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2-PRIMAL TERNARY SEMIGROUPS

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Abstract: In this paper we introduce the notion of 2-primal ternary semigroup, right symmetric, right anhilator and TRI ternary semigroup and also we give characterization of 2-primal ternary semigroup. Also we introduce the notion of intersection of factors property [IFP] and characterize the results.

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Key Words: 2-primal, right symmetric, right anhilator, intersection of factors property, super nilpotent

1. Introduction

The term of 2-primal was introduce by G.Birkenmeir, H.F.Heatherly and E.K.Lee [6] in the contest of left near rings. They called a ring R is 2-primal if its intersection of all prime ideals of R coincides with the set of all nilpotent elements of R. Historically some of the earliest results known to as about 2-primal rings (all though the name 2-primal was not popular at that time) and prime ideals due to G. Shin [7]. We notice that C.Y.Hong and T.K. Kwak [4] also showed that R is 2-primal ring if and only if the prime radical mathcal $P(R)$ of R has the IFP if and only if for each minimal prime ideal P of R, the set $N(P) = \{a \in R : aRb \subseteq \mathcal{P}(R) \quad \text{for} \quad \text{some} \quad b \in R \backslash P\}$ has the IFP.

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The introduction of ternary algebra can be dated back to 1932 when Lehmar [5] studied certain ternary system called triplexes which turn out to be a gernalization of abelian groups. Though the notion of ternary semigroup generalization of notion of semiring but it is not merely a generalization of semigroup because there are certain notions, for example the lateral ideals which have no analogue in semigroup. The concept of m system in ternary semigroup was introduced and studied in Y. Sarala [11].

In this paper we introduce the notion of 2-primal ternary semigroup. We characterize 2-primal ternary semigroups. Also we generalize the results of 2-primal ternary semigroups.

2. Main Results

Definition 2.1. A ternary semigroup T is said to be 2-primal ternary semigroup if $\mathcal{P}(T) = \mathcal{N}(T)$, where $\mathcal{P}(T)$ denotes the prime radical of T . *i.e* intersection of all prime ideals of T and $\mathcal{N}(T)$ denotes the set of all strongly nilpotent elements of T.

Proposition 2.2. *For any ternary semigroup T, $\mathcal{P}(T) \subseteq \mathcal{N}(T)$.*

Proof. Let $a \notin \mathcal{N}(T)$. Then $a^{2n+1} \neq 0$ for any positive integer n. Consider the set $S = \{a^{2n+1} : n \in N_0\}$. Then S is an m-system not containing 0. So $< 0 > \cap S = \emptyset$. Then there exist an ideal P of T maximal among all ideals disjiont from S and so P is prime and $a \notin P$. Therefore $a \notin \mathcal{P}(T)$. Thus $\mathcal{P}(T) \subseteq \mathcal{N}(T)$.

Definition 2.3.: An ideal I of a ternary semigroup T is said to be right symmetric if $xTyTz \subseteq I$ implies $xTzTy \subseteq I$ for $x, y, z \in T$.

Definition 2.4. Let T be a ternary semigroup. Let $x \in T$ then the right anhilator $x \in T$, denoted by $r_T(x)$ is defined by $r_T(x) = \{y \in T : xyt = 0 \quad \text{for} \quad \text{all} \quad t \in T\}$.

Definition 2.5. A ternary semigroup T is said to be satisfy (TRI) if for each $x \in T$, $r_T(x)$ is an ideal of T.

Lemma 2.6. *For any ternary semigroup T the following statements are equivalent:*

(i) T satisfies (TRI);

(ii) For any $x, y \in T$, $xyT = 0$ implies $xTTyT = 0$.

Proof. $(i) \Rightarrow (ii)$ Let $xyT = 0$, for $x, y \in T$. Then $y \in r_T(x)$. As $r_T(x)$ is an ideal of T, $TTy \subseteq r_T(x)$. So $xTTyT = 0$.

$(ii) \Rightarrow (i)$ Obviously $r_T(x)$ is a right ideal of T for each $x \in T$. Let $y \in r_T(x)$ then $xyT = 0$ and by (ii) $xTTyT = 0$. So $TT(r_T(x)) \subseteq r_T(x)$. Therefore $r_T(x)$ is a left ideal of T. Again $TyT \subseteq TTyTT \subseteq r_T(x)$. Since T admits identity. So $r_T(x)$ is a lateral ideal of T. Hence $r_T(x)$ is an ideal of T.

