that $f$ is a convex function as we show in the following example.

**Example 10.** let $S = [-10, 10] \subseteq \mathbb{R}, E : \mathbb{R} \to \mathbb{R}$ be a linear mapping such that $E(s) = \frac{1}{2}s$ for each $s \in \mathbb{R}$ and define a function $f : S \to \mathbb{R}$ as $f(s) = s^3$ for each $s \in S$. It is clear that $f$ is not a convex function on $S$. However,the level sets

$$S_\alpha^E[f] = \{\frac{1}{2}s \in [-5, 5] : s^3 \leq \alpha\} = \{s \in [-10, 10] : s^3 \leq \alpha\},$$

are either empty sets or intervals. In either cases, $S_\alpha^E[f]$ is convex, for all $\alpha \in \mathbb{R}$. Also, since $E(S)$ and $S_\alpha^E[f]$ are convex sets, then for each $E(s_1), E(s_2) \in S_\alpha^E[f] \cap E(S)$ such that

$$\lambda E(s_1) + (1-\lambda)E(s_2) \in E(S),$$

for every $0 \leq \lambda \leq 1$, we have

$$\lambda E(s_1) + (1-\lambda)E(s_2) \in S_\alpha^E[f],$$

for all $\alpha \in \mathbb{R}$. i.e., $S_\alpha^E[f]$ is a s. 2-convex w.r.t. $E(S)$.

The following proposition proposes a necessary and sufficient condition for $f$ to be a quasiconvex.

**Proposition 11.** *Let $f, S$, and $E$ are defined as in assumption A. If $E$ is a linear mapping, $S$ is a convex set. Then $S_\alpha^E[f]$ is a convex set, for all $\alpha \in \mathbb{R}$ if and only if $f$ is a quasiconvex on $S$.*

*Proof.* First we prove $f$ is a quasiconvex on $S$. Let $s_1, s_2 \in S$, and set $\alpha = \max\{f(s_1), f(s_2)\}$. Then $E(s_1), E(s_2) \in S_\alpha^E[f]$ which is a convex set, then

$$\lambda E(s_1) + (1-\lambda)E(s_2) \in S_\alpha^E[f], \tag{6}$$

for each $0 < \lambda < 1$. Using (6), and the linearity of $E$, we get

$$\lambda E(s_1) + (1-\lambda)E(s_2) = E(\lambda s_1 + (1-\lambda)s_2) \in S_\alpha^E[f] \subseteq E(S).$$

Then, $\lambda s_1 + (1-\lambda)s_2 \in S$ and

$$f(\lambda s_1 + (1-\lambda)s_2) \leq \alpha = \max\{f(s_1), f(s_2)\}.$$

Hence, $f$ is a quasiconvex on $S$. Let us show the other direction and obtain $S_\alpha^E[f]$ is a convex set, for all $\alpha \in \mathbb{R}$. Let $\alpha \in \mathbb{R}$ and $E(s_1), E(s_2) \in S_\alpha^E[f]$, then $s_1, s_2 \in S$ and $f(s_1) \leq \alpha, f(s_1) \leq \alpha$. Since $S$ is convex and $E$ is linear, then for each $0 \leq \lambda \leq 1$, $\lambda s_1 + (1-\lambda)s_2 \in S$ and

$$\lambda E(s_1) + (1-\lambda)E(s_2) = E(\lambda s_1 + (1-\lambda)s_2) \in E(S). \tag{7}$$

From the assumption, $f$ is quasiconvex on $S$, thus

$$f(\lambda s_1 + (1-\lambda)s_2) \leq \max\{f(s_1), f(s_2)\} \leq \alpha. \tag{8}$$

From (7) and (8), we conclude $S_\alpha^E[f]$ is a convex set. □

A necessary and sufficient condition for the level set $E - S_\alpha[f]$ of a function $f$ to be $E$-convex is given next.

**Proposition 12.** *Let $f : \mathbb{R}^n \to \mathbb{R}$, $S \subseteq \mathbb{R}^n$ is $E$-convex set, and $E : \mathbb{R}^n \to \mathbb{R}^n$. Then $foE$ is a quasi-semi $E$-convex on $S$ if and only if $E - S_\alpha[f]$ is $E$-convex set, for all $\alpha \in \mathbb{R}$.*

*Proof.* let $\alpha \in \mathbb{R}$, and $s_1, s_2 \in E - S_\alpha[f]$, then $f(E(s_1)) \leq \alpha$, and $f(E(s_2)) \leq \alpha$. We intend to show that $\lambda E(s_1) + (1-\lambda)E(s_2) \in E - S_\alpha[f]$, for each $0 \leq \lambda \leq 1$. Since $foE$ is a quasi-semi $E$-convex on the $E$-convex set $S$, then

$$\lambda E(s_1) + (1-\lambda)E(s_2) \in S,$$

and

$$(foE)(\lambda E(s_1) + (1-\lambda)E(s_2)) \leq \max\{(foE)(s_1), (foE)(s_2)\} \leq \alpha.$$

