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ON A NONLOCAL INVERSE PROBLEM WITH THE INTEGRAL OVERDETERMINATION CONDITION FOR A PARABOLIC EQUATION OF SECOND ORDER

E.I. Azizbayov[1 §], Y.T. Mehraliyev[2]

[1]Department of Computational Mathematics
Baku State University
Z. Khalilov Str. 23, Baku, AZ1148, AZERBAIJAN

[2]Department of Differential and Integral Equations
Baku State University
Z. Khalilov Str. 23, Baku, AZ1148, AZERBAIJAN

Abstract: In this paper, the inverse boundary value problem for the determination of the coefficient of parabolic equation with nonlocal boundary and integral overdetermination condition is investigated. Moreover, in the present work a time-nonlocal boundary condition is considered. The existence and uniqueness theorem for a classical solution is proved. The proof is based on the contraction mapping principle and Fourier method.

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1. Introduction

Contemporary problems of natural science leads to the necessity of formulation and investigation qualitatively new problems, a notable example of which is the class of non-local problems for differential equations in partial deriva-

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§Correspondence author

tives. Among the non-local problems can be distinguished class of problems with integral conditions. Conditions of this kind appear in the mathematical modeling of phenomena associated with physical plasmas [1], the spread of heat [2], [3], the process of moisture in capillary-simple environment [4], the problems of demography and mathematical biology. Inverse problems are actively developing branch of modern mathematics. In the inverse problems together with the initial and boundary conditions, characteristic of a direct problem, given the additional information, the need for which is due to the presence of unknown coefficients or right-hand side of the equation. However, on the study of inverse problems are dedicated to some of the research works of M.I.Ivanchov [5], Sh.Nakamura [6], A.I.Kozhanov [7], A.I.Prilepko [8], [9], and many other papers.

Additional information, which is called an overdetermination condition, can be presented in various forms. Inverse problems with the integral overdetermination conditions for parabolic equations have been investigated in the works of D.Lesnic [10], M.I.Ismailov [10], [11], A.Bouziani [12] and etc. but the statement of problems and the proof technique used in this study are different from representations in these papers.

The aim of this work is to prove the existence and uniqueness of the classical solution of the inverse boundary value problem for a second order parabolic equation with an integral overdetermination and time non-local conditions.

2. Mathematical Formulation of the Problem

Let $T > 0$ be a fixed number and denote by

$$D_T := \{(x,t) : 0 \le x \le 1, 0 \le t \le T\}.$$

We consider the equation

$$c(t)u_t(x,t) = u_{xx}(x,t) + a(t)u(x,t) + f(x,t) \tag{1}$$

in the rectangular domain D_T. The inverse problem has nonlocal initial condition

$$u(x,0) + \delta u(x,T) + \int_0^T p(t)u(x,t)dt = \varphi(x) \quad (0 \le x \le 1), \tag{2}$$

boundary condition

$$u(0,t) = u_x(1,t) = 0 \quad (0 \le t \le T), \tag{3}$$

and overdetermination condition

$$\int_0^1 H(x)u(x,t)dx = h(t) \quad (0 \le t \le T) \tag{4}$$

where $\delta \ge 0$ is fixed number, $0 < c(t)$, $f(x,t)$, $0 \le p(t)$, $\varphi(x)$, $H(x)$, $h(t)$ are given functions, $u(x,t), a(t)$ are the sought functions.

The problem of finding a pair $\{u(x,t), a(t)\}$ in (1)-(4) will be called an inverse problem.

Definition 1. The pair $\{u(x,t), a(t)\}$ from the class $C^{2,1}(D_T) \times C[0,T]$ is said to be a classical solution of problem (1) - (4), if the functions $u(x,t) \in C^{2,1}(D_T)$ and $a(t) \in C[0,T]$ satisfy: an equation (1) in D_T, the condition (2) on $[0,1]$, and the conditions (3), (4) on the interval $[0,T]$.

The following lemma is valid.

