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POISSON APPROXIMATION FOR THE COLOURFUL CARDS UNDER POISSON VARIATION OF MONTMORT'S MATCHING PROBLEM

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Abstract: We introduced poisson variation of Montmort's matching problem with colourful cards and obtained bound for approximation on the probability of the number of interested colour cards under Poisson variation of Montmort's matching problem by using the Stein-Chen coupling method.

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Key Words: the Poisson variation of Montmort's matching problem, Montmort's matching problem, Stein-Chen coupling method

1. Introduction

One of the classic problems in probability theory is The Matching Problem [3] consisting of many variations which were developed many years in the past. There are many ways to describe the problem such the description as follow: Suppose there are n cards which each one had the identified number ranging from $\{1, 2, \ldots, n\}$. So, the matching will occur when we pick up the card at the i^{th} time and the identified number of the card is i, for $i \in \{1, 2, \ldots, n\}$.

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Example 1.1. We pick up the card for the second time and the identified number of the card is 2.

Example 1.2. . The face values of a five-card deck appear in the order 52143. This could be said that there are two matching occur (2 and 4).

For each $i \in \{1, \ldots, n\}$, let

$$X_i = \begin{cases} 1 & \text{if the } i^{th} \text{ card has value } i, \\ 0 & \text{otherwise.} \end{cases}$$

The probability that $X_i = 1$ is given by

$$P(X_i = 1) = \frac{(n-1)!}{n!} = \frac{1}{n}.$$

Let

$$W = \sum_{i \in n} X_i.$$

Then W is the total number of matches. In 1981, Karoński and Ruciński [6] employed the method of moments to prove that the Poisson distribution can approximate the distribution of W with parameter

$$\lambda = \mathrm{E}W = nP(X_i = 1) = n\frac{1}{n} = 1.$$

In 1992, Barbour, Holst and Janson [2] proposed a uniform bound for approximating the distribution with parameter 1 in the form of

$$\left| P(W \leq w_0) - \frac{1}{e}\sum_{k=0}^{w_0} \frac{1}{k!} \right| \leq \frac{2(1-e^{-1})}{n},$$

where $w_0 \in \{0, 1, \ldots, n\}$

D.R. Rawlings [4] defined the Poisson variation of Montmort's matching problem: Supposing that there are n cards which each one had the identified number ranging from $\{1, 2, \ldots, n\}$. We randomly selected k cards and underlined them. So, the matching will occur when we pick up the card at the i^{th} time, $i \in \{1, 2, \ldots, n\}$ and we got the non-underlined cards with the i number. For example from a five-card deck with the sequence of marked and unmarked face values $\underline{1}5\underline{3}42$ appear, then the only one match is 4.

The colouful cards under Poisson variation of Montmort's matching problem refer to : Suppose that there are n colourful cards which each one had the identified number ranging from $\{1, 2, \ldots, n\}$. There are s colours of cards that we are interested in. From the n colourful cards, k of them, they were randomly selected and underlined. The matching will occur when we pick up the cards in the i^{th} time and we get the card having the interested colour, non-underlined, and with the i number. For example, if we are interested in the card in red and the face values of a six-card deck appear in the order $\underline{red1}$*green*5*red*3*blue*4*red*6*yellow*2 appear, then the only one match is red 3.

For each $i \in \{1, \ldots, n\}$, let

$$\tilde{X}_i = \begin{cases} 1 & \text{if the } i^{th} \text{ interested non-underlined has value } i \\ 0 & \text{otherwise.} \end{cases}$$

The probability that $\tilde{X}_i = 1$ is given by

$$P(\tilde{X}_i = 1) = \frac{\binom{n-1}{s}\binom{n-1}{k}(n-1)!}{\binom{n}{s}\binom{n}{k}n!}$$

Let

$$\tilde{W}_n = \sum_{i \in n} \tilde{X}_i.$$

Then $\tilde{W}_n$ be the total number of matches in The colourful cards under Poisson variation of Montmort's matching problem, and we have

$$\lambda = \mathrm{E}\tilde{W}_n = nP(\tilde{X}_i = 1) = \frac{(n-s)(n-k)}{n^2}$$

The purpose of this paper is to give the bound for Poisson approximation of the colourful cards under the Poisson variation of Montmort's matching problem by using the Stein-Chen coupling method which is introduced in Section 2 and the following theorem is our main result.