Therefore, $\lambda E(s_1)+(1-\lambda)E(s_2)\in E-S_\alpha[f]$. To show the reverse direction, let $s_1,s_2\in E-S_\alpha[f]$, and $0\le\lambda\le 1$. Set $\alpha=\max\{(foE)(s_1),(foE)(s_2)\}$. Since $E-S_\alpha[f]$ is $E$-convex, then $\lambda E(s_1)+(1-\lambda)E(s_2)\in E-S_\alpha[f]$ such that

$$(f(E(\lambda E(s_1)+(1-\lambda)E(s_2))\le\alpha=\max\{(foE)(s_1),(foE)(s_2)\},$$

i.e,

$$(foE)(\lambda E(s_1)+(1-\lambda)E(s_2))\le\max\{(foE)(s_1),(foE)(s_2)\}.$$

Thus, $foE$ is a quasi-semi $E$-convex on $S$. □

In general, the epigraphs defined in Definition 5 are not equal, e.g (see [22]). We start first with the below proposition which shows the relationship between $epif$ and $E-e(f)$, and $epi^Ef$, $epi_Ef$, respectively. The first part of this proposition has been proved in [23, Theorem 2.2]. We get same conclusion but under weaker condition than the one assumed in [23].

**Proposition 13.** *Let $f,S$, and $E$ are defined as in assumption A such that $f(E(s))\le f(s)\ \ \forall s\in S$, then*

1. *$epif\subset E-e(f)$.*
2. *$epi^Ef\subset epi_Ef$.*

*Proof.* To show (1), let $(s,\alpha)\in epif$, from the definition of $epif$ and the assumption, $f(E(s))\le f(s)\le\alpha$ which implies that $(s,\alpha)\in E-e(f)$. For proving (2), suppose that $(E(s),\alpha)\in epi^Ef$, then $(E(s),\alpha)\in E(S)\times\mathbb{R}$ such that $f(s)\le\alpha$. Since $f(E(s))\le\ f(s)$, then $f(E(s))\le\alpha$. Thus, $(E(s),\alpha)\in epi_Ef$ as required. □

**Proposition 14.** *Let $f,S$, and $E$ are defined as in assumption A such that $E(S)\subseteq S$, then $epi_Ef\subset epif$.*

*Proof.* Let $(E(s),\alpha)\in epi_Ef$, thus

$$(E(s),\alpha)\in E(S)\times\mathbb{R}\text{ and }f(E(s))\le\alpha.$$

Since $E(S)\subseteq S$, then $(E(s),\alpha)\in S\times\mathbb{R}$. Thus, $(E(s),\alpha)\in epif$. □

**Remark 15.** In the preceding proposition, if $E:S\to S$ then $epi_Ef\subset epif$ automatically holds [23]. However, if $E:\mathbb{R}^n\to\mathbb{R}^n$, the assumption $E(S)\subseteq S$ is essential for proving $epi_Ef\subset epif$. If we ignore this assumption, then the conclusion of Proposition 14 may not hold. For example, let $S=[-6,6]\subset\mathbb{R}$ and $E:\mathbb{R}\to\mathbb{R}$ defined as $E(s)=2s$, for all $s\in\mathbb{R}$. Let $f:S\to\mathbb{R}$ such that $f(s) = s^2$ for all $s \in S$. Clearly, $E(S) = [-12, 12] \not\subset S$. i.e., $S$ is not $E$-convex set [5]. Moreover, $epi_E f \not\subset epi f$. Indeed, let $(E(6), 150) \in epi_E f$, i.e., $f(E(6)) = 144 < 150$ but $E(6) = 12 \notin S$. Thus, $(E(6), 150) \notin epi f$.

Chen [22, proposition 9] and [23, Theorems 2.4-2.5, 2.8] provided some characterizations of semi $E$-convex function $f$ in terms of $epif, E - e(f), epi_E f$, and $epi^E f$. Duca and Lupsa [4, Theorems 3.1-3.5], on the other hand, relate $E$-convex function $f$ with $epif$ and $epi_E f$. In what follow, we give new characterizations of semi-$E$-convex functions, $E$-convex functions, and convex functions in terms of epif, $epi_E$ f and $epi^E f$. We start first with sufficient conditions for $f$ to be semi $E$-convex function using the epigraph $epi^E f$. The necessary condition, for this result, is shown in [23, Theorem 2.8].

