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EVENTUAL STABILITY WITH TWO MEASURES FOR NONLINEAR DIFFERENTIAL EQUATIONS WITH NON-INSTANTANEOUS IMPULSES

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Abstract: The integral stability of the solutions of a nonlinear differential equation with non-instantaneous impulses is studied using Lyapunov like functions. In these differential equation we have impulses, which start abruptly at some points and their action continue on given finite intervals. Sufficient conditions for integral stability are established.

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1. Introduction

In the real world life there are many processes and phenomena that are characterized by rapid changes in their state. In the literature there are two popular types of impulses: *instantaneous impulses* ([1], [4],[8]-[13], [15], [17], [18]) and *non-instantaneous impulses* ([1], [3], [5], [7], [14], [19], [20], [22], [23]).

In this paper the impulses start abruptly at some points and their action

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continue on given finite intervals. As a motivation for the study of these systems we consider the following simplified situation concerning the hemodynamical equilibrium of a person. In the case of a decompensation (for example, high or low levels of glucose) one can prescribe some intravenous drugs (insulin). Since the introduction of the drugs in the bloodstream and the consequent absorption for the body are gradual and continuous processes, we can interpret the situation as an impulsive action which starts abruptly and stays active on a finite time interval. The model of this situation is the so called noninstanteneous impulsive differential equation.

Integral stability using two different measures for the initial values and for the solutions of non-instantaneous impulsive nonlinear differential equations is defined and studied. Sufficient conditions for integral stability are obtained.

2. Preliminary Notes and Definitions

In this paper we will assume two increasing sequences of points $\{t_i\}_{i=1}^{\infty}$ and $\{s_i\}_{i=0}^{\infty}$ are given such that $0 < s_0 < t_i \le s_i < t_{i+1}$, $i = 1, 2, \dots,$ and $\lim_{k\to\infty} t_k = \infty$.

Let $t_0 \in \cup_{k=0}^{\infty}[s_k, t_{k+1})$ be a given arbitrary point. Without loss of generality we will assume that $t_0 \in [s_0, t_1)$, i.e. $0 \le t_0 < t_1$.

Consider the initial value problem for the system of *non-instantaneous impulsive differential equations* (NIDE)

$$\begin{aligned} &x' = f(t,x) \text{ for } t \in (t_k, s_k],\ k = 0,1,2,\dots \\ &x(t) = \phi_i(t, x(t_i - 0)) \quad \text{for } t \in (s_i, t_{i+1}],\ i = 0,1,2,\dots, \\ &x(t_0) = x_0, \end{aligned} \tag{1}$$

where $x, x_0 \in \mathbb{R}^n$, $f : \cup_{k=0}^{\infty}[t_k, s_k] \times \mathbb{R}^n \to \mathbb{R}^n$, $\phi_i : [s_i, t_{i+1}] \times \mathbb{R}^n \to \mathbb{R}^n$, $(i = 0, 1, 2, 3, \dots)$.

Remark 1. The intervals $(s_k, t_{k+1}]$, $k = 0, 1, 2, \dots$ are called intervals of non-instantaneous impulses and the functions $\phi_k(t, x, y)$, $k = 0, 1, 2, \dots,$ are called non-instantaneous impulsive functions.

Keeping in mind the meaning of t_0 as an initial time of the modeled the rate of change of the process, we will assume everywhere in the paper that the initial time t_0 is not in an interval of non-instantaneous impulses, i.e. we will assume $t_0 \in \cup_{k=0}^{\infty}[s_k, t_{k+1})$.

Consider the perturbed system of non-instantaneous impulsive differential equations

$$\begin{aligned} & x' = f(t,x) + F(t,x(t)) \text{ for } t \in (t_k, s_k],\ k = 0,1,2,\dots \\ & x(t) = \phi_i(t, x(t_i - 0)) + \psi_i(t, x(t_i - 0)) \quad \text{for } t \in (s_i, t_{i+1}],\ i = 0,1,2,\dots, \\ & x(t_0) = x_0, \end{aligned} \tag{2}$$

Also consider the corresponding IVP for ODE

$$x' = f(t,x) + F(t,x) \quad \text{for } t \in \ [\tau, t_p] \quad \text{with} \ \ x(\tau) = \tilde{x}_0, \tag{3}$$

where $\tau \in [s_{p-1}, t_p)$, $p = 1, 2, \dots$.