Theorem 1.3. Let $\tilde{W}_n$ be the total number of matches in The colourful cards under Poisson variation of Montmort's matching problem, for $w_0 \in \{0, 1, 2, \ldots, n-s-k\}$. Then we have,

1. $\left| P(\tilde{W}_n \leq w_0) - \frac{1}{e^{\lambda}} \sum_{k=0}^{w_0} \frac{\lambda^k}{k!} \right| \leq C_{\lambda,A} \frac{1}{n}$

2. $\left| P(\tilde{W}_n \leq w_0) - \frac{1}{e^{\lambda}} \sum_{k=0}^{w_0} \frac{\lambda^k}{k!} \right| \leq (1 - e^{-\lambda}) \frac{1}{n}$

where $C_{\lambda,A} = \min\left\{1, \lambda, \frac{\triangle(\lambda)}{M_A+1}\right\}$,

$$\triangle(\lambda) = \begin{cases} e^{\lambda} + \lambda - 1 & \text{if } \lambda^{-1}(e^{\lambda} - 1) \leq M_A, \\ 2(e^{\lambda} - 1) & \text{if } \lambda^{-1}(e^{\lambda} - 1) > M_A, \end{cases}$$

and

$$M_A = \begin{cases} \max\{w \mid C_w \subseteq A\} & \text{if } 0 \in A, \\ \min\{w \mid w \in A\} & \text{if } 0 \notin A \end{cases}$$

when $C_w = \{0, 1, \ldots, w - 1\}$.

2. Stein-Chen and Coupling Method

In 1972, Stein [9] introduced a powerful method for obtaining an explicit bound for the error in the normal approximation for dependent random variables. This method was adapted and applied to the Poisson approximation by Chen [2] in 1975. He used the Stein's method to find upper bounds for the error in approximating the distribution of a sum of dependent random indicators by the use of Poisson distribution. This method is usually referred to as the Stein-Chen method (or the Chen-Stein method). The idea of this method is based on the Stein's equation for Poisson distribution with parameter λ which says

$$I_A(j) - Poi_{\lambda}(A) = \lambda g_{\lambda,A}(j+1) - j g_{\lambda,A}(j) \tag{2.1}$$

where $\lambda > 0$, $j \in \mathbb{N} \cup \{0\}$, $A \subseteq \mathbb{N} \cup \{0\}$ and $I_A : \mathbb{N} \cup \{0\} \to \mathbb{R}$ be defined by

$$I_A(w) = \begin{cases} 1 & ; w \in A, \\ 0 & ; w \notin A. \end{cases}$$

The form of solution $g_{\lambda,A}$ in (2.1) is

$$g_{\lambda,A}(w) = \begin{cases} (w-1)!\lambda^{-w} e^{\lambda}[\mathcal{P}_{\lambda}(I_{A \cap C_{w-1}}) - \mathcal{P}_{\lambda}(I_A)\mathcal{P}_{\lambda}(I_{C_{w-1}})] & ; w \geq 1, \\ 0 & ; w = 0 \end{cases}$$

where

$$\mathcal{P}_{\lambda}(I_A) = e^{-\lambda} \sum_{l=0}^{\infty} I_A(l) \frac{\lambda^l}{l!}$$

and

$$C_{w-1} = \{0, 1, \ldots, w-1\}.$$

By substituting j and λ in (2.1) by any integer-valued random variable W and $\lambda = \mathrm{E}W$, we have

$$P(W \in A) - Poi_\lambda A = \mathrm{E}(\lambda g_{\lambda,A}(W+1)) - \mathrm{E}(W g_{\lambda,A}(W)). \tag{2.2}$$

So far W could be $\sum_{i=1}^{n} X_i$ and $\lambda = \mathrm{E}(W) = \sum_{i=1}^{n} p_i$ where $p_i = \mathrm{E}(X_i) = P(X_i = 1)$.

In 1992, Barbour, Holst and Janson [1] found the bound in Poisson approximation by assuming that we can create a random variable W_i for each i, in the same probability space as W, such that the distribution of W_i equals to the conditional distribution of $W - X_i$ given $X_i = 1$, we refer to this as coupling method. Henceforth, for each i ,

$$\begin{aligned} E[X_i g_{\lambda,A}(W)] &= \mathrm{E}\{\mathrm{E}[X_i g_{\lambda,A}(W)|X_i]\} \\ &= \mathrm{E}[X_i g_{\lambda,A}(W)|X_i = 1]P(X_i = 1) \\ &= P(X_i = 1)\mathrm{E}[g_{\lambda,A}(W_i + 1)] \\ &= p_i \mathrm{E}[g_{\lambda,A}(W_i + 1)]. \end{aligned} \tag{2.3}$$

From (2.2) and (2.3) we have,

$$\begin{aligned} &|P(W \in A) - Poi_\lambda(A)| = |\mathrm{E}(\lambda g_{\lambda,A}(W+1)) - \mathrm{E}(W g_{\lambda,A}(W))| \\ &= |\sum_{i=1}^{n} p_i \mathrm{E}(g_{\lambda,A}(W+1)) - \sum_{i=1}^{n} p_i \mathrm{E}(g_{\lambda,A}(W_i+1))| \\ &\le \sum_{i=1}^{n} p_i \mathrm{E}(|\sup_w [g_{\lambda,A}(w+1) - g_{\lambda,A}(w)][(W+1) - (W_i+1)]|) \\ &\le \sup_w [g_{\lambda,A}(w+1) - g_{\lambda,A}(w)] \sum_{i=1}^{n} p_i \mathrm{E}(|W - W_i|). \end{aligned}$$

From the estimates above, we arrive at our fundamental result.