**Proposition 16.** *Let $f, S$, and $E$ are defined as in assumption A, if $S$ is $E$-convex, $f(E(s)) \leq f(s) \quad \forall s \in S$, and $epi^E f$ is a convex set. Then $f$ is a semi $E$-convex function.*

*Proof.* Let $s_1, s_2 \in S$ such that $(E(s_1), f(s_1))$ ,$(E(s_2), f(s_2)) \in epi^E f$ which is a convex set. Thus, for each $0 \leq \lambda \leq 1$,

$$(\lambda E(s_1) + (1 - \lambda)E(s_2), \lambda f(s_1) + (1 - \lambda)f(s_2)) \in epi^E f \subseteq E(S) \times \mathbb{R}.$$

Since $\lambda E(s_1) + (1 - \lambda)E(s_2) \in E(S)$, then $\exists w \in S$ such that

$$E(w) = \lambda E(s_1) + (1 - \lambda)E(s_2) \text{ and } f(w) \leq \lambda f(s_1) + (1 - \lambda)f(s_2). \tag{9}$$

From the assumption and the inequality in (9),

$$f(\lambda E(s_1) + (1 - \lambda)E(s_2)) = f(E(w)) \leq f(w) \leq \lambda f(s_1) + (1 - \lambda)f(s_2).$$

Hence, $f$ is a semi $E$-convex function. □

**Proposition 17.** *Let $f, S$, and $E$ are defined as in assumption A, if $S$ is $E$-convex, $E(S)$ is convex, $f(E(s)) \leq f(s) \ \forall s \in S$, and $epif$ is a s. 2-convex w.r.t. $E(S) \times \mathbb{R}$. Then $f$ is a semi $E$-convex function.*

*Proof.* The conclusion directly follows from [4, Theorem 3.4] and [22, proposition 5]. Indeed, since $S$ is $E$-convex set, $E(S)$ is a convex set, and $epif$ is a s. 2-convex w.r.t. $E(S) \times \mathbb{R}$, then using [4, Theorem 3.4], $f$ is an $E$-convex function. Applying [22, proposition 5], the last conclusion with the assumption $f(E(s)) \leq \ f(s) \ \forall s \in S$ yield $f$ is a semi $E$-convex function. □

A necessary condition for $f$ to be semi $E$- convex on $S$ is given next.

**Proposition 18.** *Let $f, S$, and $E$ are defined as in assumption A. Assume that $S$ is $E$-convex set and $f$ is semi $E$- convex on $S$. Then $epif$ is a s. 2-convex w.r.t. $E(S) \times \mathbb{R}$.*

*Proof.* Let $(s_1, \alpha), (s_2, \beta) \in epif \cap (E(S) \times \mathbb{R})$ such that $\lambda(s_1, \alpha) + (1 - \lambda)(s_2, \beta) \in E(S) \times \mathbb{R}$. Let $0 \leq \lambda \leq 1$, we must prove that

$$(\lambda s_1 + (1 - \lambda)s_2, \lambda\alpha + (1 - \lambda)\beta) \in epif.$$

Because $(s_1, \alpha), (s_2, \beta) \in epif$, then $f(s_1) \leq \alpha$ and $f(s_2) \leq \beta$. We also have $(s_1, \alpha), (s_2, \beta) \in E(S) \times \mathbb{R}$ and $S$ is $E$-convex then from Proposition 2,

$$s_1, s_2 \in E(S) \subseteq S \text{ and } \lambda s_1 + (1 - \lambda)s_2 \in E(S) \subseteq S. \tag{10}$$

From the first inclusion in (10), there exists $s, w \in S$ such that

$$s_1 = E(s) \text{ and } s_2 = E(w). \tag{11}$$

Since $f$ is a semi $E$-convex on $S$, thus, from (10) and (11),

$$\begin{aligned} f(\lambda s_1 + (1 - \lambda)s_2) &= f(\lambda E(s) + (1 - \lambda)E(w)) \\ &\leq \lambda f(E(s)) + (1 - \lambda)f(E(w)) \\ &= \lambda f(s_1) + (1 - \lambda)f(s_2) \\ &\leq \lambda\alpha + (1 - \lambda)\beta. \end{aligned} \tag{12}$$

From (10) and (12), $(\lambda s_1 + (1 - \lambda)s_2, \lambda\alpha + (1 - \lambda)\beta) \in epif$. □

The next proposition provides a necessary condition for $f$ to be $E$-convex function using the set $epi_E f$. The sufficient condition is given in [4, Theorem 3.1].

