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G-RECURRENT AND G-SYMMETRIC FINSLER SPACES ADMITTING CERTAIN VECTOR FIELDS

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Abstract: In the current paper, G-recurrent and G-symmetric Finsler spaces have been introduced and the existence of certain vector fields such as concurrent vector field, special concircular vector field and torse forming vector field have been discussed. It is established that a G-recurrent Finsler space does not admit concurrent vector field, special concircular vector field or torse forming vector field, while a G-symmetric Finsler space admitting either of these vector fields is necessarily affinely connected.

AMS Subject Classification: 53B40
Key Words: Finsler space, G-recurrent, G-symmetric, special concircular vector field, torse forming vector field

1. Introduction

H.S. Ruse [3] gave the idea of recurrent curvature tensor in 3-dimensional Riemannian space in 1949. This concept was extended to an n-dimensional Riemannian space by A.G. Walker [1]. The concept of recurrent curvature tensor in Finsler space was introduced by A. Moor [2]. The recurrence of curvature tensors have been studied by several authors including Mishra and Pande [6], Sen [5] and P.N. Pandey [7]. The aim of this paper is to define G-recurrent and G-symmetric Finsler spaces and to discuss the existence of concurrent vector field, special concircular vector field and torse forming vector field therein.

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2. Preliminaries

Let F_n be an n-dimensional Finsler space having F as a metric funtion satisfying the requisite conditions [4], g_{ij} is corresponding metric tensor and G^i_{jk} Berwald connection coefficients which satisfy

$$\begin{array}{llllll} \text{(a)} & \dot{\partial}_h G^i_{jk} = G^i_{hjk} & \text{(b)} & \dot{\partial}_j G^i_k = G^i_{jk} & \text{(c)} & \dot{\partial}_j G^i = G^i_j; \\ \text{(d)} & y^j G^i_{jkh} = 0 & \text{(e)} & y^j G^i_{jk} = G^i_k & \text{(f)} & y^j G^i_j = 2G^i, \end{array} \tag{1}$$

where $\dot{\partial}_h$ denotes the partial differentiation with respect to y^h. G^i_{jkh} defined by $(1a)$, are components of a symmetric tensor.

The Berwald covariant derivative of an arbitrary tensor field T^i_j is given by

$$T^i_{j(k)} = \partial_k T^i_j - (\dot{\partial}_r T^i_j) G^r_k + T^r_j G^i_{rk} - T^i_r G^r_{jk}. \tag{2}$$

The commutation formula for the partial derivative and Berwald covariant derivative is given by

$$\dot{\partial}_k T^i_{j(h)} - (\dot{\partial}_k T^i_j)_{(h)} = T^r_j G^i_{rhk} - T^i_r G^r_{kjh}. \tag{3}$$

Definition 1. A Finsler space F_n is called affinely connected if and only if

$$G^i_{jkh} = 0. \tag{4}$$

Definition 2. A vector field $v^i(x^j)$ in a Finsler Space F_n is called concurrent, special concircular, recurrent and torse forming vector fields respectively, if it satisfies the following conditions

$$\begin{array}{llll} \text{(a)} & v^i_{(k)} = c\delta^i_k & \text{(b)} & v^i_{(k)} = \rho\delta^i_k \\ \text{(c)} & v^i_{(k)} = \mu_k v^i & \text{(d)} & v^i_{(k)} = \rho\delta^i_k + \mu_k v^i, \end{array} \tag{5}$$

where c is a nonzero constant, ρ is a scalar field and μ_k is a nonzero covariant vector field.

Definition 3. A Finsler space F_n is called G-recurrent if it satisfies the condition

$$G^i_{jkh(m)} = \lambda_m G^i_{jkh}, \qquad G^i_{jkh} \neq 0, \tag{6}$$

where λ_m be a non-null covariant vector field.

Definition 4. A Finsler space F_n is called G-symmetric if it satisfies the condition

$$G^i_{jkh(m)} = 0. \tag{7}$$

3. G-Recurrent Finsler Space

Let us consider a G-recurrent Finsler space admitting a concurrent vector field characterized by (5a).

Differentiating (5a) partially with respect to y^j and using (3), we obtain

$$v^r G^i_{rjk} = 0. \tag{8}$$

Covariant differentiation of (8) with respect to x^m gives

$$v^r_{(m)} G^i_{rjk} + v^r G^i_{rjk(m)} = 0. \tag{9}$$

Using (5a) and (6) in (9), we get

$$cG^i_{mjk} + \lambda_m v^r G^i_{rjk} = 0. \tag{10}$$

In view of (8) and (10), we have

$$G^i_{mjk} = 0, \tag{11}$$

for $c \neq 0$. This gives a contradiction. Thus, a G-recurrent Finsler space does not admit a concurrent vector field.

Theorem 5. *A G-recurrent Finsler space does not admit a concurrent vector field.*

Let us consider a G-recurrent Finsler space admitting a special concircular vector field characterized by (5b).