We will say condition (H1) and (H2) are satisfied if

(H1) The functions $f, F \in C(\cup_{k=0}^{\infty}[t_k, s_k], \mathbb{R}^n)$ is such that for any initial point $(\tau, \tilde{x}_0):\ t_p \le \tau < s_p,\ \tilde{x}_0 \in \mathbb{R}^n$, p is a non negative integer number , the IVP for the system of ODE (3) has a solution $x(t;\tau,\tilde{x}_0) \in C^1([\tau, s_p], \mathbb{R}^n)$.

(H2) The functions $\phi_k, \psi_k \in C([s_k, t_{k+1}]\times\mathbb{R}^n, \mathbb{R}^n)$ and $\phi_k(t,0) = 0,\ \psi_k(t,0) = 0,\ t \in [s_k, t_{k+1}]$.

Remark 2. If $t_k = s_k,\ k = 1, 2, \dots$ then the IVP for NIDE (1) reduces to an IVP for impulsive differential equations (for example see the monographs [8], [15] and the cited references therein). In this case at any point of instantaneous impulse t_k the amount of jump of the solution $x(t)$ is given by $I_k = \phi_k(t_k, x(t_k + 0), x(t_k - 0)) - x(t_k - 0)$.

Let $J \subset \mathbb{R}_+$ be a given interval. Introduce the following classes of functions

$$\begin{aligned} PC^1(J) \ &= \ \{u : J \to \mathbb{R}^n :\ u \in C^1(J \cap \big(\cup_{k=0}^{\infty} (s_k, t_{k+1}]\big), \mathbb{R}^n) : \\ & \quad\ u(t_k) = u(t_k - 0) = \lim_{t\uparrow t_k} u(t) < \infty, \\ & \quad\ u(t_k + 0) = \lim_{t\downarrow t_k} u(t) < \infty, \quad k :\ t_k \in J\}, \\ PC(J) \ &= \ \{u : J \to \mathbb{R}^n :\ u \in C(J \cap \big(\cup_{k=0}^{\infty} (s_k, t_{k+1}]\big), \mathbb{R}^n) : \\ & \quad\ u(t_k) = u(t_k - 0) = \lim_{t\uparrow t_k} u(t) < \infty, \\ & \quad\ u(t_k + 0) = \lim_{t\downarrow t_k} u(t) < \infty, \quad k :\ t_k \in J\}. \end{aligned}$$

Definition 1. Let $h, h_0 \in \Gamma$. We say h_0 is finer than h if there exists $\delta > 0$ and a function $\phi \in \mathcal{K}$ such that $h_0 T, x) < \delta$ implies $h(t,x) \le \phi(h_0(t,x))),\ t \ge 0, xin\mathbb{R}^n$.

Remark 3. According to the above description any solution of (1) is from the class $PC^1([t_0, b))$, $b \le \infty$, i.e. any solution might have a discontinuity at any point $t_k, k = 1, 2, \dots$.

Let $\rho, t, T > 0$ be constants, $h \in \Gamma$. Define sets:

$$\begin{aligned}
\mathcal{K} &= \{\sigma \in C(\mathbb{R}_+ \mathbb{R}_+), \text{strictly increasing and } \sigma(0) = 0\}, \\
PCK &= \{\sigma : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+, \text{strictly increasing and } \sigma(., u) \in PC(\mathbb{R}_+) \\
&\quad \text{for each } u \in \mathbb{R}_+ \text{ and } \sigma(t, .) \in \mathcal{K} \text{ for each } t \in \mathbb{R}_+\}, \\
S(h, A) &= \{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n : \ h(t, x) < A\}, \qquad A > 0, \\
S^C(h, \rho) &= \{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n : \ \ h(t, x) \geq \rho\}; \\
\Gamma(J) &= \{h : J \times \mathbb{R}^n \to \mathbb{R}_+, \ h(t, x) \in PC(J) \text{ for each } x \in \mathbb{R}^n, \\
&\qquad h(t, .) \in C(\mathbb{R}^n, \mathbb{R}_+) \text{ for each } t \in J \text{ and } \inf h(t, x) = 0\} \\
W(t, T, \rho) &= \{x \in \mathbb{R}^n : \ \ h(s, x) < \rho \ \text{ for } s \in [t, t + T]\}..
\end{aligned}$$

Definition 2. Let $h_0, h \in \Gamma(J)$. Then we say that h_0 is liner than h if there exists a $\delta > 0$ and a function $\phi \in K$ such that $h_0(t, x) < \delta$ implies $h(t, x) \leq \phi(h_0(t, x)), \ t \in J, x \in \mathbb{R}^n$.