**Proposition 19.** *Let $f, S$, and $E$ are defined as in assumption A. If $E(S)$ is a convex set and $f$ is an $E$-convex function on the $E$-convex set $S$. Then $epi_E f$ is a convex set.*

*Proof.* Assume that $(E(s_1), \alpha), (E(s_2), \beta) \in epi_E f$. From the definition of $epi_E f$, we have $f(E(s_1)) \leq \alpha, f(E(s_2)) \leq \beta$ and $E(s_1), E(s_2) \in E(S)$. Since $E(S)$ is a convex set, it follows that, for each $0 \leq \lambda \leq 1.$, we have

$$\lambda E(s_1) + (1 - \lambda)E(s_2) \in E(S). \tag{13}$$

Since $f$ is an $E$-convex function, then

$$\begin{aligned} f(\lambda E(s_1) + (1 - \lambda)E(s_2)) &\leq \lambda f E(s_1) + (1 - \lambda) f E(s_2) \\ &\leq \lambda\alpha + (1 - \lambda)\beta. \end{aligned} \tag{14}$$

From(13) and (14), we get $(\lambda E(s_1) + (1 - \lambda)E(s_2), \lambda\alpha + (1 - \lambda)\beta) \in epi_E f$. i.e., $epi_E f$ is a convex set. □

Combining the preceding Proposition and [4, Theorem 3.1], we obtain the following result.

**Proposition 20.** *Let $f, S$, and $E$ are defined as in assumption A. Assume that $E(S)$ is a convex set and $S$ is an $E$-convex set $S$. Then $epi_E f$ is a convex set if and only if $f$ is an $E$-convex function on $S$.*

Another necessary condition for $f$ to be an $E$-convex function using the set $epi_E f$ is given next.

**Proposition 21.** *Let $f, S$, and $E$ are defined as in assumption A. If $f$ is an $E$-convex on the $E$-convex set, $S$. Then $epi_E f$ is a s. 2-convex w.r.t. $E(S) \times \mathbb{R}$.*

*Proof.* Assume that $(E(s_1), \alpha), (E(s_2), \beta) \in epi_E f \cap (E(S) \times \mathbb{R})$ such that, for each $0 \leq \lambda \leq 1.$, we have

$$(\lambda E(s_1) + (1 - \lambda)E(s_2), \lambda\alpha + (1 - \lambda)\beta) \in E(S) \times \mathbb{R}.$$

From the last assertion and the $E$-convexity of $S$, we have

$$\lambda E(s_1) + (1 - \lambda)E(s_2) \in E(S) \subseteq S. \tag{15}$$

Since $f$ is $E$-convex,then

$$\begin{aligned} f(\lambda E(s_1) + (1 - \lambda)E(s_2)) &\leq (\lambda f E(s_1) + (1 - \lambda) f E(s_2)) \\ &\leq \lambda\alpha + (1 - \lambda)\beta. \end{aligned} \tag{16}$$

From (15) and (16), it follows that

$$(\lambda E(s_1) + (1 - \lambda)E(s_2), \lambda\alpha + (1 - \lambda)\beta) \in epi_E f,$$

as required. □

The converse of the preceding proposition is satisfied when $E(S)$ is a convex set (see [4, Theorem 3.2]). Consequently, the following proposition follows.

**Proposition 22.** *Let $f, S$, and $E$ are defined as in assumption A. Assume that $E(S)$ is a convex set and $S$ is an $E$-convex set $S$. Then $epi_E f$ is a s. 2-convex w.r.t. $E(S) \times \mathbb{R}$ if and only if $f$ is an $E$-convex function on $S$.*

The following two propositions give necessary conditions for $f$ to be a convex function using the convexity of the set $epi^E f$.

**Proposition 23.** *Let $f, S$, and $E$ are defined as in assumption A. If $E(S)$ is a convex set, $f$ is a convex function on the convex set $S$, and $E$ is a linear mapping. Then $epi^E f$ is a convex set.*

*Proof.* suppose that $(E(s_1),\alpha),(E(s_2),\beta)\in epi^E f$ and $0\le\lambda\le 1$. We must show that $(\lambda E(s_1)+(1-\lambda)E(s_2),\lambda\alpha+(1-\lambda)\beta)\in epi^E f$. From the definition of $epi^E f$, we have $f(s_1)\le\alpha, f(s_2)\le\beta$ and $E(s_1),E(s_2)\in E(S)$ Since $E(S)$ is a convex set and $E$ is a linear mapping, then

$$\lambda E(s_1)+(1-\lambda)E(s_2)=E(\lambda(s_1)+(1-\lambda)(s_2))\in E(S), \tag{17}$$

where $\lambda s_1+(1-\lambda)s_2\in S$ . Since $f$ is a convex function on $S$, then

$$f(\lambda(s_1)+(1-\lambda)(s_2))\le\lambda f(s_1)+(1-\lambda)f(s_2)\le\lambda\alpha+(1-\lambda)\beta. \tag{18}$$