Lemma 2. *Assume that* $\delta \ge 0$, $0 < c(t) \in C[0,T]$, $a(t) \in C[0,T]$ *and* $0 \le p(t) \in C[0,T]$ *hold. Then the problem*

$$c(t)y'(t) = a(t)y(t) \quad (0 \le t \le T), \tag{5}$$

$$y(0) + \delta y(T) + \int_0^T p(\tau)y(\tau)d\tau = 0 \tag{6}$$

has a unique trivial solution.

Proof. It's clear that the general solution of equation (5) has the form:

$$y(t) = ce^{\int_0^t \frac{a(\tau)}{c(\tau)}d\tau}. \tag{7}$$

Using (6) we get

$$c\left(1 + \delta e^{\int_0^T \frac{a(t)}{c(t)}dt} + \int_0^T p(t)e^{\int_0^t \frac{a(\tau)}{c(\tau)}d\tau} dt\right) = 0.$$

By $\delta \ge 0$ and $p(t) \ge 0$, from the foregoing relation it is obvious that $c = 0$. Substituting the value of $c = 0$ in (7), we find that $y(t) \equiv 0 \ \ (0 \le t \le T)$; i.e., the problem (5), (6) has only the trivial solution. The proof is complete. □

Theorem 3. *Suppose that* $0 < c(t) \in C[0,T]$, $f(x,t) \in C(D_T)$, $\varphi(x) \in C[0,1]$, $H(x) \in C[0,1]$, $h(t) \in C^1[0,T]$, $h(t) \neq 0$ $(0 \leq t \leq T)$ *and the compatibility condition*

$$h(0) + \delta h(T) + \int_0^T p(t)h(t)dt = \int_0^1 H(x)\varphi(x)dx. \tag{8}$$

hold. Then the problem of finding a classical solution of (1)-(4) is equivalent to the problem determining of functions $u(x,t) \in C^{2,1}(D_T)$, *and* $a(t) \in C[0,T]$ *satisfying equation (1), conditions (2) and (3), and the condition*

$$c(t)h'(t) = \int_0^1 H(x)u_{xx}(x,t)dx+$$

$$+a(t)h(t) + \int_0^1 H(x)f(x,t)dx \quad (0 \leq t \leq T). \tag{9}$$

Proof. Let $\{u(x,t), a(t)\}$ be a classical solution of (1) - (4). Multiplying the equation (1) by $H(x)$ and integrating both sides of (1) with respect to x from 0 to 1 yields

$$c(t)\frac{d}{dt}\int_0^1 H(x)u(x,t)dx = \int_0^1 H(x)u_{xx}(x,t)dx+$$

$$+a(t)\int_0^1 H(x)u(x,t)dx + \int_0^1 H(x)f(x,t)dx \quad (0 \leq x \leq 1). \tag{10}$$

Under the assumption $h(t) \in C^1[0,T]$ we differentiating (4) and get

$$\int_0^1 H(x)u_t(x,t)dx = h'(t) \quad (0 \leq t \leq T).$$

From (10), by (4), we conclude that the relation (9) is fulfilled.

Now, assume that the pair of functions $\{u(x,t), a(t)\}$ is the solution of (1)-(3),(9). Then from (9) and (10) we have

$$c(t)\frac{d}{dt}\left(\int_0^1 H(x)u(x,t)dx - h(t)\right) =$$

$$= a(t)\left(\int_0^1 H(x)u(x,t)dx - h(t)\right) \quad (0 \le t \le T). \tag{11}$$

By (2) and the compatibility condition (8), it is easy to see that

$$\int_0^1 H(x)u(x,0)dx - h(0) + \delta\left(\int_0^1 H(x)u(x,T)dx - h(T)\right) +$$

$$+\int_0^T p(t)\left(\int_0^1 H(x)u(x,t)dx - h(t)\right)dt =$$

$$= \int_0^1 H(x)\left[u(x,0) + \delta u(x,T) + \int_0^T p(t)u(x,t)dt\right]dx -$$

$$-\left(h(0) + \delta h(T) + \int_0^T p(t)h(t)dt\right) =$$

$$= \int_0^1 H(x)\varphi(x)dx - \left(h(0) + \delta h(T) + \int_0^T p(t)h(t)dt\right) = 0. \tag{12}$$

From (11), (12), by virtue of Lemma 2, we conclude that the condition (4) is satisfied. Theorem has been proved. □

3. Existence and Uniqueness of the Solution of the Inverse Problem

The theorem proved above allowed the development of the transition to the justification of the solvability of the considered problem, which is as follows. First we prove the existence and uniqueness of solution of the problem (1) - (3), (9), and then show that under certain conditions on the input data, it is a classical solution of problem (1) - (4).