We give a definition for integral stability of (1). In the definition below we denote by $x(t; t_0, x_0) \in PC^1([t_0, \infty), \mathbb{R}^n)$ any solution of (1).

Definition 3. . Let $h, h_0 \in \Gamma$. The system of non-instantaneous impulsive differential equations (1), is said to be (h_0, h)-*integrally stable* if for every $\alpha > 0$ and for any $t_0 \geq 0$, there exists a e function $\beta = \beta(\alpha) \in K$ such that for every solution $y(t; t_0, x_0)$ of the perturbed system of non-instantaneous impulsive differential equations (2) the inequality

$$h(t, y(t; t_0, x_0)) < \beta, \quad t \geq t_0$$

holds provided that the initial value $x_0 \in \mathbb{R}^n$ satisfies

$$h_0(t_0, x_0) < \alpha,$$

and for every $T > 0$ the perturbations $F(t, x)$ and $\psi_k(t, x), \ k = 1, 2, \dots$ of the right side parts of the system (2) satisfy

$$\sum_{i=0}^{p} \int_{s \in \Omega_k} sup_{x \in W(t_0, T, \beta)} ||F(s, x)|| ds \\
+ \sum_{k=0}^{p-1} sup_{(t,x) \in (s_k, t_{k+1}] \times \mathbb{R}^n : \ h(t,x) < \beta} ||\psi_k(t, x)|| < \alpha.$$

where $p : \ t_0 + T \in (t_p, s_p]$, $\Omega_k = (t_k, s_k]$, and $\Omega_p = (t_p, \min\{t_0 + T, s_p\}]$.

Remark 4. We note that in the case when $h_0(t,x) \equiv ||x||$ and $h(t,x) \equiv ||x||$ the (h_0,h)-integral stability reduces to integral stability, studied in [21].

In our further investigations we will use following comparison scalar non-instantaneous impulsive differential equation

$$\begin{aligned} &u' = g_1(t,u) \text{ for } t \in (t_k, s_k],\ k = 0,1,2,\dots \\ &u(t) = \xi_i(t, u(s_i - 0)) \quad \text{for } t \in (s_i, t_{i+1}],\ i = 0,1,2,\dots. \end{aligned} \tag{4}$$

the scalar non-instantaneous impulsive differential equation

$$\begin{aligned} &w' = g_2(t,w) \text{ for } t \in (t_k, s_k],\ k = 0,1,2,\dots \\ &w(t) = \eta_i(t, w(s_i)) \quad \text{for } t \in (s_i, t_{i+1}],\ i = 0,1,2,\dots, \end{aligned} \tag{5}$$

and its perturbed scalar non-instantaneous impulsive differential equation

$$\begin{aligned} &w' = g_2(t,w) + q(t,w) \text{ for } t \in (t_k, s_k],\ k = 0,1,2,\dots \\ &w(t) = \eta_i(t, w(s_i)) + \gamma_i(t, w(s_i)) \quad \text{for } t \in (s_i, t_{i+1}],\ i = 0,1,2,\dots. \end{aligned} \tag{6}$$

where $u, w \in \mathbb{R}$, $g_i(t,0) \equiv 0,\ i = 1,2,\ \xi_k(t,0) = 0,\ \eta_k(t,0) = 0,\ k = 1,2,\dots.$

In our further investigations we will assume that solutions of the scalar impulsive equations (4), (5), and (6) exist on $[t_0, \infty)$ for any initial values.

We now introduce the class Λ of Lyapunov-like functions which will be used to investigate the stability of the zero solution of the system IFrDE (1).

Definition 4. Let $J \in \mathbb{R}_+$ be a given interval, and $\Delta \subset \mathbb{R}^n$, $0 \in \Delta$ be a given set. We will say that the function $V(t,x) : J \times \Delta \to \mathbb{R}_+$, $V(t,0) \equiv 0$ belongs to the class $\Lambda(J,\Delta)$ if

1. The function $V(t,x)$ is continuous on $J/\{s_k \in J\} \times \Delta$ and it is locally Lipschitzian with respect to its second argument;

2. For each $s_k \in J$ and $x \in \Delta$ there exist finite limits

$$V(s_k - 0, x) = \lim_{t \uparrow s_k} V(t,x), \quad \text{and} \quad V(s_k + 0, x) = \lim_{t \downarrow s_k} V(t,x)$$

   and $V(s_k - 0, x) = V(s_k, x)$.