Proposition 2.7. *If T satisfies (TRI) then T is a 2-primal ternary semigroup.*

Proof. By Proposition 2.2 $\mathcal{P}(T) \subseteq \mathcal{N}(T)$. Suppose $x \in \mathcal{N}(T)$, then $x^{2n+1} = 0$ for some positive integer n. If possible, let $x \notin \mathcal{P}(T)$. Then $x \notin P$ for some prime ideal P of T. As P is prime, $xTTxTTx \nsubseteq P$. Again since $xTTxTTx \nsubseteq P$ and $x \notin P$, we have $xTTxTTxTTxTTx \nsubseteq P$. Continuing this process we have $xTTxTTx...TTx \nsubseteq P$ [(2n+1) number of x]. Thus $TxTTxTTx...TTxT \nsubseteq P$ as T has identity. Since T satisfies (TRI), by lemma 2.6, $x^{2n+1} = 0$. i.e $Txx^{2n}T = 0$ implies $TxTTx^{2n}T = 0 \Rightarrow (TxTTx)x^{2n-1}T = 0 \Rightarrow (TxTTxTTx)x^{2n-2} = 0$ by lemma 2.6 continuing this process we get $TxTTxTTx...TTxT = 0 \subseteq P$ [(2n+1) number of x], a contradiction. Thus $x \in \mathcal{P}(T)$. So $\mathcal{P}(T) = \mathcal{N}(T)$. i.e T is 2-primal ternary semigroup.

Definition 2.8. A ternary semigroup T is said to be super nilpotent if each $a \in \mathcal{N}(T)$ is super nilpotent.

Definition 2.9. An ideal I of a ternary semigroup T is said to have the intersection of factors property or simply IFP if $xyz \in I$ implies $xTTyTTz \subseteq I$ for $x, y, z \in T$.

Definition 2.10. A ternary semigroup T admits an identity provided that there exist elements $\{(e_i, f_i) \in T \times T (i = 1, 2, ..., n)\}$ such that

$$\sum_{i=1}^{n} e_i f_i x = \sum_{i=1}^{n} e_i x f_i = \sum_{i=1}^{n} x e_i f_i = x,$$

for all $x \in T$. In this case, the ternary semigroup T is said to be a ternary semigroup with identity $\{(e_i, f_i) : i = 1, 2, ..., n\}$.

Definition 2.11. A non empty subset X of a ternary semigroup T with identity is called an m_{id}- system if for each $x, y, z \in X$ there exists an elements a_1, a_2, a_3, a_4 of T such that $xa_1a_2ya_3a_4z \in X$.

Theorem 2.12. *For a super nilpotent ternary semigroup T the following statements are equivalent:*

(i) *T is a 2 Primal ternary semigroup.*

(ii) *$\mathcal{P}(T)$ is a completely semiprime ideal of T.*

(iii) *$\mathcal{P}(T)$ is a left and right symmetric ideal of T.*

(iv) *$aTb \in \mathcal{P}(T)$ implies $bTTaT \subseteq \mathcal{P}(T)$ for $a, b \in T$.*

Proof. $(i) \Rightarrow (ii)$ Let T be a 2-primal ternary semigroup and $x^3 \in \mathcal{P}(T)$ where $x \in T$ then $x^3 \in \mathcal{N}(T)$ [Since $\mathcal{P}(T) = \mathcal{N}(T)$] which implies that $(x^3)^{2n+1} = 0$, for some positive integer n. i.e. $x^{6n+3} = 0$. So $x \in \mathcal{N}(T) = \mathcal{P}(T)$. Therefore $\mathcal{P}(T)$ is a completely semiprime ideal of T.

$(ii) \Rightarrow (iii)$ Let $xTyTz \subseteq \mathcal{P}(T)$ where $x, y, z \in T$. Now

$$(zTxTy)^3 = (zTxTy)T(zTxTY)T(zTxTy) = zT(xTyTz)T(xTyTz)TxTy \subseteq \mathcal{P}(T) \Rightarrow (zTxTy)^3 \subseteq \mathcal{P}(T),$$

as T admits identity. Since $\mathcal{P}(T)$ is completely semiprime, $zTxTy \in \mathcal{P}(T)$. Again $(xTyTxTz)T(xTyTxTz)T(xTyTxTz) \subseteq \mathcal{P}(T) \Rightarrow (xTyTxTz)^3 \subseteq \mathcal{P}(T)$, as T admits identity. Since $\mathcal{P}(T)$ is completely semiprime, $xTyTxTz \subseteq \mathcal{P}(T)$ Again

$$(yTxTzTyTx)^3 = (yTxTzTyTx)T(yTxTzTyTx)T(yTxTzTyTx) \subseteq \mathcal{P}(T) \Rightarrow (yTxTzTyTx)^3 \subseteq \mathcal{P}(T),$$

as T admits identity. Since $\mathcal{P}(T)$ is completely semiprime, $yTxTzTyTx \subseteq \mathcal{P}(T)$. Now

$$(xTzTy)T(xTzTy)T(xTzTy) = xTzT(yTxTzTyTx)TzTy \in \mathcal{P}(T) \Rightarrow (xTzTy)^3 \in \mathcal{P}(T) \Rightarrow xTzTy \subseteq \mathcal{P}(T).$$

