International Journal of Pure and Applied Mathematics
Volume 116 No. 3 2017, 689-693
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v116i3.14

ijpam.eu

ON ω-MIXING FOR TOPOLOGICAL DYNAMICAL SYSTEMS

N. Rajesh[1 §], S. Shanthi[2]

[1]Department of Mathematics
Rajah Serfoji Govt. College
Thanjavur, 613005, Tamilnadu, INDIA

[2]Department of Mathematics
Arignar Anna Govt. Arts College
Namakkal, 637 001, Tamilnadu, INDIA

Abstract: In this paper, we study the properties of ω-mixing for discrete dynamical systems defined on ω-open topological spaces.

AMS Subject Classification: 54C10, 54C08, 54C05
Key Words: ω-open sets, topological dynamical system

1. Introduction

The concept of topological transitivity goes back to G.D. Brikhoff. According to [2], he used it in 1920, [3]. We will consider a discrete dynamical system (X, f) given by a metric space (phase space) x and a continuous map $f : X \to X$. A point $x \in X$ "moves", its trajectory being the sequence $x, f(x), f^2(x), f^3(x), ...$ where f^n is the nth iteration of f. The point $f^n(x)$ is the position of x after n units of time. The set of points of the trajectory of x under f is called the orbit of x, denoted by or $bf(x)$. As a motivation for the notion of topological transitivity of (X, f) one may think of a real physical system, where a state is

Received: May 21, 2017
Revised: July 16, 2017
Published: October 26, 2017

© 2017 Academic Publications, Ltd.
url: www.acadpubl.eu

§Correspondence author

never given or measured exactly, but always up to a certain error. So instead of points one should study (small) open subsets of the phase space and describe how they move in that space. If for instance the minimality of (X, f). Is defined by requiring that every point $x \in X$ visit every open set V in X (that is, $f^n(x) \in V$ for some $n \in N$) then, instead one may wish to study the following concept: every nonempty open subset U of X visits every nonempty open subset V if X in the following sense: $f^n(U) \cap V = \emptyset$ for some $n \in N$. If the system (X, f) has this property, then it is called topologically transitive. We also say that f itself is topologically transitive if no misunderstanding can arise concerning the underlying phase space. Intuitively, a topologically transitive map f has points which eventually move under iteration from one arbitrarily small neighborhood to any other. Consequently, the dynamical system cannot be broken down or decomposed into two subsystems (disjoint sets with nonempty interiors) which do not interact under f, that is, are invariant under the map ($A \subset X$ is invariant if $f(A) \subset A$) [4].

2. Preliminaries

Definition 2.1. [1] Let (X, τ) topological dynamical system, and a point $x \in X$ is called a periodic point of f if for each $n \in N$ for which $f^n(x) = x$ and n is called its period.

Definition 2.2. [5] A subset S of a topological space (X, τ) is called a semiopen set if $S \subset \mathrm{Cl}(\mathrm{Int}(S))$.

Definition 2.3. [6] A subset A of a topological space (X, τ) is called an ω-closed set if $\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and U is semiopen in (X, τ). The complement of an ω-closed set is called an ω-open set. The family of all ω-open sets of (X, τ) is denoted by τ_ω. The intersection of all ω-closed sets containing $A \subset X$ is called the ω-closure of A and is denoted by $\omega\,\mathrm{Cl}(A)$. The union of all ω-open sets contained in $A \subset X$ is called the ω-interior of A and is denoted by $\omega\,\mathrm{Int}(A)$.

Definition 2.4. [7] A topological dynamical system (X, τ) is said to be ω-compact if every cover of X by ω-open sets has a finite subcover. A topological dynamical system (X, τ) is said to be ω-locally compact if every point in X has ω-neighborhood with ω-compact ω-closure sets in X. A topological dynamical system (X, τ) is said to be ω-separable if it has a countable ω-dense set.

Definition 2.5. [7] Let (X, τ) be a topological dynamical system. A set $D \subset X$ is said to be ω-dense in X if $\omega\,\mathrm{Cl}(D) = X$.

Definition 2.6. Let (X, τ) be a topological dynamical system and $f : X \to X$ a continuous function on X. Then f is called topologically ω-transitively if there exists $x \in X$ such that its orbit $O_f(x) = \{f^n(x)\}_{n \in Z}$ is ω-dense in X.

Definition 2.7. [7] A function $f : (X, \tau) \to (Y, \sigma)$ is said to be ω-irresolute if $f^{-1}(V) \in \tau_\omega$ for every $V \in \sigma_\omega$.

3. Topological ω-Transitivity

Definition 3.1. Let $f : X \to X$ be an ω-irresolute map of ω-compact metric space, we say that f is topologically ω-transitive if for any pair of nonempty ω-open sets U and V in X there is an $n \in N$ such that $f^n(U) \cap V \neq \emptyset$.