Definition 5. Let $V \in \Lambda(J, \mathbb{R}^n)$ and $h_0, h \in \Gamma(J)$. Then $V(t,x)$ is said to be

- h-positive definite if there exists $\rho > 0$ and a function $b \in K$ such that $h(t,x) < \rho$ implies $b(h(t,x)) \le V(t,x)$;

- h_0-descrescent if there exists a $\delta > 0$ and a function $a \in K$ such that $h_0(t,x) < \delta$ implies $V(t,x) < a(h_0(t,x))$;

- weakly h_0-descrescent if there exists a $\delta > 0$ and a function $a \in PCK$ such that $h_0(t,x) < \delta$ implies $V(t,x) < a(h_0(t,x))$

In this paper we will use piecewise continuous Lyapunov functions from the introduced above class $\Lambda([t_0,T),\Delta)$. We will define the *generalized Dini derivative* of the function $V(t,x) \in \Lambda([t_0,T),\Delta)$ along trajectories of solutions of IVP for the system NIDE (1) by:

$$\begin{aligned} {}_{(1)}D_+V(t,x) &= \limsup_{h\to 0^+} \frac{1}{h}\Big\{V(t,x) - V(t-h, x-hf(t,x))\Big\} \\ &\text{for } t \in (t_k,s_k),\ k = 0,1,2,\dots, \end{aligned} \tag{7}$$

where $x \in \Delta$, and for any $t \in (t_k,s_k)$ there exists $h_t > 0$ such that $t-h \in (t_k,s_k)$, $x - hf(t,x) \in \Delta$ for $0 < h \le h_t$.

In our further study we will use the following comparison result ([3]).

Lemma 1. *[3] (Comparison result for NIDE). Assume the following conditions are satisfied:*

1. *The function $x^*(t) = x(t;t_0,x_0) \in PC^1([t_0,T],\Delta)$ is a solution of the NIDE (1) where $\Delta \subset \mathbb{R}^n$, $0 \in \Delta$, $x_0 \in \Delta$ and t_0, $T \in \mathbb{R}_+$, $t_0 < T$, $0 \le t_0 < t_1$ are given numbers.*

2. *The function $V \in \Lambda([t_0,T],\Delta)$ and*

   (i) the inequality ${}_{(1)}D_+V(t,x^*(t)) \le g_1(t,V(t,x^*(t))$ for $t \in \Big(\cup_{k=0}^{\infty} (s_k,t_{k+1})\Big) \cap [t_0,T]$ holds;

   (ii) for any $k = 1,2\dots$ the inequalities

   $V(t,\phi_k(t,x^*(t))) \le \xi_k(t,V(s_k-0,x^*(t_k-0)))$ *for* $t \in [t_0,T] \cap (s_k,t_{k+1}]$

   hold.

Then for $t \in [t_0,T]$ the inequality

$$V(t,x^*(t)) \le u^*(t) \tag{8}$$

holds where $u^*(t)$ is the maximal solution of NIDE (4) on $[t_0,T]$.

Remark 5. Note the claims of Lemma 1 is true if in Condition 1 the initial time t_0 is in $[s_{p-1},t_p)$ where p is any natural number.

3. Main Results

First we study the stability properties of the zero solution of nonlinear differential equations with non-instantaneous impulses.

Theorem 1. *Let the following conditions be satisfied:*

1. *Conditions (H1), (H2) are satisfied.*
2. *Functions* $g_1, g_2 \in C(\cup_{k=0}^{\infty}[t_k, s_k], \mathbb{R})$, $g_i(t,0) = 0,\ i = 1,2]$
3. *Functions* $\xi_k, \eta_k \in C([s_k, t_{k+1}] \times \mathbb{R}^n, \mathbb{R}^n)$ *and* $\xi_k(t,0) = 0,\ \eta_k(t,0) = 0,\ t \in [s_k, t_{k+1}]$.
4. $h_0, h \in \Gamma(\mathbb{R}_+)$, h_0 *is finer than* h *and there exists a* $\rho_0 :\ 0 < \rho_0 < \rho$ *such that* $h(s_k, x) < \rho_0$ *implies* $h(s_k + 0, \phi_k(s_k, x)) < \rho$, $k = 0, 1, 2 \dots$.
5. *There exists a function* $V_1 :\ [0,\infty) \times \mathbb{R}^n \to \mathbb{R}_+$, $V_1 \in \Lambda$ *that is* h_0*-decrescent and*