Differentiating (5b) partially with respect to y^j and using (3), we get

$$v^r G^i_{rjk} = 0. \tag{12}$$

Differentiating (12) covariantly with respect to x^m, we get

$$v^r_{(m)} G^i_{rjk} + v^r G^i_{rjk(m)} = 0. \tag{13}$$

From (5b) and (6), (13) becomes

$$\rho G^i_{mjk} + \lambda_m v^r G^i_{rjk} = 0. \tag{14}$$

In view of (12) and (14), we obtain

$$G^i_{mjk} = 0, \tag{15}$$

for $\rho \neq 0$. This gives a contradiction. Thus, a G-recurrent Finsler space does not admit a special concircular vector field. This leads to

Theorem 6. *A G-recurrent Finsler space does not admit a special concircular vector field.*

Let us consider a G-recurrent Finsler space admitting a recurrent vector field characterized by (5c).

Differentiating (5c) partially with respect to y^j and using (3), we obtain

$$v^r G^i_{rjk} = (\dot{\partial}_j \mu_k) v^i. \tag{16}$$

Covariant differentiation of (16) with respect to x^m gives

$$v^r_{(m)} G^i_{rjk} + v^r G^i_{rjk(m)} = (\dot{\partial}_j \mu_k)_{(m)} v^i + v^i_{(m)} (\dot{\partial}_j \mu_k). \tag{17}$$

Using (5c) and (6) in (17), we have

$$\mu_m v^r G^i_{rjk} + \lambda_m v^r G^i_{rjk} = (\dot{\partial}_j \mu_k)_{(m)} v^i + \mu_m v^i (\dot{\partial}_j \mu_k). \tag{18}$$

In view of (16) and (18), we get

$$\{(\dot{\partial}_j \mu_k)_{(m)} - \lambda_m \dot{\partial}_j \mu_k\} v^i = 0, \tag{19}$$

which implies

$$(\dot{\partial}_j \mu_k)_{(m)} - \lambda_m \dot{\partial}_j \mu_k = 0. \tag{20}$$

Thus we have

Theorem 7. *A G-recurrent Finsler space admitting a recurrent vector field satisfies (20).*

Let us consider a G-recurrent Finsler space admitting a torse forming vector field characterized by (5d).

Differentiating (5*d*) partially with respect to y^j and using (3), we get

$$v^r G^i_{rjk} = (\dot{\partial}_j \mu_k) v^i. \tag{21}$$

Differentiating (21) covariantly with respect to x^m, we get

$$v^r_{(m)} G^i_{rjk} + v^r G^i_{rjk(m)} = (\dot{\partial}_j \mu_k)_{(m)} v^i + (\dot{\partial}_j \mu_k) v^i_{(m)}. \tag{22}$$

Substituting (5*d*) and (6) in (22), we obtain

$$\rho G^i_{mjk} + \mu_m v^r G^i_{rjk} + v^r \lambda_m G^i_{rjk} = (\dot{\partial}_j \mu_k)_{(m)} v^i + \rho \delta^i_m (\dot{\partial}_j \mu_k) + \mu_m v^i (\dot{\partial}_j \mu_k). \tag{23}$$

In view of (21) and (23), we get

$$\rho G^i_{mjk} + v^r \lambda_m G^i_{rjk} = (\dot{\partial}_j \mu_k)_{(m)} v^i + \rho \delta^i_m (\dot{\partial}_j \mu_k). \tag{24}$$

Transvecting (24) with y^m and using (1d), we have

$$\lambda_m y^m v^r G^i_{rjk} = y^m(\dot{\partial}_j \mu_k)_{(m)} v^i + \rho y^i (\dot{\partial}_j \mu_k). \tag{25}$$

On simplifying , (25) becomes

$$[y^m(\dot{\partial}_j \mu_k)_{(m)} - y^m \lambda_m (\dot{\partial}_j \mu_k)]v^i + \rho y^i (\dot{\partial}_j \mu_k) = 0. \tag{26}$$

Let f^{jk} be an arbitrary tensor, transvecting (26) with f^{jk} yields

$$f^{jk}[y^m(\dot{\partial}_j \mu_k)_{(m)} - y^m \lambda_m (\dot{\partial}_j \mu_k)]v^i + f^{jk}\rho(\dot{\partial}_j \mu_k) y^i = 0, \tag{27}$$

which is of the form av^i + by^i =0.

Since P.N. Pandey [8] has proved a lemma, if $v^i(x^j)$ are components of a non-null vector then the equation $av^i + by^i = 0$ implies $a = b = 0$.

By this lemma, (27) implies

$$f^{jk}[y^m(\dot{\partial}_j \mu_k)_{(m)} - y^m \lambda_m (\dot{\partial}_j \mu_k)] = 0 \quad and \quad f^{jk}\rho(\dot{\partial}_j \mu_k) = 0.$$

Since f^{jk} be an arbitrary tensor and $\rho \neq 0$, we have

$$\dot{\partial}_j \mu_k = 0. \tag{28}$$

In view of (24) and (28), we get

$$G^i_{mjk} = 0, \tag{29}$$

for $\rho \neq 0$. Which is a contradiction, hence we have

Theorem 8. *A G-recurrent Finsler space does not admit a torse forming vector field.*

4. G-Symmetric Finsler Space

Let us consider a G-symmetric Finsler space admitting a concurrent vector field characterized by (5a).