Proposition 3.2. *Let (X, f) a topological dynamical system, let f is topologically ω-transitive if and only if $\bigcup_{n=0}^{\infty} f^n(U)$ is ω-dense in X for every nonempty ω-open set U of X.*

Proof. Assume, $\bigcup_{n=0}^{\infty} f^n(U)$ is ω-dense in X, there exists a nonempty ω-open set V such that, $\bigcup_{n=0}^{\infty} f^n(U) \cap V \neq \emptyset$. This implies, $f^n(U) \cap V \neq \emptyset$ for all $n \in N$. This is a contradiction to the ω-transitivity of f, hence $\bigcup_{n=0}^{\infty} f^n(U)$ is ω-dense in X. Conversely, let U and V be two nonempty ω-open sets in X. $\bigcup_{n=0}^{\infty} f^n(U)$ is ω-dense in X. $\bigcup_{n=0}^{\infty} f^n(U) \cap V \neq \emptyset$. This implies that $f^k(U) \cap V \neq \emptyset$ for some k. Hence f is ω-transitive. □

Proposition 3.3. *Let (X, f) a topological dynamical system, let f be topologically ω-transitive if and only if $\bigcup_{n=0}^{\infty} f^{-n}(U)$ is ω-dense in X for every nonempty ω-open set U of X.*

Proof. Since $\bigcup_{n=0}^{\infty} f^{-n}(U)$ is ω-open and f is ω-transitive, tt has to meet every ω-open set in X and hence is ω-dense. Conversely, let U and V be nonempty ω-open subsets of X. Then $\bigcup_{n=0}^{\infty} f^{-n}(U)$ is ω-dense in X. As $U \cap \bigcup_{n=0}^{\infty} f^{-n}(V) \neq \emptyset$, there exists $m \in N$ such that $U \cap f^{-m}(V) \neq \infty$. We further have $f^{-m}(U) \cap f^{-m}(V) \neq \infty$, hence f is ω-transitive. □

Definition 3.4. A subset A of a topological spaces (X, τ) is said to be nowhere ω-dense if $\omega\,\mathrm{Cl}(\omega\,\mathrm{Int}(A)) = \emptyset$.

Proposition 3.5. *Let (X, f) a topological dynamical system and f be topologically ω-transitive if and only if $E \subset X$ is ω-closed and $f(E) \subset E$, then $E = X$ or E is nowhere ω-dense in X.*

Proof. Suppose f is ω-transitive, $E \subset X$ is ω-closed and $f(E) \subset E$. Assume that $E \neq X$ and has a nonempty ω-interior. Define $U = X \backslash E$. Clearly U is ω-open since E is ω-closed. Let $V \subset E$ be ω-open since E has a nonempty ω-interior. We have $f^n(V) \cap E$ is invariant, then $f^n(V) \cap E = \emptyset$ for all $n \in N$. This is a contradiction to ω-transitivity, hence $E = X$ or E is nowhere ω-dense. Conversely, let U be a nonempty ω-open set in X. Suppose f is not ω-transitive, then from Proposition 3.3, $\overset{\infty}{\underset{n=0}{\cup}} f^{-n}(U)$ is not ω-dense, but ω-open. Define $E = X \backslash \overset{\infty}{\underset{n=0}{\cup}} f^{-n}(U)$. Clearly E is ω-closed and $E \neq X$. We need to prove that $f(E) \subset E$. Suppose $f(E)$ is not subset of E. This implies $f(E) \cap \overset{\infty}{\underset{n=0}{\cup}} f^{-n}(U) \neq \emptyset$. Then $f^{-1}(f(E) \cap \overset{\infty}{\underset{n=0}{\cup}} f^{-n}(U)) = E \cap \overset{\infty}{\underset{n=0}{\cup}} f^{-n}(U) \neq \emptyset$. This is a contradiction to the definition of E. Since $\overset{\infty}{\underset{n=0}{\cup}} f^{-n}(U)$ is not ω-dense, there exists a nonempty ω-open W in X such that $\overset{\infty}{\underset{n=0}{\cup}} f^{-n}(U) \cap W = \emptyset$. This implies $V \subset E$. This is a contradiction to the fact that E is now here ω-dense, hence f is ω-transitive □

Proposition 3.6. *Let (X, f) be a topological dynamical system, let f is topologically ω-transitive if and only if $U \subset X$ is ω-open and $f^{-1}(U) \subset U$ then $U = \emptyset$ or U is ω-dense in X.*

Proof. Suppose f is ω-transitive, $U \subset X$ is ω-open and $f^{-1}(U) \subset U$. Assume that $U \neq \emptyset$ and U is not ω-dense in X. Then there exist a nonempty ω-open V in X such that $U \cap V = \emptyset$. Further $f^n(U) \cap V = \emptyset$ for all $n \in N$. This implies $U \cap f^{-n}(V) = \emptyset$ for all $n \in N$, a contradiction to transitivity of f, hence $U = \emptyset$ or U is ω-dense in X. Conversely, suppose f is not ω-transitive. For a nonempty ω-open set U in X, let $\overset{\infty}{\underset{n=0}{\cup}} f^{-n}(U)$ is nonempty ω-open and not ω-dense. Clearly $f^{-1}(\overset{\infty}{\underset{n=0}{\cup}} f^{-n}(U))$. This is a contradiction since $\overset{\infty}{\underset{n=0}{\cup}} f^{-n}(U) \neq \emptyset$ is ω-dense. This proves that f is ω-transitive. □

References

[1] G. Buck and J. Simon, Knote as dynamical systems, Topology Appl., 51(1993), 229-246.

[2] W.H. Gottschalk and G.A. Hedlund, Topological Dynamics, Amer. Math. Soc. Colloq. Publ. 36, Providence, RI, 1995.

[3] G.D. Birkhoff, Collected Mathematical Papers, Vols. 1, 2, 3, New York, 1950.

[4] S. Kolyada and L. Snoha, A survey some aspects of topological transitivity,2005.

[5] N. Levine, Semiopen sets and semicontinuity in topological spaces, Amer. Math. Monthly, 70 (1963), 36-41.

[6] P. Sundaram and M. Sheik John, Weakly closed sets and weakly continuous maps in topological spaces, Proc. 82^{nd} session of the Indian Science Congress, Calcutta, 1995 (P-49).

[7] M. Sheik John, A study on generalizations of closed sets and continuous maps in topological spaces, *Ph. D. Thesis*, Bharathiyar University, Coimbatore, India (2002).