   (i) for any number $t \geq 0$*the inequality*

   $$D^+_{(1)} V_1(t, \psi(t)) \leq g_1(t, V_1(t, \psi(t))),\ t \in \Big([0, s_0] \cup_{k=1}^{\infty} (t_k, s_k)\Big),\ (t,x) \in S(h, \rho),$$

   holds, where $\rho > 0$ *is a constant.*

   (ii) $V_1(t, \phi_k(t,x)) \leq \xi_k(V_1(s_k, x)),\ \ t \in (s_k, t_{k+1}]$ *for* $(s_k, x) \in S(h, \rho),\ k = 1, 2, \dots$.
6. *For any number* $\mu > 0$ *there exists a function* $V_2^{(\mu)} :\ [-r, \infty) \times \mathbf{R}^n \to \mathbf{R}_+$, $V_2^{(\mu)} \in \Lambda$ *such that*

   (iii) $b(h(t,x)) \leq V_2^{(\mu)}(t,x) \leq a(h_0(t,x))$ *for* $(t,x) \in [-r, \infty) \times \mathbf{R}^n$, *where* $a, b \in K$ *and* $lim_{u\to\infty} b(u) = \infty$.

   (iv) for any number $t \geq 0$ *the inequality*

   $$D^+_{(1)} V_1(t,x) + D^+_{(1)} V_2^{(\mu)}(t,x) \leq g_2\Big(t, V_1(t,x) + V_2^{(\mu)}(t,x)\Big)$$
   $$t \in \Big([0, s_0] \cup_{k=1}^{\infty} (t_k, s_k)\Big),\ (t,x) \in S(h, \rho),$$

   holds;

   (v) $V_1(t, \phi_k(t,x)) + V_2^{(\mu)}(t, \phi_k(t,x)) \leq \eta_k\Big(V_1(s_k, x) + V_2^{(\mu)}(s_k, x)\Big),\ \ t \in (s_k, t_{k+1}]$
   for $(s_k, x) \in S(h, \rho) \bigcap S^C(h_0, \mu),\ k = 1, 2, \dots$.

7. *Zero solution of the scalar impulsive differential equation (4) is equi-stable.*

8. *Scalar impulsive differential equation (5) is uniform-integrally stable.*

Then system of non-instantaneous impulsive differential equations (1) is (h_0, h)-uniform-integrally stable.

Proof. Since function $V_1(t, x)$ is h_0-decrescent, there exist a constant $\rho_1 \in (0, \rho)$ and a function $\psi_1 \in K$ such that $h_0(t, x) < \rho_1$ implies that

$$V_1(t, x) \leq \psi_1(h_0(t, x)). \tag{9}$$

Since $h_0(t, x)$ is uniformly finer than $h(t, x)$, there exist a constant $\rho_0 \in (0, \rho_1)$ and a function $\psi_2 \in K$ such that $h_0(t, x) < \rho_0$ implies that $h(t, x) \leq \psi_2(h_0(t, x))$ where $\psi_2(\rho_0) < \rho_1$.

According to Condition 4 inequality $h_0(t_0, x_0) < \rho_0$ implies

$$h(t_0, x_0) \leq \psi_2(h_0(t_0, x_0)). \tag{10}$$

Let $t_0 \geq 0$ be a fixed point. Choose a number $\alpha > 0$ such that $\alpha < \rho_0$.

According to condition 6 of Theorem 1 there exists function $V_2^{(\alpha)}(t, x)$ with Lipshitz constant M_2. Let M_1 be Lipshitz constant of function $V(t, x)$.

Denote $(M_1 + M_2)\alpha = \alpha_1$. Without loss of generality we assume $\alpha_1 < b(\rho)$.

Since the zero solution of the scalar non-instantaneous impulsive differential equation (4) is equi-stable, there exists a function $\delta_1 = \delta_1(t_0, \alpha_1) > 0$ such that the inequality $|u_0| < \delta_1$ implies

$$|u(t; t_0, u_0)| < \frac{\alpha_1}{2}, \quad t \geq t_0, \tag{11}$$

where $u(t; t_0, u_0)$ is a solution of (4).

Since the function $\psi_1 \in K$ there exists $\delta_2 = \delta_2(\delta_1) > 0$, $\delta_2 < \rho_1$ such that for $|u| < \delta_2$ the inequality

$$\psi_1(u) < \delta_1 \tag{12}$$

holds.