Now consider the problem (1) - (3), and suggesting that $a(t)$ is known function. Since for $\lambda_k = \frac{\pi}{2}(2k-1)$, the system $\{\sin \lambda_k x\}_{k=1}^{\infty}$ form a basis in $L_2(0,1)$, then

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t) \sin \lambda_k x \quad \left(\lambda_k = \frac{\pi}{2}(2k-1)\right), \tag{13}$$

where

$$u_k(t) = 2\int_0^1 u(x,t)\sin\lambda_k x dx \quad (k = 1, 2, ...).$$

By applying the formal scheme of the Fourier method, from (1) and (2) we obtain

$$c(t)u_k'(t) + \lambda_k^2 u_k(t) = f_k(t) + a(t)u_k(t)(k = 1, 2, ...; 0 \le t \le T), \tag{14}$$

$$u_k(0) + \delta u_k(T) + \int_0^T p(t)u_k(t)dt = \varphi_k \quad (k = 1, 2, \dots) \tag{15}$$

where

$$f_k(t) = 2\int_0^1 f(x,t)\sin\lambda_k x dx \quad (k = 1, 2, \dots),$$

$$\varphi_k = 2\int_0^1 \varphi(x)\sin\lambda_k x dx \quad (k = 1, 2, \dots).$$

Solving problem (14), (15) we find

$$u_k(t) = \frac{e^{-\int_0^t \frac{\lambda_k^2}{c(s)}ds}}{1 + \delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}ds}}\left(\varphi_k - \int_0^T p(t)u_k(t)dt\right) -$$

$$- \frac{\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}ds}}{1 + \delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}ds}}\int_0^T \frac{1}{c(\tau)}F_k(\tau; u, a)e^{-\int_\tau^t \frac{\lambda_k^2}{c(s)}ds}d\tau +$$

$$+ \int_0^t \frac{1}{c(\tau)}F_k(\tau; u, a)e^{-\int_\tau^t \frac{\lambda_k^2}{c(s)}ds}d\tau \quad (k = 1, 2, \dots), \tag{16}$$

where $F_k(\tau; u, a) = f_k(t) + a(t)u_k(t) \quad (k = 1, 2, \dots)$.

After substituting expressions $u_k(t)(k = 1, 2, ...)$ into (13), for determining the component $u(x,t)$ of the solution of problem (1) - (3), (9) we get

$$u(x,t) = \sum_{k=1}^{\infty}\left\{\frac{e^{-\int_0^t \frac{\lambda_k^2}{c(s)}ds}}{1 + \delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}ds}}\left(\varphi_k - \int_0^T p(t)u_k(t)dt\right) -\right.$$

$$-\frac{\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}ds}}{1+\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}ds}} \int_0^T \frac{1}{c(\tau)} F_k(\tau;u,a) e^{-\int_\tau^t \frac{\lambda_k^2}{c(s)}ds} d\tau +$$

$$+ \int_0^t \frac{1}{c(\tau)} F_k(\tau;u,a) e^{-\int_\tau^t \frac{\lambda_k^2}{c(s)}ds} d\tau \Bigg\} \sin \lambda_k x. \tag{17}$$

Now, using (13) from (9) we have

$$a(t) = [h(t)]^{-1} \Bigg\{ c(t)h'(t) - \int_0^1 H(x) f(x,t)dx +$$

$$+ \sum_{k=1}^{\infty} \lambda_k^2 u_k(t) \int_0^1 H(x) \sin \lambda_k x dx \Bigg\}. \tag{18}$$

In order to obtain a second component of the pair $\{u(x,t), a(t)\}$ we substitute the expression (16) into (18).

