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ON PRIME AND SEMIPRIME BI-IDEALS IN A PARTIALLY ORDERED Γ-SEMIGROUP

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Abstract: For this paper, we define the concepts of prime and semiprime bi-ideals in a partially ordered Γ-semigroup. And several properties of them are presented. Indeed, we give a characterization when a bi-ideal of a partially ordered Γ-semigroup is prime, and prove that a partially ordered Γ-semigroup is regular if and only if every bi-ideals of the partially ordered Γ-semigroup is semiprime.

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1. Preliminaries

In 1983, A.P.J. van der Walt [1] introduced the interesting concepts of prime and semiprime bi-ideals for an associative ring with unity. In 1995, using the concepts defined by A. P. J. van der Walt, the structure of a ring containing prime and semiprime bi-ideals were studied by H. J. le Roux [2]. In the line of H. J. le Roux, for this paper, we shall define the concepts of prime and semiprime bi-ideals in a partially ordered Γ-semigroup. And then several

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properties are considered. For instance, a characterization when a bi-ideal of a partially ordered Γ-semigroup is prime will be given. Moreover, we prove also that a partially ordered Γ-semigroup is regular if and only if every bi-ideals of the partially ordered Γ-semigroup is semiprime.

The notion of Γ-semigroups credited to M. K. Sen [3], see also [4]–[6], is defined as a generalization of semigroups by: let S and Γ be non-empty sets. Then S is called a Γ-*semigroup* if there is a mapping from $S \times \Gamma \times S$ into S, written as $(a, \alpha, b) \mapsto a\alpha b$, such that

$$(x\alpha y)\beta z = x\alpha(y\beta z)$$

for all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$.

Example 1. Let $(S, \cdot)$ be a semigroup with identity 1. Setting $\Gamma = \{1\}$; then S is a Γ-semigroup under the mapping from $S \times \Gamma \times S$ into S defined by:

$$(x, 1, y) \mapsto x \cdot 1 \cdot y \ (= x \cdot y).$$

Example 2. Let $(S, \cdot)$ be a semigroup. Choose $a_0 \in S$, and set $\Gamma = \{a_0\}$; then S is a Γ-semigroup under the mapping from $S \times \Gamma \times S$ into S defined by:

$$(x, a_0, y) \mapsto x \cdot a_0 \cdot y.$$

Example 3. Let S be a set of all negative rational numbers. Clearly, under the usual product of national numbers, S is not a semigroup. Let

$$\Gamma := \{\tfrac{-1}{p} \mid p \text{ is prime}\}.$$

Hence S is a Γ-semigroup.

A Γ-semigroup (S, Γ) with a partial order on S is called a *partially ordered* Γ-*semigroup* (or simply a *po*-Γ-*semigroup*) if

$$x \leq y \Rightarrow z\alpha x \leq z\alpha y,\ x\alpha z \leq y\alpha z$$

for any $x, y, z \in S$ and $\alpha \in \Gamma$.

An element 0 of a partially ordered Γ-semigroup $(S, \Gamma, \leq)$ is said to be a *zero element* of S if

(i) $0\alpha x = x\alpha 0 = 0$ for all $x \in S$ and $\alpha \in \Gamma$;

(ii) $0 \leq x$ for all $x \in S$.

Throughout this paper we deal with a partially ordered Γ-*semigroup with zero.*

Example 4. Consider, respectively, the sets S and Γ as the sets of $m \times n$ and of $n \times m$ matrices over the set of all non-negative integers. Define a mapping from $S \times \Gamma \times S$ into S by: for $A, B \in S$, $C \in \Gamma$,

$$(A, C, B) \mapsto ACB$$

Here, ACB is the usual matrix multiplication. Then S is a Γ-semigroup.

Moreover, define a partial order on S by: for $A, B \in S$,

$$A \preceq B \text{ if and only if } a_{ij} \leq b_{ij}$$

for all i, j. Then S is a partially ordered Γ-semigroup. The zero matrix acts as a zero element of S.

For non-empty subsets A and B of a partially ordered Γ-semigroup $(S, \Gamma, \leq)$, the set product $A\Gamma B$ of A and B, and the subset $(A]$ of S are defined by:

$$\begin{aligned} A\Gamma B &:= \{a\alpha b \in S \mid a \in A, b \in B, \alpha \in \Gamma\}; \\ (A] &:= \{x \in S \mid \exists a \in A (x \leq a)\}. \end{aligned}$$

It is observed that the following hold:

(1) $A \subseteq (A]$ (hence, $S = (S]$);

(2) $A \subseteq B \Rightarrow (A] \subseteq (B]$;

(3) $(A]\Gamma(B] \subseteq (A\Gamma B]$;

(4) $((A]] = (A]$;

(5) $((A]\Gamma(B]] = (A\Gamma B]$.