Since the scalar non-instantaneous impulsive differential equation (5) is uniform-integrally stable, there exists $\beta_1 = \beta_1(\alpha_1) \in K$, $b(\rho) > \beta_1 \geq \alpha_1$ such that for every solution $w(t; t_0, w_0)$ of the perturbed equation (6) the inequality

$$|w(t; t_0, w_0))| < \beta_1, \quad t \geq t_0, \tag{13}$$

holds, provided that

$$|w_0| < \alpha_1 \tag{14}$$

and for every $T > 0$

$$\int_{t_0}^{t_0+T} sup_{w:\ |w|<\beta_1}|q(s,w)|ds + \sum_{k:\ t_0<\tau_k\leq t_0+T} sup_{w:\ |w|<\beta_1}|\gamma_k(w)| < \alpha_1. \quad (15)$$

Since the function $b \in K$, $lim_{s\to\infty}b(s) = \infty$, and $\psi_2(\alpha) < \psi_2(\rho_0) < \rho_1 < \rho$ we could choose $\beta = \beta(\beta_1) > 0$, $\rho > \beta > \alpha$, $\beta > \psi_2(\alpha)$ such that

$$b(\beta) \geq \beta_1. \quad (16)$$

Since the functions $a \in K$, $\psi_2 \in K$, and $\beta > \psi_2(\alpha)$ we can find $\delta_3 = \delta_3(\alpha_1, \beta) > 0$, $\alpha < \delta_3 < \min(\delta_2, \rho_0)$ such that the inequalities

$$a(\delta_3) < \frac{\alpha_1}{2}, \quad \psi_2(\delta_3) < \beta \quad (17)$$

hold.

From (10) and (17) follows that $h_0(t_0, x_0) < \alpha$ implies

$$h(t_0, x_0) \leq \psi_2(h_0(t_0, x_0)) < \psi_2(\alpha) < \psi_2(\delta_3) < \beta,$$

i.e. $h(t, x) < \beta$ for $t \in [t_0 - r, t_0]$.

Now let the initial value $x_0 \in \mathbb{R}^n$ be such that

$$h_0(t_0, x_0) < \alpha \quad (18)$$

and the perturbations $F(t, x)$ and $\psi_k(t, x)$, $k = 1, 2, \ldots$ in NIDE (2) be such that

$$\sum_{i=0}^{p} \int_{s\in\Omega_k} sup_{x\in W(t_0,T,\beta)}||F(s,x)||ds + \sum_{k=0}^{p-1} sup_{(t,x)\in(s_k,t_{k+1}]\times\mathbb{R}^n:\ h(t,x)<\beta}||\psi_k(t,x)|| < \alpha \quad (19)$$

for every $T > 0$.

Let $y(t) = y(t; t_0, \phi)$ be a solution of (2), where the initial value and the perturbations satisfy (18) and (19).

We will prove that if inequalities (18) and (19) are satisfied then

$$h(t, y(t; t_0, \phi)) < \beta, \quad t \geq t_0. \quad (20)$$

Suppose it is not true. Therefore there exists a point $t^* > t_0$ such that

$$h(t^*, y(t^*; t_0, \phi)) \geq \beta, \quad h(t, y(t; t_0, \phi)) < \beta, \quad t \in [t_0, t^*). \tag{21}$$

Case 1. Let there exists a number $k: \ t^* \in (t_k, s_k]$. Then from the continuity of the solution $y(t; t_0, \phi)$ at point t^* follows that $h(t^*, y(t^*; t_0, \phi)) = \beta$.

If we assume that $h_0(t^*, y(t^*)) \leq \delta_3$ then from the choice of δ_3 and inequality (17) it follows $h(t^*, y(t^*)) \leq \psi_2(h_0(t^*, y(t^*)) \leq \psi_2(\delta_3) < \beta$ that contradicts (21).

Therefore

$$h_0(t^*, y(t^*)) > \delta_3, \quad h_0(t_0, x_0) < \alpha < \delta_1. \tag{22}$$

Case 1.1. Let there exists a point $t_0^* \in (t_0, t^*)$, $t_0^* \neq s_k, k = 1, 2, \ldots$ such that $\delta_3 = h_0(t_0^*, y(t_0^*))$ and $(t, y(t)) \in S(h, \beta) \bigcap S^c(h_0, \delta_3)$. Since $\beta < \rho$ and $\delta_3 > \alpha$ it follows that

$$(t, y(t)) \in S(h, \rho) \bigcap S^c(h_0, \alpha), \quad t \in [t_0^*, t^*). \tag{23}$$