Also $(yTxTz)T(yTxTz)T(yTxTz) = yT(xTzTy)TxTz \in \mathcal{P}(T) \Rightarrow (yTxTz)^3 \subseteq \mathcal{P}(T) \Rightarrow yTxTz \subseteq \mathcal{P}(T)$. Therefore $\mathcal{P}(T)$ is a left and right symmetric ideal of T.

$(iii) \Rightarrow (iv)$ Let $aTb \in \mathcal{P}(T)$ where $a, b \in T$. Suppose $t \in T$, then $tTaTb \subseteq \mathcal{P}(T)$. As $\mathcal{P}(T)$ is right symmetric $tTbTa \subseteq \mathcal{P}(T)$. Also since $\mathcal{P}(T)$ is left symmetric $bTtTa \subseteq \mathcal{P}(T)$. As t is arbitrary element of T, $bTTTa \subseteq \mathcal{P}(T) \Rightarrow$

$bTTTaTT \subseteq \mathcal{P}(T) \Rightarrow bTTaT \subseteq \mathcal{P}(T)$ as $\mathcal{P}(T)$ is an ideal of T and T admits identity.

$(iv) \Rightarrow (i)$ We know that $\mathcal{P}(T) \subseteq \mathcal{N}(T)$. Let $a \in \mathcal{N}(T)$. Since T is super nilpotent ternary semigroup, $(aT)^n a = 0$ for some positive integer n. Hence $(aT)^n a = aT(aT)^{n-1}a \subseteq \mathcal{P}(T)$. By (iv), $(aT)^{n-1}aTTaT \subseteq \mathcal{P}(T)$. Also

$$(aT)^{n-1}aTTaT = ((aT)^{n-2}a)T(aTTaT)) \subseteq \mathcal{P}(T)$$

implies $aTTaTTT(aT)^{n-2}aT \subseteq \mathcal{P}(T)$. Continuing this process we get

$$aTTaTTa...aTTa \subseteq \mathcal{P}(T)$$

as $\mathcal{P}(T)$ is an ideal of T. Since $\mathcal{P}(T)$ is a semiprime ideal, $a \in \mathcal{P}(T)$. Thus $\mathcal{N}(T) \subseteq \mathcal{P}(T)$ and consequently T is a 2 Primal ternary semigroup.

Theorem 2.13. *The following statements are equivalent for a super nilpotent ternary semigroup T:*

(i) T is a 2 Primal ternary semigroup.

(ii) $\mathcal{P}(T)$ has the IFP.

(iii) $N(P)$ has the IFP for each prime ideal P of T.

(iv) $N(P) = N_P = \overline{N_P}$ for each prime ideal P of T.

Proof. $(i) \Rightarrow (ii)$ Let T be a 2 Primal ternary semigroup. Let $a, b, c \in \mathcal{P}(T)$ then $(bca)^3 = bc(abc)abca \in \mathcal{P}(T) \Rightarrow bca \in \mathcal{P}(T)$ as $\mathcal{P}(T)$ is completely semiprime by theorem 2.12 (ii). Again $bcaTT \subseteq \mathcal{P}(T)$ as $\mathcal{P}(T)$ is an ideal of T. Repeating above argument, we get $caTTb \subseteq \mathcal{P}(T)$ and hence $aTTbTTc \subseteq \mathcal{P}(T)$ as $\mathcal{P}(T)$ is an ideal of T. Thus $\mathcal{P}(T)$ has the IFP.

$(ii) \Rightarrow (iii)$ Let $abc \in N(P)$ where P is a prime ideal of T. So $abcTTyT \subseteq \mathcal{P}(T)$ for some $y \in T\backslash P$. Since $\mathcal{P}(T)$ has the IFP, $aTTbTTcTTyT \subseteq \mathcal{P}(T)$. Therefore $aTTbTTc \subseteq N(P)$. Thus $N(P)$ has the IFP for each prime ideal P of T.