$$a(t) = [h(t)]^{-1} \Bigg\{ c(t)h'(t) - \int_0^1 H(x) f(x,t)dx +$$

$$+ \sum_{k=1}^{\infty} \lambda_k^2 \left[\frac{e^{-\int_0^t \frac{\lambda_k^2}{c(s)}ds}}{1+\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}ds}} \left(\varphi_k - \int_0^T p(t) u_k(t) dt \right) - \right.$$

$$-\frac{\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}ds}}{1+\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}ds}} \int_0^T \frac{1}{c(\tau)} F_k(\tau;u,a) e^{-\int_\tau^t \frac{\lambda_k^2}{c(s)}ds} d\tau +$$

$$\left. + \int_0^t \frac{1}{c(\tau)} F_k(\tau;u,a) e^{-\int_\tau^t \frac{\lambda_k^2}{c(s)}ds} d\tau \right] \int_0^1 H(x) \sin \lambda_k x dx \Bigg\}. \tag{19}$$

Consequently, the solution of (1) - (3), (9) reduced to the solution of the system (17), (19), with respect to the unknown functions of $u(x,t)$ and $a(t)$.

To study the question of uniqueness of the solution of problem (1) - (3), (9), the following lemma play an essential role.

Lemma 4. *If* $\{u(x,t), a(t)\}$ *is an arbitrary solution of (1)-(3), (9), then the functions*

$$u_k(t) = 2\int_0^1 u(x,t)\sin\lambda_k x dx \quad (k = 1, 2, ...)$$

satisfy the system (16) on the interval $[0,T]$.

Proof. Suppose that $\{u(x,t), a(t)\}$ is a solution of (1)-(3), (9), then multiplying both sides of equation (1) by the function $2\sin\lambda_k x \quad (k = 1, 2, \ldots)$, integrating obtained equality with respect to x from 0 to 1, and using the following relations

$$2\int_0^1 u_t(x,t)\sin\lambda_k x dx =$$

$$= \frac{d}{dt}\left(2\int_0^1 u(x,t)\sin\lambda_k x dx\right) = u_k'(t) \quad (k = 1, 2, \ldots),$$

$$2\int_0^1 u_{xx}(x,t)\sin\lambda_k x dx =$$

$$= -\lambda_k^2\left(2\int_0^1 u(x,t)\sin\lambda_k x dx\right) = -\lambda_k^2 u_k(t) \quad (k = 1, 2, \ldots)$$

we get that satisfies the equation (14).

Similarly, from (2) we find that the condition (15) holds true.

Thus, $u_k(t) \quad (k = 1, 2, ...)$ are the solutions of problem (14), (15). Hence, directly it follows that the functions $u_k(t) \quad (k = 1, 2, ...)$ satisfy the system (16) on $[0,T]$. The lemma is thus proved. □

It's clear that, if $u_k(t) = 2\int_0^1 u(x,t)\sin\lambda_k x dx \quad (k = 1, 2, ...)$ are the solution of system (16), then the functions

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t)\sin\lambda_k x \quad \left(\lambda_k = \frac{\pi}{2}(2k-1)\right)$$

and $a(t)$ are the solution of the systems (17), (19) too.

From Lemma 4 it follows that

Corollary 5. *Suppose that system (17), (19) has a unique solution. Then the problem (1) - (3), (9), couldn't have more than one solution, in other words, if problem (1)-(3), (9) has a solution, then it is unique.*

With the purpose to study, consider the following spaces:

Let $B_{2,T}^3$ [13] denote the set of all functions of the form

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t) \sin \lambda_k x \quad \left(\lambda_k = \frac{\pi}{2}(2k-1)\right),$$

considered in domain D_T, where the function $u_k(t)\ \ (k = 1, 2, ...)$, is continuous on $[0,T]$ and satisfy the condition

$$J(u) = \left(\sum_{k=1}^{\infty} \left(\lambda_k^3 \, \|u_k(t)\|_{C[0,T]} \right)^2 \right)^{\frac{1}{2}} < +\infty.$$

The norm in the space $B_{2,T}^3$ is defined as follows

$$\|u(x,t)\|_{B_{2,T}^3} = J(u).$$

It is known that $B_{2,T}^3$ is Banach space.