Differentiating (5a) partially with respect to y^j and using (3), we get

$$v^r G^i_{rjk} = 0. \tag{30}$$

Differentiating (30) covariantly with respect to x^m, we have

$$v^r_{(m)} G^i_{rjk} + v^r G^i_{rjk(m)} = 0. \tag{31}$$

Using (5a) and (7) in (31), we obtain

$$G^i_{mjk} = 0, \tag{32}$$

for $c \neq 0$. This leads to

Theorem 9. *A G-symmetric Finsler space admitting a concurrent vector field is affinely connected.*

Let us consider a G-symmetric Finsler space admitting a special concircular vector field characterized by (5b).

Differentiating (5b) partially with respect to y^j and using (3), we get

$$v^r G^i_{rjk} = 0. \tag{33}$$

Covariant differentiation of (33) with respect to x^m yields

$$v^r_{(m)} G^i_{rjk} + v^r G^i_{rjk(m)} = 0. \tag{34}$$

Substituting (5b) and (7) in (34), we get

$$G^i_{mjk} = 0, \tag{35}$$

for $\rho \neq 0$. This leads to

Theorem 10. *A G-symmetric Finsler space admitting a special concircular vector field is affinely connected.*

Let us consider a G-symmetric Finsler space admitting a recurrent vector field characterized by (5c).

Differentiating (5c) partially with respect to y^j and using (3), we get

$$v^r G^i_{rjk} = (\dot{\partial}_j \mu_k) v^i. \tag{36}$$

Differentiating (36) covariantly with respect to x^m, we have

$$v^r_{(m)} G^i_{rjk} + v^r G^i_{rjk(m)} = (\dot{\partial}_j \mu_k)_{(m)} v^i + (\dot{\partial}_j \mu_k) v^i_{(m)}. \tag{37}$$

Substituting (5c) and (7) in (37), we get

$$\mu_m v^r G^i_{rjk} = (\dot{\partial}_j \mu_k)_{(m)} v^i + \mu_m v^i (\dot{\partial}_j \mu_k). \tag{38}$$

Using (36) in (38), we obtain

$$(\dot{\partial}_j \mu_k)_{(m)} v^i = 0. \tag{39}$$

This implies

$$(\dot{\partial}_j \mu_k)_{(m)} = 0. \tag{40}$$

Thus we have

Theorem 11. *A G-symmetric Finsler space admitting a recurrent vector field satisfies (40).*

Let us consider a G-symmetric Finsler space admitting a torse forming vector field characterized by (5d).

Differentiating (5d) partially with respect to y^j and using (3), we get

$$v^r G^i_{rjk} = (\dot{\partial}_j \mu_k) v^i. \tag{41}$$

Differentiating (41) covariantly with respect to x^m , we have

$$v^r_{(m)} G^i_{rjk} + v^r G^i_{rjk(m)} = (\dot{\partial}_j \mu_k)_{(m)} v^i + (\dot{\partial}_j \mu_k) v^i_{(m)}. \tag{42}$$

Using (5d) and (7) in (42), we get

$$\rho G^i_{mjk} + \mu_m v^r G^i_{rjk} = (\dot{\partial}_j \mu_k)_{(m)} v^i + \rho \delta^i_m (\dot{\partial}_j \mu_k) + (\dot{\partial}_j \mu_k) \mu_m v^i. \tag{43}$$

In view of (41) and (43), we obtain

$$\rho G^i_{mjk} = (\dot{\partial}_j \mu_k)_{(m)} v^i + \rho \delta^i_m (\dot{\partial}_j \mu_k). \tag{44}$$

Transvecting (44) with y^m and using (1d), we get

$$y^m (\dot{\partial}_j \mu_k)_{(m)} v^i + \rho y^i (\dot{\partial}_j \mu_k) = 0. \tag{45}$$

Transvecting (45) with an arbitrary tensor f^{jk} , we have

$$f^{jk} \{ y^m (\dot{\partial}_j \mu_k)_{(m)} \} v^i + \rho f^{jk} (\dot{\partial}_j \mu_k) y^i = 0, \tag{46}$$

which is of the form av^i + by^i =0.

Since P. N. Pandey [8] has proved that if $v^i(x^j)$ are components of a non-null vector then the equation $av^i + by^i = 0$ implies $a = b = 0$.

Then (46) implies

$$\rho f^{jk} (\dot{\partial}_j \mu_k) = 0. \tag{47}$$

Since f^{jk} is an arbitrary tensor and $\rho \neq 0$, (47) gives

$$(\dot{\partial}_j \mu_k) = 0. \tag{48}$$

In view of (44) and (48), we obtain

$$G^i_{mjk} = 0. \tag{49}$$

This leads to

Theorem 12. *A G-symmetric Finsler space admitting a torse forming vector field is necessarily an affinely connected space.*

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