International Journal of Pure and Applied Mathematics
Volume 116 No. 3 2017, 695-699
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v116i3.15

ijpam.eu

GROUPS WITH PRIMES ORDER CLASSES

Bilal N. Al-Hasanat[1 §], Mohanad N. Al-Hasanat[2], Asad A. Al-Hasanat[3]
[1,2]Al Hussein Bin Talal University
Ma'an, JORDAN
[3]Yarmouk University
Irbid, JORDAN

Abstract: The order classes of finite groups can give certain and valuable properties of these groups. Unfortunately, not many research have illustrated these classes, it depends on many factors, such as the group description and the exponent of the group. In addition to the order of the group itself. Although, some of the previous research determined the order classes for certain groups, but the group structure has been used to show these classes. Conversely, in this paper we will configure the groups description using their order classes. This will introduce a new notion in finite groups called POC-group "Primes Order Classes group".

AMS Subject Classification: 20B05, 20E34
Key Words: element order, order classes, prime order

1. Introduction

A partition of a finite set introduces the set in such useful way to realize the structure of this set. Some useful partition of finite groups is the order classes. These classes can perform the classification of some finite groups, see [4]. The order classes of some groups are found, such as dihedral groups [1], symmetric groups [2] and non-abelian 2-generators $p-$groups of nilpotency class 2 [3]. These researches and other can help to identify the elements classification of a finite group using the group structure. Conversely, this research interest to give

Received: May 23, 2017
Revised: June 3, 2017
Published: October 26, 2017

§Correspondence author

the structure description of a group using the classification of its order classes. That is by introducing some cases of order classes, then give the description of these groups. This research will establish a new notion in groups theory, which will be called prime order classes group and denoted by POC-group.

2. The Order Classes

The order classes of a group G determines the orders of the group elements and denoted by $OC(G)$, theses classes can be introduced in two ways; once by the set of all available orders of all elements $x \in G$, or by the set of all order pairs $[i, j]$, where i is an order of $x \in G$ and j is the number of such elements in G of order i. In this paper, it is enough to use the first representation.

For a group G of prime order n, it is easy to check that $OC(G) = \{1, n\}$. By Lagrange's Theorem, any $k \in OC(G)$ should be a divisor of n. Therefore, this group is cyclic of order n. Otherwise, the order classes of a group G will be $OC(G) = \{1, k_1, k_2, \cdots, k_m\}$, such that k_i is an order of some elements in G for all $i = 1, \cdots, m$. This research will classify finite groups when each k_i is prime. That are the groups of order classes consists of primes numbers only.

3. Primes Order Classes Groups

For a finite group G of order n, the factorization of n into primes is $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$. The order classes of G only contains the orders of the group elements, each order should divides n. If we choose the groups where the order classes contains only primes, then we have the following definition:

Definition 1. A POC-group G is a finite group in which each $i \in OC(G)$ is a prime factor of $|G|$.

The previous definition introduced a new notion of finite groups. That is, if G is a finite group of order $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, $p_i \neq p_j$ for distinct $i, j = 1, \cdots, r$. Then, G is POC-group if $OC(G) \subseteq \{1, p_1, p_2, \cdots, p_r\}$. to clarify this sort of groups we start with the next elementary result.

Remark 1. A group G of order $n = p^k$ (p is prime) is a POC-group if and only if G isomorphic to $\underbrace{\mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p}_{k-times}$.

Proof. Let G be a POC-group of order $n = p^k$ (p is prime). Then $OC(G) = \{1, p\}$. This implies that any $x \in G \setminus \{e\}$ is of order p. Hence, $\langle x \rangle$ is a cyclic subgroup of G of order p for all $x \in G\backslash\{e\}$. Therefore, G breaks down as a direct product of groups of order p. Since these groups are all of prime orders, then each is cyclic group of order p. Thus, G is isomorphic to $\underbrace{\mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p}_{k-times}$.

Conversely, let G be isomorphic to a direct product of k cyclic groups each of prime order p. Then, $|G| = p^k$, and each element $x \in G$ can be written as $x = (m_1, m_2, \cdots, m_k)$ with $m_i \in \mathbb{Z}_p$ for all $i = 1, \cdots, k$. So, the order of x is

$$o(x) = \text{lcm}\{o(m_i) \mid m_i \in \mathbb{Z}_p \text{ for all } i = 1, \cdots, k\}.$$

Since $m_i \in \mathbb{Z}_p$ for all $i = 1, \cdots, k$, then $o(m_i) = 1$ or p for all $i = 1, \cdots, k$. Therefore, $o(x) = 1$ or p, this implies that $OC(G) = \{1, p\}$. Hence, G is a POC-group. □

The smallest non-cyclic POC-group is S_3. To configure out the other groups by knowing the structure of its order classes, we will follow the next theorems.

Theorem 1. *A group G of order $n = 2p$ (p is an odd prime) is a POC-group with $OC(G) = \{1, 2, p\}$ if and only if $G \cong D_n$.*

Proof. Let G be a POC-group of order $n = 2p$ with $OC(G) = \{1, 2, p\}$ where p is an odd prime. Then, G is a 2-generators group. That is, $G = \langle a, b \rangle$, with $o(a) = 2$ and $o(b) = p$. Since p is an odd prime, then $o(ab) = 2$. Hence, $G \cong \left\langle a\,,\ b \mid a^2 = b^p = (ab)^2 \right\rangle$. Therefore, $G \cong D_{2p} = D_n$.