As in the (classical) semigroups, a non-empty subset A of a partially ordered Γ-semigroup $(S, \Gamma, \leq)$ is called a *left* (resp. *right*) *ideal* of S if

(i) $S\Gamma A \subseteq A$ (resp. $A\Gamma S \subseteq A$);

(ii) $(A] = A$.

And A is called a *two-sided ideal* (or simply an *ideal*) of S if it is both a left and a right ideal of S. The condition $A = (A]$ is equivalent to the condition: for any $x \in A$ and $y \in S$, $y \leq x$ implies $y \in A$. Note that the intersection of left ideals of S, if it is non-empty, is a left ideal of S. Thus, for a non-empty subset A of S, the intersection of all left ideals of S containing A is a left ideal of S containing A, and it is of the form

$$L(A) = (A \cup S\Gamma A].$$

Similarly, the intersection of all right ideals of S containing A is a right ideal of S containing A, and it is of the form

$$R(A) = (A \cup A\Gamma S].$$

What is more, the intersection of all ideals of S containing A is an ideal of S containing A; this will be denoted by $I(A)$.

For a partially ordered Γ-semigroup $(S, \Gamma, \leq)$, the following generalizes the notion of one-sided ideals of S. A non-empty subset A of S is called a *quasi-ideal* of S if

(i) $S\Gamma A \cap A\Gamma S \subseteq A$;

(ii) $(A] = A$.

A subsemigroup A of a partially ordered Γ-semigroup $(S, \Gamma, \leq)$ (i.e., $A \neq \emptyset$ and $A\Gamma A \subseteq A$) is called a *bi-ideal* of S if

(i) $A\Gamma S\Gamma A \subseteq A$;

(ii) $(A] = A$.

Let $(S, \Gamma, \leq)$ be a partially ordered Γ-semigroup. It is observed that every quasi-ideal of S is a bi-ideal of S, and, for any $a \in S$, $(a\Gamma S\Gamma a]$ is a bi-ideal of S.

An ideal A of a partially ordered Γ-semigroup $(S, \Gamma, \leq)$ is said to be *prime* if for any ideals B, C of S,

$$B\Gamma C \subseteq A \text{ implies } B \subseteq A \text{ or } C \subseteq A.$$

Equivalently, for any $x, y \in S$, $I(x)\Gamma I(y) \subseteq A$ implies $x \in A$ or $y \in A$. To see this, assume that for any $x, y \in S$, $I(x)\Gamma I(y) \subseteq A$ implies $x \in A$ or $y \in A$. Let B, C be ideals of S such that $B\Gamma C \subseteq A$. Suppose $B \nsubseteq A$; then there exists $b \in B \setminus A$. If $c \in C$, then by $I(b)\Gamma I(c) \subseteq B\Gamma C \subseteq A$ we have $I(b) \subseteq A$ or $I(c) \subseteq A$. But $b \notin A$; then $I(c) \subseteq A$. The opposite direction is clear.

An ideal A of a partially ordered Γ-semigroup $(S, \Gamma, \leq)$ is said to be *semiprime* if for any ideal B of S,

$$B\Gamma B \subseteq A \text{ implies } B \subseteq A.$$

Equivalently, for any $x \in S$, $I(x)\Gamma I(x) \subseteq A$ implies $x \in A$. Clearly, every prime ideal of S is semiprime.

2. Main Results

As in [2] (Definitions 2, 3), we begin this section with the definition of prime and semiprime bi-ideals of a partially ordered Γ-semigroup by:

Definition 2.1. Let B be a bi-ideal of a partially ordered Γ-semigroup $(S, \Gamma, \leq)$. Then B is said to be

- *prime* if, for any $x, y \in S$, $x\Gamma S\Gamma y \subseteq B$ implies $x \in B$ or $y \in B$;
- *semiprime* if, for any $x \in S$, $x\Gamma S\Gamma x \subseteq B$ implies $x \in B$.

It is observed that every prime bi-ideal of a partially ordered Γ-semigroup is semiprime.

We now give a characterization when a bi-ideal of a partially ordered Γ-semigroup is prime.

Theorem 2.2. *Let B be a bi-ideal of a partially ordered Γ-semigroup $(S, \Gamma, \leq)$. Then B is prime if and only if, for any right ideal R and for any left ideal L of S,*

$$R\Gamma L \subseteq B \text{ implies } R \subseteq B \text{ or } L \subseteq B.$$

Proof. Assume first that B is prime. Let R and L be a right ideal and a left ideal of S, respectively, such that $R\Gamma L \subseteq B$. Suppose that $R \nsubseteq B$; then there exists $x \in R \setminus B$. If $y \in L$, then

$$x\Gamma S\Gamma y \subseteq R\Gamma S\Gamma L \subseteq R\Gamma L \subseteq B.$$

Since $x \notin B$, it follows by assumption that $y \in B$. Hence $L \subseteq B$.