We denote by E_T^3, the Cartesian product of $B_{2,T}^3 \times C[0,T]$. Clearly, the set of E_T^3, equipped with the norm

$$\|u(x,t)\|_{E_T^3} = \|u(x,t)\|_{B_{2,T}^3} + \|a(t)\|_{C[0,T]}$$

will be Banach space.

Let's introduce the following denotation

$$\Phi_1(u,a) = \tilde{u}(x,t) \equiv \sum_{k=1}^{\infty} \tilde{u}_k(t) \sin \lambda_k x, \quad \Phi_2(u,a) = \tilde{a}(t)$$

where $\tilde{u}_k(t)\ \ (k = 1, 2, ...)$ and $\tilde{a}(t)$ are equal to the right-hand sides of (16) and (19) respectively.

Now consider the operator

$$\Phi(u,a) = \{\Phi_1(u,a), \Phi_2(u,a)\}$$

in the space E_T^3.

Assume that the data of problem (1) - (3) and (9); i.e., $c(t)$, $f(x,t)$, $\varphi(x)$, $p(t)$, $h(t)$, and δ satisfy the following conditions:

A) $\varphi(x) \in C^2[0,1], \varphi^{(3)}(x) \in L_2(0,1), \varphi(0) = \varphi'(1) = \varphi''(0) = 0$;

B) $f(x,t), f_x(x,t), \quad f_{xx}(x,t) \in C(D_T), \quad f_{xxx}(x,t) \in L_2(D_T)$,
$f(0,t) = f_x(1,t) = f_{xx}(0,t) = 0 \quad (0 \le t \le T)$;

C) $\delta \ge 0, \quad 0 \le p(t) \in C[0,T], \quad H(x) \in C[0,1]$,
$h(t) \in C^1[0,T], \quad h(t) \ne 0 \quad (0 \le t \le T)$.

Then using simple transformations from (16) and (19) we get that

$$\left(\sum_{k=1}^{\infty} \left(\lambda_k^3 \|\tilde{u}_k(t)\|_{C[0,T]} \right)^2 \right)^{\frac{1}{2}} \le \sqrt{6} \left(\sum_{k=1}^{\infty} \left(\lambda_k^3 |\varphi_k| \right)^2 \right)^{\frac{1}{2}} +$$

$$+\sqrt{6} T \|p(t)\|_{C[0,T]} \left(\sum_{k=1}^{\infty} \left(\lambda_k^3 \|u_k(t)\|_{C[0,T]} \right)^2 \right)^{\frac{1}{2}} +$$

$$+(1+\delta) \left\| \frac{1}{c(t)} \right\|_{C[0,T]} \sqrt{6} T \|a(t)\|_{C[0,T]} \times$$

$$\times \left(\sum_{k=1}^{\infty} \left(\lambda_k^3 \|u_k(t)\|_{C[0,T]} \right)^2 \right)^{\frac{1}{2}}, \tag{20}$$

$$\|\tilde{a}(t)\|_{C[0,T]} \le$$

$$\le \left\| [h(t)]^{-1} \right\|_{C[0,T]} \left\{ \left\| c(t)h'(t) - \int_0^1 H(x) f(x,t) dx \right\|_{C[0,T]} + \right.$$

$$+ \left(\sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} \|H(x)\|_{C[0,1]} \left[\left(\sum_{k=1}^{\infty} \left(\lambda_k^3 |\varphi_{ik}| \right)^2 \right)^{\frac{1}{2}} + \right.$$

$$+T \|p(t)\|_{C[0,T]} \left(\sum_{k=1}^{\infty} \left(\lambda_k^3 \|u_k(t)\|_{C[0,T]} \right)^2 \right)^{\frac{1}{2}} +$$

$$+(1+\delta)\sqrt{T} \left\| \frac{1}{c(t)} \right\|_{C[0,T]} \left(\int_0^T \sum_{k=1}^{\infty} \left(\lambda_k^3 |f_k(\tau)| \right)^2 d\tau \right)^{\frac{1}{2}} +$$

$$+(1+\delta) \left\| \frac{1}{c(t)} \right\|_{C[0,T]} T \|a(t)\|_{C[0,T]} \times$$

$$\times\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left\|u_k(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\Bigg]\Bigg\}. \tag{21}$$