Conversely, let $G = D_n$ with $n = 2p$ and p is prime. Then, the order classes of G is $OC(G) = \{1, 2, p\}$ (see Proposition 7 in [1]). Hence, G is a POC-group. □

This theorem confirms that, only dihedral groups of orders double of prime numbers are POC-groups. So, D_{30} is not POC-group, because $30 = 2 \times 15$ and 15 not a prime.

Corollary 1. *A POC-group G of order $n = 2p$ (p is an odd prime) with $OC(G) = \{1, 2, p\}$ is not simple group.*

Proof. Let G be a POC-group, with $OC(G) = \{1, 2, p\}$. Then, G has a proper subgroup of order p. This subgroup is of index 2. Therefore, it is normal subgroup of G. That is G is not simple. □

It is clear that a POC-group G with $OC(G) = \{1, 2, p\}$ has only one normal subgroup of index 2, that is the cyclic subgroup N of order p. Therefore, $G/N = \{gN \mid g \in G\} = \{N, G \setminus \{N\}\}$.

Corollary 2. *The exponent of $G = D_{2p}$ (p is an odd prime) is $Exp(G) = |G| = 2p$.*

Proof. Let $G = D_{2p}$, where p is an odd prime. Then, G is a POC-group, with $OC(G) = \{1, 2, p\}$. Thus:

$$Exp(G) = lcm\{o(x) \mid x \in G\} = lcm\{1, 2, p\} = 2p$$

□

Theorem 2. *If G is a POC-group of order $n = p_1^{k_1} p_2^{k_2}$ (p_1 and p_2 are odd primes) with $OC(G) = \{1, p_1, p_2\}$, then $G \cong H \rtimes K$ where H (K) is a direct product of k_1 (k_2) cyclic groups each of order p_1 (p_2).*

Proof. Let G be POC-group of order $n = p_1^{k_1} p_2^{k_2}$ and $OC(G) = \{1, p_1, p_2\}$. Then each element of $G \setminus \{e\}$ is of order p_1 or p_2, and there is no elements of order divides both p_1 and p_2. Therefore, G breaks down as a semidirect product of two groups say H and K of orders $p_1^{k_1}$ and $p_2^{k_2}$. Without lose of generality, assume that the groups H and K are of orders $p_1^{k_1}$ and $p_2^{k_2}$, respectively. Then $H \cong \underbrace{\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_1}}_{k_1-times}$ and $K \cong \underbrace{\mathbb{Z}_{p_2} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_2}}_{k_2-times}$. Therefore,

$$G \cong \left(\underbrace{\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_1}}_{k_1-times} \right) \rtimes \left(\underbrace{\mathbb{Z}_{p_2} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_2}}_{k_2-times} \right)$$

□

The alternating groups A_4 and A_5 are POC-groups. Note that, $|A_4| = 2^2 \times 3$ and $|A_5| = 2^2 \times 3 \times 5$, in both 2 is of the factors but with exponent > 1. So, A_4 or A_5 does not satisfy Theorem 1. Similarly, since not all of the factors are odd primes, then A_4 or A_5 does not satisfy Theorem 2.

Certainly, not any group size associated with POC-group. Such an example, there are 14 groups of order $36 = 2^2 \times 3^2$ (using `GAP`), but none of them is a POC-group. That is because there is no group of order 36 can be produced as a semidirect product of $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$.

Moreover, it is clear that any POC-group of order $n = p^k$ is nilpotent of class 1. Otherwise, it is easy to show that POC-group of order $n = p^k q^m$ is not nilpotent. So, any group G of order $n = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$ for distinct primes p_i with $OC(G) \subseteq \{p_i \mid i = 1, 2, \cdots, m\}$ is not nilpotent group.

The order classes of 2-generator p-group G of nilpotency class 2 and order p^n (p is an odd prime), was given in Theorem 3.3 [3]. Which is $OC(G) = \{1, p, p^2, \cdots, p^w\}$, where w is the max of the generators orders. This implies, these groups are POC-groups only for $n = 0, 1$. Otherwise, these groups are not POC-groups. Conversely, no POC-group will be a 2-generator p-group of class 2 and order p^n (p an odd prime and $n > 1$).

References

[1] Bilal Al-Hasanat, Ahmad, A. and Sulaiman, H. , The order classes of dihedral groups. Proceeding of Mathematical Sciences National Conference (SKSM21), AIP CONF PROC, **1605**, No. 1 (2014), 551-556, **doi:** 10.1063/1.4887648.

[2] Bilal Al-Hasanat, Ahmad, A. and Sulaiman, H., Order classes of symmetric groups. *International Journal of Applied Mathematics*, **26**, No. 4 (2013), 501-510, **doi:** 10.1155/2016/8435768.

[3] Bilal Al-Hasanat, E. Al-Sarairah and Mahmoud Alhasanat, The order classes of 2-generator p-groups, *Journal of Applied Mathematics* (2017), to appear.

[4] A.V. Vasilev, M.A. Grechkoseeva, V.D. Mazurov, Characterization of the finite simple groups by spectrum and order, *Algebra Logic*, **48** (2009), 385-409, **doi:** 10.1007/s10469-009-9074-9.