For the opposite direction, we assume that for any right ideal R and for any left ideal L of S, $R\Gamma L \subseteq B$ implies $R \subseteq B$ or $L \subseteq B$. Let $x, y \in S$ be such that $x\Gamma S\Gamma y \subseteq B$. By

$$(x\Gamma S]\Gamma(S\Gamma y] \subseteq (x\Gamma S\Gamma S\Gamma y] \subseteq (x\Gamma S\Gamma y] \subseteq B,$$

it follows by assumption that $(x\Gamma S] \subseteq B$ or $(S\Gamma y] \subseteq B$. Assume that $(x\Gamma S] \subseteq B$ (for $(S\Gamma y] \subseteq B$ can be proved in the same manner). Then $x\Gamma x \subseteq B$, and hence

$$\begin{aligned} R(x)\Gamma L(x) &= (x \cup x\Gamma S]\Gamma(x \cup S\Gamma x] \\ &\subseteq (x\Gamma x \cup x\Gamma S\Gamma x] \\ &\subseteq (x\Gamma x \cup x\Gamma S] \\ &\subseteq B. \end{aligned}$$

Thus, $R(x) \subseteq B$ or $L(x) \subseteq B$. Hence $x \in B$. $\square$

Theorem 2.3. *If B is a prime bi-ideal of a partially ordered Γ-semigroup $(S,\Gamma,\leq)$, then B is a left ideal S or B is a right ideal of S.*

Proof. Assume that B is a prime bi-ideal of a partially ordered Γ-semigroup $(S,\Gamma,\leq)$. It is observed that $(S\Gamma B]$ is a left ideal of S and $(B\Gamma S]$ is a right ideal of S. By

$$(B\Gamma S]\Gamma(S\Gamma B] \subseteq (B\Gamma S\Gamma S\Gamma B] \subseteq (B\Gamma S\Gamma B] \subseteq (B] = B$$

it follows by Theorem 2.2 that $(S\Gamma B] \subseteq B$ or $(B\Gamma S] \subseteq B$. Thus $S\Gamma B \subseteq B$ or $B\Gamma S \subseteq B$. □

Hereafter, for a bi-ideal B of a partially ordered Γ-semigroup $(S,\Gamma,\leq)$, let

$$\begin{aligned} L(B) &:= \{x \in B \mid S\Gamma x \subseteq B\}; \\ H(B) &:= \{y \in L(B) \mid y\Gamma S \subseteq L(B)\}. \end{aligned}$$

Firstly, we have $L(B)$ is a left ideal of a partially ordered Γ-semigroup $(S,\Gamma,\leq)$ with zero:

Lemma 2.4. *Let B be a bi-ideal of a partially ordered Γ-semigroup $(S,\Gamma,\leq)$ with zero 0. Then $L(B)$ is a left ideal of S.*

Proof. We have $0 \in L(B)$, and $L(B) \neq \emptyset$. Let $x \in L(B)$ and $y \in S$. We have

$$y\Gamma x \subseteq S\Gamma x \subseteq B.$$

Moreover,

$$S\Gamma(y\Gamma x) = (S\Gamma y)\Gamma x \subseteq S\Gamma x \subseteq B.$$

Let $x \in L(B)$ and $y \in S$ be such that $y \leq x$. If $z \in S\Gamma y$, then $z = s\alpha y$ for some $s \in S$ and $\alpha \in \Gamma$. By

$$z = s\alpha y \leq s\alpha x \in S\Gamma x \subseteq B,$$

it follows that $z \in B$. Hence $y \in B$, and $L(B)$ is a left ideal of S. □

And we have:

Theorem 2.5. *Let B be a bi-ideal of a partially ordered Γ-semigroup $(S,\Gamma,\leq)$ with zero 0. Then $H(B)$ is the (unique) largest ideal of S contained B.*

Proof. We have $0 \in H(B)$. Let $x, x' \in S$, $y \in H(B)$, and $\alpha, \beta \in \Gamma$. By Lemma 2.4, $x\alpha y \in L(B)$. Since $y \in H(B)$, we have $y\beta x' \in L(B)$. Hence

$$(x\alpha y)\beta x' = x\alpha(y\beta x') \in L(B).$$

This shows that $S\Gamma H(B) \subseteq H(B)$.

By

$$y\alpha x \in L(B) \text{ and } (y\alpha x)\beta x' \in y\Gamma S \subseteq L(B),$$

it follows that $H(B)\Gamma S \subseteq H(B)$.