From the inequalities (20) and (21) we find

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left\|\tilde{u}_k(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \le \sqrt{6}\left\|\varphi^{(3)}(x)\right\|_{L_2(0,1)} +$$

$$+\sqrt{6}T\left\|p(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left\|u_k(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} +$$

$$+(1+\delta)\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\sqrt{6T}\left\|f_{xxx}(x,t)\right\|_{L_2(D_T)} +$$

$$+(1+\delta)\left\|\frac{1}{c(t)}\right\|_{C[0,T]} T\left\|a(t)\right\|_{C[0,T]}\times$$

$$\times\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left\|u_k(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}, \tag{22}$$

$$\left\|\tilde{a}(t)\right\|_{C[0,T]} \le$$

$$\le \left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|c(t)h'(t)-\int_0^1 H(x)f(x,t)dx\right\|_{C[0,T]} +\right.$$

$$+\left\|H(x)\right\|_{C[0,1]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left[\left\|\varphi^{(3)}(x)\right\|_{L_2(0,1)} +\right.$$

$$+T\left\|p(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left\|u_k(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} +$$

$$+(1+\delta)\sqrt{T}\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\left\|f_{xxx}(x,t)\right\|_{L_2(D_T)} +$$

$$+(1+\delta)\left\|\frac{1}{c(t)}\right\|_{C[0,T]} T\left\|a(t)\right\|_{C[0,T]}\times$$

$$\times\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left\|u_k(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\Bigg]\Bigg\}. \tag{23}$$

Now we introduce the following denotations

$$A_1(T)=\sqrt{6}\left\|\varphi^{(3)}(x)\right\|_{L_2(0,1)}+$$

$$+\sqrt{6T}(1+\delta)\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\left\|f_{xxx}(x,t)\right\|_{L_2(D_T)},$$

$$B_1(T)=\sqrt{6}T\left\|p(t)\right\|_{C[0,T]},$$

$$C_1(T)=\sqrt{6}(1+\delta)\left\|\frac{1}{c(t)}\right\|_{C[0,T]}T,$$

$$A_2(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|c(t)h'(t)-\int_0^1 H(x)f(x,t)dx\right\|_{C[0,T]}+\right.$$

$$+\left\|H(x)\right\|_{C[0,1]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\times$$

$$\left.\times\left[\left\|\varphi^{(3)}(x)\right\|_{L_2(0,1)}+(1+\delta)\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\sqrt{T}\left\|f_{xxx}(x,t)\right\|_{L_2(D_T)}\right]\right\},$$

$$B_2(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^{-2})\right)^{\frac{1}{2}}\left\|H(x)\right\|_{C[0,1]}\left\|p(t)\right\|_{C[0,T]}T,$$

$$C_2(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^{-2})\right)^{\frac{1}{2}}\times$$

$$\times\left\|H(x)\right\|_{C[0,1]}(1+\delta)\left\|\frac{1}{c(t)}\right\|_{C[0,T]}T.$$

The estimations (22) and (23) may be written in the form

$$\left\|\tilde{u}(x,t)\right\|_{B^3_{2,T}}\le A_1(T)+B_1(T)\left\|u(x,t)\right\|_{B^3_{2,T}}+$$

$$+C_1(T)\left\|a(t)\right\|_{C[0,T]}\left\|u(x,t)\right\|_{B^3_{2,T}}, \tag{24}$$

$$\|\tilde{a}(t)\|_{C[0,T]} \leq A_2(T) + B_2(T)\,\|u(x,t)\|_{B^3_{2,T}} +$$
$$+C_2(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^3_{2,T}}\,. \tag{25}$$

Further, from (24) and (25) it follows that

$$\|\tilde{u}(x,t)\|_{B^3_{2,T}} + \|\tilde{a}(t)\|_{C[0,T]} \leq$$
$$\leq A(T) + B(T)\,\|u(x,t)\|_{B^3_{2,T}} + C(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^3_{2,T}}\,, \tag{26}$$

where

$$A(T) = A_1(T) + A_2(T), B(T) = B_1(T) + B_2(T), C(T) = C_1(T) + C_2(T).$$

Now we can prove the following theorem.