Thus, from (17) and (18), $(\lambda E(s_1)+(1-\lambda)E(s_2),\lambda\alpha+(1-\lambda)\beta)\in\ epi^E f$, and hence, $epi^E f$ is a convex set. □

**Proposition 24.** *Let $f,S$, and $E$ are defined as in assumption A. Assume that $f$ is a convex on the convex set $S,E$ is a linear mapping, and $f(E(s))\le f(s)$ for all $s\in S$. Then $epi^E f$ is a s. 2-convex w.r.t. $E(S)\times\mathbb{R}$.*

*Proof.* $(E(s_1),\alpha),(E(s_2),\beta)\in epi^E f\cap(E(S)\times\mathbb{R})$ such that, for $0\le\lambda\le 1.$, we have $(\lambda E(s_1)+(1-\lambda)E(s_2),\lambda\alpha+(1-\lambda)\beta)\in E(S)\times\mathbb{R}$, i.e.,

$$\lambda E(s_1)+(1-\lambda)E(s_2)\in E(S). \tag{19}$$

Since $(E(s_1),\alpha),(E(s_2),\beta)\in epi^E f$, then $f(s_1)\le\alpha$ and $f(s_2)\le\ \beta$. Because $E$ is linear mapping, hence, $\lambda E(s_1)+(1-\lambda)E(s_2)=E(\lambda(s_1)+(1-\lambda)(s_2))\in E(S)$. Thus, $\lambda s_1+(1-\lambda)s_2\in S$ Using the last two assertions and the assumption $f(E(s))\le f(s)$ for all $s\in S$, we get

$$\begin{aligned} f(\lambda E(s_1)+(1-\lambda)E(s_2)) &= f(E(\lambda s_1+(1-\lambda)s_2)) \\ &\le f(\lambda s_1+(1-\lambda)s_2) \\ &\le \lambda\alpha+(1-\lambda)\beta. \end{aligned} \tag{20}$$

By (19) and (20), we obtain $(\lambda E(s_1)+(1-\lambda)E(s_2),\lambda\alpha+(1-\lambda)\beta)\in\ epi^E f$, as we want to prove. □

The next proposition suggests a sufficient condition for $f$ to be a convex function using the set $epi^E f$.

**Proposition 25.** *Let $f,S$, and $E$ are defined as in assumption A. If $E$ is a linear mapping, $S$ is a convex set, and $epi^E f$ is a s. 2-convex w.r.t. $E(S)\times\mathbb{R}$. Then $f$ is a convex function.*

*Proof.* Let $s_1, s_2 \in S$ and $0 \le \lambda \le 1$. Let $(E(s_1), f(s_1)), (E(s_2), f(s_2)) \in (E(S) \times \mathbb{R}) \cap epi^E f$ such that whenever

$$(\lambda E(s_1) + (1-\lambda)E(s_2), \lambda f(s_1) + (1-\lambda)f(s_2)) \in E(S) \times \mathbb{R},$$

then $(\lambda E(s_1)+(1-\lambda)E(s_2), \lambda f(s_1)+(1-\lambda)f(s_2)) \in epi^E f$. Since $E$ is a linear mapping, the last statement yields
$(\lambda E(s_1) + (1-\lambda)E(s_2), \lambda f(s_1) + (1-\lambda)f(s_2)) = (E(\lambda s_1 + (1-\lambda)s_2), \lambda f(s_1) + (1-\lambda)f(s_2)) \in epi^E f$. This means

$$f(\lambda s_1 + (1-\lambda)s_2) \le \lambda f(s_1) + (1-\lambda)f(s_2).$$

From the convexity of $S$, $\lambda s_1 + (1-\lambda)s_2 \in S$. Therefore, $f$ is a convex function. □

From Propositions 24 and 25, the following result deduces.