Theorem 2.1. If W and W_i are defined as above, then

$$|P(W \in A) - Poi_\lambda(A)| \le \| g_{\lambda,A} \| \sum_{i=1}^{n} p_i \mathrm{E}(|W - W_i|) \tag{2.4}$$

where $\| g_{\lambda,A} \| := \sup_w [g_{\lambda,A}(w+1) - g_{\lambda,A}(w)]$.

In order to justify the Poisson approximation, we, therefore, have to

1. bound $\| g_{\lambda,A} \|$ and

2. find couplings (W, W_i) such that $\mathrm{E}(|W - W_i|)$ is small.

Many authors would like to determine a bound of $\| g_{\lambda,A} \|$. For $A \subseteq \mathbb{N} \cup \{0\}$, Chen ([2], 1975) prove that

$$\| g_{\lambda,A} \| \leq \min\{1, \lambda^{-1}\}$$

and Janson ([5], 1994) showed that

$$\| g_{\lambda,A} \| \leq \lambda^{-1}(1 - e^{-\lambda}). \tag{2.5}$$

In case of non-uniform bound, Neammanee ([7], 2003) showed that

$$\| g_{\lambda,A} \| \leq \min\left\{\frac{1}{w_0}, \lambda^{-1}\right\}$$

and Teerapabolarn and Neammanee ([10], 2005) gave bound of $\| g_{\lambda,A} \|$ where $A = \{0, 1, \ldots, w_0\}$ in the terms of

$$\| g_{\lambda,A} \| \leq \lambda^{-1}(1 - e^{-\lambda}) \min\left\{1, \frac{e^{\lambda}}{w_0 + 1}\right\}.$$

In general case for any subset A of $\{0, 1, \ldots, n\}$, Santiwipanont and Teerapabolarn ([8], 2006) gave a bound in the form of

$$\| g_{\lambda,A} \| \leq \lambda^{-1} \min\left\{1, \lambda, \frac{\triangle(\lambda)}{M_A + 1}\right\} \tag{2.6}$$

where

$$\triangle(\lambda) = \begin{cases} e^{\lambda} + \lambda - 1 & \text{if } \lambda^{-1}(e^{\lambda} - 1) \leq M_A, \\ 2(e^{\lambda} - 1) & \text{if } \lambda^{-1}(e^{\lambda} - 1) > M_A, \end{cases}$$

and

$$M_A = \begin{cases} \max\{w \mid C_w \subseteq A\} & \text{if } 0 \in A, \\ \min\{w \mid w \in A\} & \text{if } 0 \notin A. \end{cases}$$

The difficult part in applying Theorem 2.1 is to construct W_i which makes $\mathrm{E}|W - W_i|$ small. This has not solution in general. For the case of $X_1, \ldots, X_n$ which are independent, we let $W_i = W - X_i$. Then $\mathrm{E}|W - W_i| = p_i$ and from (2.4), we have $|P(W \in A) - Poi_{\lambda}(A)| \leq \| g_{\lambda,A} \| \sum_{i=1}^{n} p_i^2$.

In next section, we will use Theorem 2.1 to prove our main result by constructing the random variable W_i.

3. Proof of Main Results

Proof of Theorem 1.3. Let s be the number of interested coloured cards. We construct $\tilde{W}_{n,i}$ is the total number of matches in the colourful cards under Poisson variation of Montmort's matching problem, by dropping the card of i number, $\{i \in 0, 1, 2, \ldots, n\}$ is at the i^{th} place under Poisson variation of Montmort's matching problem. For $w_0 \in \{0, 1, 2, \ldots, n-s-k\}$. That is,

$$P(\tilde{W}_{n,i} = w_0) = \frac{(n-1-w_0)!\binom{n-1-w_0}{k}\binom{n-1-w_0}{s}}{(n-1)!\binom{n-1}{k}\binom{n-1}{s}},$$

and we have

$$\begin{aligned} P(\tilde{W}_n - \tilde{X}_i = w_0 \mid X_i = 1) &= \frac{P(\tilde{W}_n - \tilde{X}_i = w_0, \tilde{X}_i = 1)}{P(\tilde{X}_i = 1)} \\ &= \frac{P(\tilde{W}_n = w_0 + 1, \tilde{X}_i = 1)}{P(\tilde{X}_i = 1)} \\ &= \frac{\dfrac{(n-1-w_0)!\binom{n-1-w_o}{k}\binom{n-1-w_o}{s}}{n!\binom{n}{k}\binom{n}{s}}}{\dfrac{(n-1)!\binom{n-1}{k}\binom{n-1}{s}}{n!\binom{n}{k}\binom{n}{s}}} \\ &= \frac{(n-1-w_0)!\binom{n-1-w_0}{k}\binom{n-1-w_0}{s}}{(n-1)!\binom{n-1}{k}\binom{n-1}{s}}. \end{aligned}$$