Theorem 6. *If conditions A)-C) and the inequality*

$$((A(T) + 2)C(T) + B(T))(A(T) + 2) < 1, \tag{27}$$

hold, then problem (1)-(3), (9) has a unique solution in the ball $K = K_R(\|z\|_{E^3_T} \leq R \leq A(T) + 2)$ *of the space* E^3_T.

Proof. We write the system of equations (17) and (19) in the form

$$z = \Phi z, \tag{28}$$

where $z = \{u, a\}$, $\Phi = \{\Phi_1(u,a), \Phi_2(u,a)\}$, and $\Phi_i(u,a)$ $(i = 1,2)$, defined by the right side of equations (17) and (19).

Consider the operator $\Phi(u,a)$, in the ball $K = K_R$ of the space E^3_T. We show that the operator Φ mapping the elements of the ball $K = K_R$ into itself. Similarly, with the aid of (26) we get that for any $z \in K_R$ the following inequalities hold

$$\|\Phi z\|_{E^3_T} \leq A(T) + B(T)\,\|u(x,t)\|_{B^3_{2,T}} +$$
$$+C(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^3_{2,T}} \leq$$
$$\leq A(T) + B(T)(A(T) + 2) + C(T)(A(T) + 2)^2 =$$
$$= A(T) + (B(T) + C(T)(A(T) + 2))(A(T) + 2) < A(T) + 2.$$

Hence, taking into account (27), the operator Φ acts in the ball $K = K_R$. We will show that the operator Φ is contracted.

Indeed, for any $z_1, z_2 \in K_R$ the following estimation is valid

$$\|\Phi z_1 - \Phi z_2\|_{E_T^3} \leq (C(T)(A(T)+2) + B(T))\times$$

$$\times(\|u_1(x,t) - u_2(x,t)\|_{B^3_{2,T}} + \|a_1(t) - a_2(t)\|_{C[0,T]}).$$

This means that by virtue of (27), the operator Φ is a contracted.

Consequently, it is clear that the operator Φ on the set $K = K_R$ satisfy the conditions of the contraction mapping principle. Therefore the operator Φ has a unique fixed point $\{z\} = \{u, a\}$, in the ball $K = K_R$, which is a solution of equation (28), i.e. in the ball $K = K_R$ is the unique solution of the systems (17) and (19).

Then the function $u(x,t)$, as an element of space $B^3_{2,T}$, is continuous and has continuous derivatives $u_x(x,t)$ and $u_{xx}(x,t)$ in D_T.

From the equation (14) it is clear that

$$\left(\sum_{k=1}^{\infty} \left(\lambda_k \left\|u_k'(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \leq$$

$$\leq \sqrt{2}\left\|\frac{1}{c(t)}\right\|_{C[0,T]} \left(\sum_{k=1}^{\infty}\left(\lambda_k^3 \|u_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} +$$

$$+\sqrt{2}\left\|\frac{1}{c(t)}\right\|_{C[0,T]} \left\|\|f_x(x,t) + a(t)u_x(x,t)\|_{C[0,T]}\right\|_{L_2(0,1)} < +\infty.$$

Hence we conclude that the function $u_t(x,t)$ is continuous in the domain D_T.

Further, it is possible to verify that equation (1) and conditions (2), (3) and (9) are satisfied in the usual sense. Consequently, $\{u(x,t), a(t)\}$ is the solution of problem (1) - (3), (9). By virtue of the corollary to Lemma 4, it is unique in the ball $K = K_R$. The theorem is thus proved. □

From Theorem 6 and Theorem 3, it follows directly the uniquely solvability of the original problem (1)-(4).

Theorem 7. *Suppose that all assumptions of Theorem 6, and the compatibility condition (8) hold. Then problem (1)-(4) has a unique classical solution in the ball* $K = K_R(\|z\|_{E_T^3} \leq R \leq A(T)+2)$ *of space* E_T^3.

4. Conclusions

The inverse boundary value problem which requires determining of the coefficient in the parabolic equation under nonlocal boundary and integral overdetermination conditions has been investigated. First, the existence and uniqueness of the solution of auxiliary equivalent problem have been established. However, the existence and uniqueness theorem for the classical solution of original problem is proved.

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