Define a function $\phi^*(t) = y(t)$ for $t \in [t_0^* - r, t_0^*]$ and let $r_1(t; t_0^*, u_0)$ be the maximal solution of impulsive scalar differential equation (4) where $u_0 = V_1(t_0^*, x_0)$. Let $x^*(t) \equiv x^*(t; t_0^*, x_0^*)$ be the solution of the NIDE (1). From conditions (i), (ii) of Theorem 1 according to Lemma 1 follows that

$$V_1(t, x^*(t)) \leq r_1(t; t_0^*, u_0), \quad t \in [t_0^*, t^*]. \tag{24}$$

From the choice of the point t_0^* it follows that $h_0(t_0^*, \phi^*(t_0^*)) = h_0(t_0^*, y(t_0^*)) = \delta_3 < \delta_2$. According to inequalities (9) and (12) we obtain

$$u_0 = V_1(t_0^*, x_0^*) \leq \psi_1(h_0(t_0^*, x_0^*)) < \delta_1.$$

From inequalities (11) and (24) it follows that $V_1(t, x^*(t)) \leq r_1(t; t_0^*, u_0) < \frac{\alpha_1}{2}$ for $t \in [t_0^*, t^*]$, or

$$V_1(t_0^*, x_0^*) = V_1(t_0^*, x_0^*) < \frac{\alpha_1}{2}. \tag{25}$$

From inequality (17) and condition (iii) of Theorem 1 follows that

$$V_2^{(\alpha)}(t_0^*, y(t_0^*)) < a(h_0(t_0^*, y(t_0^*))) = a(\delta_1) < \frac{\alpha_1}{2}. \tag{26}$$

Consider function $V: \ [-r, \infty) \times \mathbf{R}^n \to \mathbf{R}_+$, $V \in \Lambda$ defined by equality

$$V(t, x) = V_1(t, x) + V_2^{(\alpha)}(t, x). \tag{27}$$

Function $V(t, x)$ satisfies the conditions of Lemma 1. Indeed, let point $t \in [t_0^*, t^*]$, $t \in (t_k, s_k]$ and $(t, x) \in S(h, \beta) \bigcap S^c(h_0, \alpha)$. Then using the Lipshitz conditions for functions $V_1(t,x)$ and $V_2^{(\alpha)}(t,x)$, and condition (iv) of Theorem 1 we obtain

$$\begin{aligned}
D^+_{(2)}V(t,\psi(t)) &= D^+_{(2)}V_1(t,x) + D^+_{(2)}V_2^{(\alpha)}(t,x)\\
&= \limsup_{\epsilon\to 0+}\frac{1}{\epsilon}\Big\{\{V(t+\epsilon, x+\epsilon\Big(f(t,x)+F(t,x)\Big) - V(t,x)\Big\}\\
&\le \limsup_{\epsilon\to 0+}\frac{1}{\epsilon}\{V(t+\epsilon, x+\epsilon f(t,x)\} + \limsup_{\epsilon\to 0+}\frac{1}{\epsilon}\{V(t+\epsilon, x+\epsilon[f(t,x)\\
&\qquad +F(t,x) - V(t+\epsilon, x+\epsilon f(t,x)\}\\
&\le g_2(t,V(t,x)) + (M_1+M_2)||F(t,x)||\\
&\le g_2(t,V(t,x)) + (M_1+M_2)sup_{x\in W(t_0^*-r,T^*,\beta)}||F(t,x)||, \qquad (28)
\end{aligned}$$

where $T^* = t^* - t_0^* + r$.

Let $s_k \in (t_0^*, t^*)$, $x\in\mathbb{R}^n$ be such that $(s_k,x)\in S(h,\beta)\bigcap S^c(h_0,\alpha)$. According to condition (v) of Theorem 1 we have

$$\begin{aligned}
&V(t,\phi_k(t,x)+\psi_k(t,x))\\
&\quad = V(t,\phi_k(t,x)) + \Big\{V(t,\phi_k(t,x)+\psi_k(t,x)) - V(t,\phi_k(t,x))\Big\}\\
&\quad \le \eta_k(V(s_k,x)) + (M_1+M_2)||\psi_k(t,x)||\\
&\quad \le \eta_k(V(s_k,x)) + (M_1+M_2)sup_{t,x:\ h(t,x)<\beta}||\psi_k(t,x)||. \qquad (29)
\end{aligned}$$