It clearly shows that the constructed $\tilde{W}_{n,i}$ is distributed as $\tilde{W}_n - 1$ in condition of $\tilde{X}_i = 1$.

In order to bound $E \mid \tilde{W}_n - \tilde{W}_{n,i} \mid$, we observed that

- In case $\tilde{X}_i = 1$, we have the i^{th} card is matched. Thus the total number of cards which are matched after dropping the card numbered i, $i \in \{0, 1, 2, \ldots, n\}$ equals to the number of matched cards minus 1, that is

$$\tilde{W}_{n,i} = \tilde{W}_n - 1. \tag{3.7}$$

- In case $\tilde{X}_i = 0$, the total number of matched cards after dropping the card numbered i and we drawn them again, equals to the matched card minus the sum of number of the j^{th} card, where $i \neq j$. Let

$$\tilde{W}_{n,i} = \tilde{W}_n - \sum_{j=1, i\neq j}^{n} \tilde{X}_i \tilde{Y}_j. \tag{3.8}$$

For each $j \in \{0, 1, 2, \ldots, n\}$, such that we define the indicator random variable $\tilde{Y}_j$ as follow:

$$\tilde{Y}_j = \begin{cases} 1 & \text{if the } j^{th} \text{ card is not matched in The colouful cards} \\ & \text{under Poisson variation of Montmort's matching problem} \\ & \text{after we drawn the cards again,} \\ 0 & \text{otherwise.} \end{cases}$$

We know that

$$E \mid \tilde{W}_n - \tilde{W}_{n,i} \mid = E(\tilde{W}_n - \tilde{W}_{n,i})^+ + E(\tilde{W}_n - \tilde{W}_{n,i})^-,$$

where

$$(\tilde{W}_n - \tilde{W}_{n,i})^+ = \max\{\tilde{W}_n - \tilde{W}_{n,i}, 0\},$$

$$(\tilde{W}_n - \tilde{W}_{n,i})^- = -\min\{\tilde{W}_n - \tilde{W}_{n,i}, 0\}.$$

Form (3.7) and (3.8).

In case $\tilde{X}_i = 1$, we have $(\tilde{W}_n - \tilde{W}_{n,i})^+ = 1$ and $(\tilde{W}_n - \tilde{W}_{n,i})^- = 0$
In case $\tilde{X}_i = 0$, we have $(\tilde{W}_n - \tilde{W}_{n,i})^+ = \sum_{i,j=1, i\neq j}^{n} \tilde{X}_i \tilde{Y}_j$ and $(\tilde{W}_n - \tilde{W}_{n,i})^- = 0$.
Therefore,

$$(\tilde{W}_n - \tilde{W}_{n,i})^+ = \sum_{i,j=1, i\neq j}^{n} \tilde{X}_i \tilde{Y}_j$$

and $(\tilde{W}_n - \tilde{W}_{n,i})^- = 0$.

$$E(\tilde{W}_n - \tilde{W}_{n,i})^+ \leq E\{\sum_{i,j=1, i\neq j}^{n} \tilde{X}_i \tilde{Y}_j\}$$

$$
\begin{aligned}
&= \sum_{i,j=1,i\neq j}^{n} E\{\tilde{X}_i\tilde{Y}_j\} \\
&= \sum_{i,j=1,i\neq j}^{n} P(\tilde{X}_i = 1, \tilde{Y}_j = 1) \\
&= \sum_{i,j=1,i\neq j}^{n} P(\tilde{X}_i = 1)P(\tilde{Y}_j = 1) \\
&= \sum_{i,j=1,i\neq j}^{n} \frac{\binom{n-1}{s}\binom{n-1}{k}(n-1)!}{\binom{n}{s}\binom{n}{k}n!}\left(1 - \frac{\binom{n-2}{s}\binom{n-2}{k}(n-2)!}{\binom{n-1}{s}\binom{n-1}{k}(n-1)!}\right) \\
&\leq \frac{1}{n} \qquad (3.9)
\end{aligned}
$$

hence, by (2.4), (2.5), (2.6) and (3.9), the proof is completed □

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