$(iii) \Rightarrow (i)$ Always $\mathcal{P}(T) \subseteq \mathcal{N}(T)$. Let $x \in \mathcal{N}(T)$ then $x^{2n+1} = 0$ for some positive integer n. If possible suppose $x \notin \mathcal{P}(T)$, then there exists a prime ideal P of T such that $x \notin P$. As P is prime ideal of T, $T\backslash P$ is an m_{id}- system of T. So $xTTxTTx...xTTx \nsubseteq P$. Now $x^{2n+1} = 0 \in N(P)$. Since $N(P)$ has the IFP, $xTTxTTx...xTTx \subseteq N(P)$. As $N(P) \subseteq P$, $xTTxTTx...xTTx \subseteq P$, a contradiction. So $x \in \mathcal{P}(T)$. Hence $\mathcal{P}(T) = \mathcal{N}(T)$. i.e T is a 2-primal ternary semigroup.

$(i) \Rightarrow (iv)$ Let P be a prime ideal of T and $a \in N(P)$. Then there exists $b \in T\backslash P$ such that $aTTbT \subseteq \mathcal{P}(T)$. Since T contains the identity element, $abT \subseteq \mathcal{P}(T)$. So $a \in N_P \subseteq \overline{N_P}$. Thus $N(P) \subseteq N_P \subseteq \overline{N_P}$. Conversely, let $x \in \overline{N_P}$. Then $(xT)^n x \in N_P$ for some positive integer n. So there exists $y \in T\backslash P$ such that $(xT)^n xyT \subseteq \mathcal{P}(T)$ implies $x^{2n+1}yT \subseteq \mathcal{P}(T)$. Now by(ii), $\mathcal{P}(T)$ has the IFP, so $(x)^{2n+1}yT \subseteq \mathcal{P}(T) \Rightarrow xTTTTxTTTTx...xTTyTTT \subseteq \mathcal{P}(T)$. Thus in particular, $(xTTyT)^{2n+1} \subseteq \mathcal{P}(T)$ as T admits identity. Since T is a 2 Primal ternary semigroup by theorem 2.12 (ii), $\mathcal{P}(T)$ is a completely semiprime ternary semigroup. Therefore $xTTyT \subseteq \mathcal{P}(T)$ which implies that $x \in N(P)$. So $\overline{N_P} \subseteq N(P)$. Hence $N(P) = N_P = \overline{N_P}$ for each prime ideal P of T.

$(iv) \Rightarrow (i)$ Always $\mathcal{P}(T) \subseteq \mathcal{N}(T)$. Let $x \in \mathcal{N}(T)$. Since T is super nilpotent ternary semigroup, $(xT)^n x = 0$ for some posiotive integer n. If possible, let $x \notin \mathcal{P}(T)$. Then $x \notin P$ for some prime ideal P of T. Now $(xT)^n x = 0 \in N_P \Rightarrow x \in \overline{N_P}$. Since $N(P) = N_P = \overline{N_P}$, $x \in N(P) \subseteq P$, a contradiction. So $x \in \mathcal{P}(T)$. Therefore $\mathcal{N}(T) \subseteq \mathcal{P}(T)$. Hence $\mathcal{P}(T) = \mathcal{N}(T)$. i.e T is a 2-primal ternary semigroup.

Theorem 2.14. *The following statements are equivalent for a super nilpotent ternary semigroup T:*

(i) T is a 2-primal ternary semigroup.

(ii) $N(P)$ is a completely semiprime ideal of T for each prime ideal P of T.

(iii) $N(P)$ is a left and right symmetric ideal of T for each prime ideal P of T.

(iv) $aTb \in N(P)$ implies $bTTaT \subseteq N(P)$ for $a, b \in T$ and for each prime ideal P of T.

Proof. $(i) \Rightarrow (ii)$ Let P be a prime ideal of T and $x^3 \in N(P)$. So there exists $y \in T\backslash P$ such that $x^3TTyT \subseteq \mathcal{P}(T)$. Now as T is a 2 Primal ternary semigroup by theorem 2.13 (ii), $\mathcal{P}(T)$ has the IFP. So $x^3TTyT \subseteq \mathcal{P}(T)$ implies $xTTTTxTTTTxTTyT \subseteq \mathcal{P}(T)$. Thus in particular $(xTTyT)^3 \subseteq \mathcal{P}(T)$. Now by theorem 2.12 (ii), $\mathcal{P}(T)$ is completely semiprime ideal of T. So $(xTTyT) \subseteq \mathcal{P}(T)$. Thus $x \in N(P)$ for each prime ideal P of T. Hence $N(P)$ is a completely semiprime ideal of T for each prime ideal P of T.

The proof of $(ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i)$ are similar to theorem 2.12.

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