**Proposition 26.** *Let $f, S$, and $E$ are defined as in assumption A. If $E$ is a linear mapping, $S$ is a convex set, and $f(E(s)) \le f(s)$ for all $s \in S$. Then $epi^E f$ is a s. 2-convex w.r.t. $E(S) \times \mathbb{R}$ if and only if $f$ is a convex function.*

## 3. Some Results of Generalized Convex Programming

In this section, we consider some applications for $E$-convex (strictly $E$-convex) functions, strictly quasi semi $E$-convex functions, $E$-quasiconvex (strictly $E$-quasiconvex) functions in optimization programming problem. Namely, we give some characterizations of the optimal solutions of a nonlinear optimization problem using the generalized convex functions mentioned above. To start, consider the following non-linear constrained optimization problem $(NLP_E)$

$$\begin{gathered}\min(f \circ E)(s)\\ \text{subject to } s \in S,\end{gathered}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ be a real valued function, $S \subseteq \mathbb{R}^n$ be a non-empty set, and $E : \mathbb{R}^n \to \mathbb{R}^n$ is a given mapping. Equivalently, problem $(NLP_E)$ can be expressed as

$$\begin{gathered}\min f(Es)\\ \text{s.t } s \in S.\end{gathered}$$

**Definition 27.** The set of all optimal solutions (or global minimum) of problem $(NLP_E)$ is denoted by $argmin_S foE$ and is defined as

$$argmin_S foE = \{s^* \in S : f(Es^*) \le f(Es) \quad \forall s \in S\}.$$

A global minimum $s^* \in S$ is said to be strict when

$$f(Es^*) < f(Es) \quad \forall s \in S, s^* \neq s.$$

A point $s^* \in \mathbb{R}^n$ is called a local minimizer for problem $(NLP_E)$ if there is exists $r > 0$ such that $f(Es^*) \leq f(Es) \quad \forall s \in B(s^*, r) \cap S$, where $B(s^*, r) = \{s \in \mathbb{R}^n : ||s - s^*|| < r\}$ is the neighborhood of $s^*$ with radius $r$.

Definition 27 can be extended to the case when the optimization problem $(NLP_E)$ is to maximize the objective function by reversing the inequalities above. For the rest of this section, the function $f$, the set $S$, and the mapping $E$ are defined as in problem $(NLP_E)$.

**Remark 28.** For the rest of this paper and wherever it is needed, we assume that the set of minimum (respectively, maximum) optimal solutions is a non-empty set.

The following result provides conditions under which each local minimum of problem $(NLP_E)$ is a global minimum.

**Theorem 29.** *Let $s^*$ is a local minimum of problem $(NLP_E)$ where $f$ is E-convex on $S$ which is E-convex set, $E(S)$ is a convex set, and $E$ is a linear mapping. Then $s^*$ is a global minimum.*

*Proof.* Assume that $s^* \in S$ is a local minimum, then there exists $r > 0$ such that

$$f(Es^*) \leq f(Es) \quad \forall s \in W = S \cap B(s^*, r). \tag{21}$$

It is enough to show that $f(Es^*) \leq f(Es) \quad \forall s \in S \backslash W$. Consider any $s \in S \backslash W$ such that $s$ lies on the extended line formed from $s^*$ and $y$. In other words, $s \notin B(s^*, r)$ and for a fixed value of $\lambda$, we define $y = \lambda Es + (1 - \lambda)Es^*$ where $\lambda = r/(s - s^*) < 1$. Since $S$ is $E$-convex and $E$ is linear, then

$$y = \lambda Es + (1 - \lambda)Es^* = E(\lambda s + (1 - \lambda)s^*) \in S.$$

On the other hand, using the convexity of $E(S)$, we have $y = E(\lambda s + (1 - \lambda)s^*) \in E(S)$. Thus, there exists $z \in S$ such that

$$z = \lambda s + (1 - \lambda)s^* \in S. \tag{22}$$

Using the expressions for $z$ and $\lambda$, we get $z - s^* = r$. i.e., $z \in B(s^*, r)$. The last conclusion together with (22) yields $z \in W = S \cap B(s^*, r)$. Since $s^*$ is a local minimum then from (21)

$$\begin{aligned} f(Es^*) \leq f(Ez) = f(y) &= f(\lambda Es + (1 - \lambda)Es^*) \\ &\leq \lambda f(Es) + (1 - \lambda)f(Es^*), \end{aligned}$$

where the last inequality follows because $f$ is $E$-convex function. By re-arranging last inequality, we get $\lambda f(Es^*) \leq \lambda f(Es)$ which yields

$$f(Es^*) \leq f(Es) \quad \forall s \in S \backslash W. \tag{23}$$

From (21) and (23), $f(Es^*) \leq f(Es) \quad \forall s \in S$. Therefore, $s^*$ is a global minimum of problem $(NLP_E)$. □

Next we give a sufficient condition to obtain unique optimal solution of $(NLP_E)$ using strictly E-convex function $f$.