Assume that $x \leq y$. Since $y \in L(B)$, $x \in L(B)$. Since $x\alpha x' \leq y\alpha x' \in L(B)$, $x\alpha x' \in L(B)$. Hence $H(B)$ is an ideal of S. Note that

$$H(B) \subseteq L(B) \subseteq B.$$

Finally, let A be an ideal of S such that $A \subseteq B$. If $x \in A$, then $x \in B$ and $S\Gamma x \subseteq A \subseteq B$. Then $x \in L(B)$, and $A \subseteq L(B)$. Let $x \in A$. Then $x\Gamma S \subseteq A \subseteq L(B)$; hence $x \in H(B)$, and $A \subseteq H(B)$. □

Moreover, we have the following:

Theorem 2.6. *If B is a prime bi-ideal of a partially ordered Γ-semigroup $(S, \Gamma, \leq)$ with zero 0, then $H(B)$ is prime.*

Proof. Let B be a prime bi-ideal of a partially ordered Γ-semigroup S. If $x, y \in S$ such that $I(x)\Gamma I(y) \subseteq H(B)$, then by Theorem 2.2, $I(x) \subseteq H(B)$ or $I(y) \subseteq H(B)$; hence $x \in H(B)$ or $y \in H(B)$. □

Theorem 2.7. *If B is a semiprime bi-ideal of a partially ordered Γ-semigroup $(S, \Gamma, \leq)$, then:*

(1) *for any left ideal L of S, $L\Gamma L \subseteq B$ implies $L \subseteq B$;*

(2) *for any right ideal R of S, $R\Gamma R \subseteq B$ implies $R \subseteq B$.*

Proof. Assume that B is a semiprime bi-ideal of a partially ordered Γ-semigroup $(S, \Gamma, \leq)$. Let L be a left ideal of S such that $L\Gamma L \subseteq B$. If $x \in L$, then

$$x\Gamma B\Gamma x \subseteq L\Gamma S\Gamma L \subseteq L\Gamma L \subseteq B.$$

By assumption, $x \in B$. The second assertion can be proved similarly. □

Theorem 2.8. *If B is a semiprime bi-ideal of a partially ordered Γ-semigroup $(S, \Gamma, \leq)$, then $H(B)$ is semiprime.*

Proof. Let B be a semiprime bi-ideal of a partially ordered Γ-semigroup $(S, \Gamma, \leq)$. If $x \in S$ such that $I(x)\Gamma I(x) \subseteq H(B)$, then by Theorem 2.2, $I(x) \subseteq H(B)$; hence $x \in H(B)$. □

Theorem 2.9. *If B is a semiprime bi-ideal of a partially ordered Γ-semigroup $(S, \Gamma, \leq)$, then B is a quasi-ideal of S.*

Proof. Assume that B is a semiprime bi-ideal of a partially ordered Γ-semigroup $(S, \Gamma, \leq)$. It suffices to show that $B\Gamma S \cap S\Gamma B \subseteq S$. Now, if $x \in B\Gamma S \cap S\Gamma B$, then

$$x\Gamma S\Gamma x \subseteq (B\Gamma S)\Gamma S\Gamma(S\Gamma B) \subseteq B\Gamma S\Gamma B \subseteq B.$$

Hence $x \in B$, and B is a quasi-ideal of S. □

We recall the following definition: a partially ordered Γ-semigroup $(S, \Gamma, \leq)$ is said to be *regular* if for any $a \in S$, $a \in (a\Gamma S\Gamma a]$. Equivalently, for any $a \in S$ there exists $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a \leq a\alpha x\beta a$.

The following shows that a regular partially ordered Γ-semigroup $(S, \Gamma, \leq)$ can be characterized by using semiprime bi-ideals of S. Indeed, we show that S is regular if and only if every bi-ideals of S is semiprime.

Theorem 2.10. *Let $(S, \Gamma, \leq)$ be a partially ordered Γ-semigroup. Then S is regular if and only if every bi-ideal of S is semiprime.*

Proof. Assume that S is regular. Let B be a bi-ideal of S. Let $a \in S$ be such that $a\Gamma S\Gamma a \subseteq B$; then $(a\Gamma S\Gamma a] \subseteq (B] = B$. By assumption, $a \in (a\Gamma S\Gamma a] \subseteq B$. Hence B is semiprime.

Conversely, assume that every bi-ideal of S is semiprime. To show that S is regular, let $a \in S$. It is observed that $(a\Gamma S\Gamma a]$ is a bi-ideal of S. Thus by assumption $(a\Gamma S\Gamma a]$ is semiprime. But $a\Gamma S\Gamma a \subseteq (a\Gamma S\Gamma a]$, then $a \in (a\Gamma S\Gamma a]$. Hence S is regular. □

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