Consider the scalar impulsive differential equation (6) where the perturbations of the right parts depend only on t and they are given by the equalities

$$q(t) = (M_1+M_2)sup_{x\in W(t_0^*-r,T^*,\beta)}||F(t,x)||,$$

$$\gamma_k = (M_1+M_2)sup_{t,x:\ h(t,x)<\beta}||\psi_k(t,x)||.$$

According to above notations and inequality (19) for $T = t^* - t_0^*$ we obtain

$$\sum_{i=0}^{p}\int_{s\in\Omega_k} q(s)ds + \sum_{k=0}^{p-1}\gamma_k < (M_1+M_2)\alpha = \alpha_1. \qquad (30)$$

Let $r^*(t;t_0^*,w_0^*)$ be the maximal solution of (6) through the point (t_0^*,w_0^*), where $w_0^* = V_1(t_0^*,y(t_0^*)) + V_2^{(\alpha)}(t_0^*,y(t_0^*))$, and perturbations $q(t)$ and γ_k are defined above and satisfy inequality (30). According to inequalities (33), (34), and Lemma 1 the inequality

$$V(t,y(t)) = V_1(t,y(t)) + V_2^{(\alpha)}(t,y(t)) \le r^*(t;t_0^*,w_0^*),\quad t\in[t_0^*,t^*] \qquad (31)$$

holds.

From inequalities (25) and (26), the definition of point w_0^*, and inequality (30) follows the validity of (13) for the solution $r^{**}(t;t_0^*,w_0^*)$, i.e.

$$r^{**}(t;t_0^*,w_0^*) < \beta_1, \quad t \geq t_0^*. \tag{32}$$

From inequalities (31), (32), the choice of point t^*, and condition (iii) of Theorem 1 we obtain

$$\begin{aligned} b(\beta) &\geq \beta_1 > r^{**}(t^*;t_0^*,w_0^*) = r^*(t^*;t_0^*,w_0^*) \\ &\quad \geq V(t^*,y(t^*)) = V_1(t^*,y(t^*)) + V_2^{(\alpha)}(t^*,y(t^*)) \\ &\geq V_2^{(\alpha)}(t^*,y(t^*)) \geq b(h(t^*,y(t^*))) = b(\beta). \end{aligned}$$

The obtained contradiction proves the validity of the inequality (20) for $t \geq t_0$.

Case 1.2. Let there exist a point $s_k \in (t_0,t^*)$ such that $\delta_3 < h_0(s_k+0,y(s_k+0;t_0,x_0))$, $\delta_3 > h_0(\tau_k,y(s_k;t_0,x_0))$ and (23) is true.

We choose a number $\tilde{\delta_3}$: $\delta_3 < \tilde{\delta_3} < \beta$ such that $\tilde{\delta_3} = h_0(t_0^*,y(t_0^*;t_0,x_0))$ and $t_0^* \in (t_0,t^*)$. We repeat the proof of Case 1.1, where instead of δ_3 we use $\tilde{\delta_3}$ and obtain a contradiction.

Case 2. Let there exists a natural number k such that $h(t,y(t)) < \beta$ for $t \leq s_k$ and $h(s_k,y(s_k+0)) = h(s_k,\phi_k(s_k,y(s_k)) + \psi_k(s_k,y(s_k))) > \beta$.

We repeat the proof of case 1 as in this case we choose $\beta = \beta(\beta_1) > 0$, such that $b(\beta) \geq sup_k\{\eta_k(\beta_1)\}$.

As in the proof of case 1 we obtain the validity of inequalities (32) and (31). We apply conditions (iii) and (v) of Theorem 1 and obtain

$$\begin{aligned} b(\beta) &\geq \eta_k(\beta_1) > \eta_k(r^*(s_k;t_0^*,w_0^*)) \geq \eta_k(V(s_k,y(s_k))) \\ &= \eta_k(V_1(s_k,y(s_k)) + V_2^{(\alpha)}(s_k,y(s_k)))) \\ &\geq V_1(s_k,\phi_k(s_k,y(s_k)) + \psi_k(s_k,y(s_k))) \\ &\quad +V_2^{(\alpha)}(s_k,\phi_k(s_k,y(s_k)) + \psi_k(s_k,y(s_k))) \\ &\geq V_2^{(\alpha)}(s_k,\phi_k(s_k,y(s_k)) + \psi_k(s_k,y(s_k))) \\ &\geq b(h(s_k,\phi_k(s_k,y(s_k)) + \psi_k(s_k,y(s_k)))) > b(\beta). \end{aligned}$$

The obtained contradiction proves the validity of the inequality (20) in this case.

Inequality (20) proves (h_0,h)-uniform-integral stability of the considered system of non-instantaneous impulsive equations. □

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