**Theorem 30.** *Assume that $f$ is strictly E-convex on an E-convex set $S$, $E(S)$ is a convex set, and $E$ is a linear mapping. Then the global optimal solution of problem $(NLP_E)$ is unique.*

*Proof.* Let $s_1^*, s_2^* \in S$ be two different global optimal solutions of problem $(NLP_E)$, then $f(Es_1^*) = f(Es_2^*)$. Since $S$ is $E$-convex, $E(S)$ is convex and $E$ is linear, then for each $0 < \lambda < 1$, the $E$-convex combination

$$\lambda Es_1^* + (1-\lambda)Es_2^* = E(\lambda s_1^* + (1-\lambda)s_2^*) \in E(S) \subseteq S,$$

and hence, there exists $z \in S$ such that $z = \lambda s_1^* + (1-\lambda)s_2^* \in S$ where $z \neq s_1^*$ and $z \neq s_2^*$. Since $f$ is strictly $E$-convex on the $E$-convex set $S$, we have

$$f(Ez) = f(\lambda Es_1^* + (1-\lambda)Es_2^*) < \lambda f(Es_1^*) + (1-\lambda)f(Es_2^*) = f(Es_2^*).$$

This means, $z$ is a global optimal solution which is a contradiction. Thus, the global minimum is unique. □

Another two sufficient conditions for a unique optimal solution of problem $(NLP_E)$ are given next.

**Theorem 31.** *Assume that $f$ is strictly quasi semi E-convex, $E(S)$ is a convex set, $E$ is linear and fixed with respect to the global optimal solution. Then the global optimal solution of $(NLP_E)$ is unique.*

*Proof.* Let $s_1^*, s_2^* \in S$ be two global optimal solutions of problem $(NLP_E)$ such that $s_1^* \neq s_2^*$, then $f(Es_1^*) = f(Es_2^*)$. Since $E(s_1^*) = s_1^*$ and $E(s_2^*) = s_2^*$, the last equality yields $f(s_1^*) = f(s_2^*)$. Now $f$ is strictly quasi semi $E$-convex on the $E$-convex set $S$, then for each $0 < \lambda < 1$, we have

$$f(\lambda Es_1^* + (1-\lambda)Es_2^*) < \max\{f(s_1^*), f(s_2^*)\} = f(s_1^*) = f(E(s_1^*)).$$

By the linearity of $E$, the left-hand side of the inequality above can be written as

$$f(E(\lambda s_1^* + (1-\lambda)s_2^*)) = f(\lambda E s_1^* + (1-\lambda)E s_2^*) < f(E(s_1^*)). \tag{24}$$

The expression above entails a contradiction. Indeed, because $E(S)$ is convex and $S$ is $E$-convex, then $E(\lambda s_1^* + (1-\lambda)s_2^*) \in E(S) \subseteq S$. i.e., there exists $z = \lambda s_1^* + (1-\lambda)s_2^* \in S$ where $z \neq s_1^*$ and $z \neq s_2^*$. Using this fact with the inequality in (24), we conclude $z$ is a global optimal solution. Thus, $s_1^* = s_2^*$ as required. □

**Theorem 32.** *Consider problem ($NLP_E$) in which $S$ is $E$-convex set, $E(S)$ is a convex set, and $E$ is a linear mapping.*

*(i) If $f$ is $E$-quasiconvex on $S$, then the set of optimal solutions $argmin_S foE$ is a convex set.*

*(ii) If $f$ is strictly $E$-quasiconvex and $E$ is injective, then $argmin_S foE$ is singleton (i.e., optimal solution is unique).*

*Proof.* To prove (i), let $s_1^*, s_2^* \in argmin_S foE$ such that $s_1^* \neq s_2^*$. Thus, $f(Es_1^*) = f(Es_2^*) \leq f(Es), \ \forall s \in S$. Since $S$ is $E$-convex, $f$ is $E$-quasiconvex on $S$ and $E$ is linear, then for each $0 \leq \lambda \leq 1$, we have

$$\begin{aligned} f(E(\lambda s_1^* + (1-\lambda)s_2^*)) &= f(\lambda E s_1^* + (1-\lambda)E s_2^*) \\ &\leq \max\{f(Es_1^*), f(Es_2^*)\} \\ &= f(Es_1^*) \leq f(Es). \end{aligned} \tag{25}$$

Since $s_1^*, s_2^* \in S$ which is $E$-convex, $E(S)$ is convex, and $E$ is linear. Then, for each $0 \leq \lambda \leq 1$, we get $E(\lambda s_1^* + (1-\lambda)s_2^*) = \lambda E(s_1^*) + (1-\lambda)E(s_2^*) \in E(S) \subseteq S$, Hence,

$$\lambda s_1^* + (1-\lambda)s_2^* \in S. \tag{26}$$

From (25) and (26), $\lambda s_1^* + (1-\lambda)s_2^* \in argmin_S foE$. Thus, $argmin_S foE$ is a convex set. To show (ii). Let $s_1^*, s_2^* \in S$ be two global optimal solutions of problem ($NLP_E$) such that $s_1^* \neq s_2^*$, then $f(Es_1^*) = f(Es_2^*)$. The fact that E is injective implies $Es_1^* \neq Es_2^*$. Since S is $E$-convex, $E$ is linear, and $f$ is strictly $E$-quasiconvex, then for each $0 < \lambda < 1$ we have

$$\begin{aligned} f(E(\lambda s_1^* + (1-\lambda)s_2^*)) &= f(\lambda E s_1^* + (1-\lambda)E s_2^*) \\ &< \max\{f(Es_1^*), f(Es_2^*)\} \\ &= f(Es_1^*) < f(Es). \end{aligned} \tag{27}$$

The rest of the proof follows as in the Theorem 31 where (27) and the convexity of $E(S)$ provide an optimal solution $z = \lambda s_1^* + (1-\lambda)s_2^* \in S$ such that $z \neq s_1^*$

and $z \neq s_2^*$ (i.e., $Ez \neq Es_1^*$ and $Ez \neq Es_2^*$) which contradicts the optimality of $s_1^*$ of problem $(NLP_E)$. □

The conclusions of the preceding Theorem can be also obtained if the function $f$ is $E$-convex (respectively, strictly $E$-convex) as we show next.

**Corollary 33.** *Consider problem $(NLP_E)$ in which $S$ is $E$-convex set, $E(S)$ is a convex set, and $E$ is a linear mapping.*

*(i) If $f$ is $E$-convex on $S$, then the set of optimal solutions $argmin_S foE$ is a convex set.*

*(ii) If $f$ is strictly $E$-convex and $E$ is injective, then $argmin_S foE$ is singleton.*

*Proof.* From [18, p.3339], every $E$-convex (respectively, strictly $E$-convex) function is $E$-quasiconvex (respectively, strictly $E$-quasiconvex). Thus, the conclusions of (i)-(ii) directly follow. □

**Theorem 34.** *Consider the following maximization non-linear problem $(M\text{-}NLP_E)$*

$$\max\,(foE)(s)$$
$$s.t.\ s \in S,$$

*where $S \subseteq \mathbb{R}^n$ be a non-empty $E$-convex set, $f : \mathbb{R}^n \to \mathbb{R}$ is $E$-convex on $S$, and $E : \mathbb{R}^n \to \mathbb{R}^n$ is a given linear mapping. Assume that $E(S)$ is a convex set $f(E(s)) \leq f(s)\ \ \forall s \in S$, and the set of optimal solutions of problem $(M\text{-}NLP_E)$ is a non-empty. i.e., $argmax_S foE = \{s^* \in S : f(Es^*) \leq f(Es)\ \ \forall s \in S\} \neq \phi$. Then, the maximal optimal solutions of $foE$ occur on the boundary of $S$.*

*Proof.* By a contrary, assume that the maximum exists at a point $s^*$ belongs to the interior of $S$. That is, $f(Es^*) \geq f(Es)\ \ \forall s \in S$ and $s^* \in S$. Draw a line passing through $s^*$ and cutting the boundary of $S$ at $s_1$ and $s_2$. Since $S$ is $E$-convex, then for some $0 < \lambda < 1$, we have $s^* = \lambda E(s_1) + (1-\lambda)E(s_2) \in S$. We also have $E(S)$ is convex and $E$ is linear, then

$$s^* = \lambda E(s_1) + (1-\lambda)E(s_2) = E(\lambda s_1 + (1-\lambda)s_2) \in E(S).$$

Thus, there exists $z \in S$ such that $z = \lambda s_1 + (1-\lambda)s_2 \in S$. Since $f$ is $E$-convex on $S$, then

$$\begin{aligned} f(s^*) = f(Ez) &\leq \lambda f(Es_1) + (1-\lambda)f(Es_2) \\ &\leq \lambda f(s_1) + (1-\lambda)f(s_2). \end{aligned} \tag{28}$$

where in the right most inequality, we used the assumption $f(E(s)) \leq f(s)$ for all $s \in S$. Now, we have two possibilities. If $f(Es_1) \leq f(Es_2)$, then

$$\lambda f(Es_1) + (1-\lambda) f(Es_2) \leq \lambda f(Es_2) + (1-\lambda) f(Es_2) = f(Es_2).$$

Using (28), we get $f(s^*) = f(Ez) \leq f(Es_2)$, yielding $s^*$ is not a global maximum which is a contradiction. Similarly, if $f(Es_2) \leq f(Es_1)$, we get $f(s^*) \leq f(Es_1)$, a contradiction. Hence, the maximum point must occur at the boundary of $S$. □

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