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ZUMKELLER CORDIAL LABELING OF CYCLE RELATED GRAPHS

B.J. Murali[1], K. Thirusangu[2], B.J. Balamurugan[3 §]

[1]Research and Development Centre
Bharathiar University
Coimbatore, 641 046, Tamil Nadu, INDIA

[2]Department of Mathematics
SIVET College
Gowrivakkam, Chennai, 600 073, Tamil Nadu, INDIA

[3]School of Advanced Sciences
VIT University
Chennai Campus
Vandalur-Kelambakkam Road, Chennai, 600 127, Tamil Nadu, INDIA

Abstract: Let $G = (V, E)$ be a graph with vertex set V and edge set E. A Zumkeller cordial labeling of the graph G can be defined as an injective function $f : V \to N$ such that the induced function $f^* : E \to \{0, 1\}$ defined by $f^*(xy) = f(x)f(y)$ is 1 if $f(x)f(y)$ is a Zumkeller number and 0 otherwise with the condition $|e_{f^*}(0) - e_{f^*}(1)| \leq 1$ where $e_{f^*}(0)$ and $e_{f^*}(1)$ denote respectively the number of edges of G with label 0 and the number of edges of G with label 1 under f^*. We make use of a technique of generating Zumkeller numbers and the concept of cordiality in the labeling of graphs. In this paper we show the existence of Zumkeller cordial labeling of cycle related graphs.

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Key Words: graphs, Zumkeller numbers, Zumkeller labeling, Zumkeller cordial labeling

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[§]Correspondence author

1. Introduction

In the theory of graph labeling, the labels are mathematical objects such as integers, prime numbers, modular integer or elements of a group. The mathematical properties of such objects are used through an evaluating function which assigns a value to a vertex or an edge or a face. The graph labeling has found its origin in the paper [13] by Alex Rosa in the late 1960's. A large number of papers have been devoted to the topic of labeling of graphs, which are updated by Galian [10]. For all notations and terminology in graph theory, we follow Harary [11]. Labelled graphs have wide applications in coding theory, X-ray crystallography, radar, astronomy, circuit design and communication network addressing.

Balamurugan et al. [4] introduced Zumkeller labeling using Zumkeller numbers, which is defined as an injective function $f : V \to N$ such that the induced function $f^* : E \to N$ defined by $f^*(xy) = f(x)f(y)$ is a Zumkeller number. The concept of Zumkeller labeling, strongly multiplicative Zumkeller labeling and k-Zumkeller labeling of graphs have been introduced and investigated in the literature [2, 3, 4, 5, 6, 7]. The concept of cordial labeling was introduced by Cahit [8].

In [12] we have introduced Zumkeller cordial labeling of graphs and proved the existence of Zumkeller cordial labeling for paths, cycles and star graphs.

In this paper we discuss the existence of Zumkeller cordial labeling for some cycle related graphs such as helm, wheel, flower graph, crown graph, etc.

2. Preliminaries

In this section we recall the definition of Zumkeller numbers, few properties of Zumkeller numbers [14], the concept of cordial labeling of a graph [8] and definitions of few cycle related graphs.

Definition 1. A positive integer n is said to be a Zumkeller number if all the positive factors of n can be partitioned into two disjoint parts so that the sum of the two parts are equal.

We shall call such a partition as Zumkeller partition.

For example 20 is a Zumkeller number since we can partition its factors into $A = \{1, 20\}$ and $B = \{2, 4, 5, 10\}$ with sum 21.

Few Zumkeller numbers are listed below

$$6, 12, 20, 24, 28, 30, 40, 42, 48, 54, 56, 60, ..., 80, 84, 88, ..., 150, 156, 160, ...,$$

$$220, 222, 224, ..., 270, 272, 276, ..., 1180, 1182, ...$$

Properties of Zumkeller Numbers:

1. Let the prime factorization of an even Zumkeller number n be $2^k p_1^{k_1} p_2^{k_2} \ldots p_m^{k_m}$. Then at least one of k_i must be an odd number.
2. If n is a Zumkeller number and p is a prime with $(n, p) = 1$, then np^ℓ is a Zumkeller number for any positive integer ℓ.
3. Let n be a Zumkeller number and $p_1^{k_1} p_2^{k_2} \ldots p_m^{k_m}$ be the prime factorization of n. Then for any positive integers
$$\ell_1, \ell_2, \ldots, \ell_m \text{ and } p_1^{k_1+\ell_1(k_1+1)} p_2^{k_2+\ell_2(k_2+1)} \ldots p_m^{k_m+\ell_m(k_m+1)}$$
are Zumkeller number.
4. For any prime $p \neq 2$ and a positive integer k with $p \leq 2^{k+1} - 1$, $2^k p$ is a Zumkeller number.

We use a programming language slightly adopted to C, given by Frank Buss in [9] to identify the Zumkeller numbers and Zumkeller partitions.

We recall the following cycle related graphs (see [1]).

(i) The wheel $W_n = C_n + K_1$, where C_n is a cycle of length n.

(ii) The helm H_n, is the graph obtained from the wheel W_n by attaching a pendant edge at each vertex of the n-cycle.

(iii) A flower graph is a graph obtained from a helm graph by joining each pendant vertex to the central vertex of the helm graph.

(iv) The crown $C_n \odot K_1$, is the graph obtained from a cycle C_n by attaching a pendant edge at each vertex of the cycle.

Definition 2. Let $G = (V, E)$ be a graph. A function $f : V \to \{0, 1\}$ is said to be a cordial labeling of the graph G if there exists an induced function $f^* : E \to \{0, 1\}$ such that $f^*(uv) = |f(u) - f(v)|$ satisfies the conditions that the number of zeros and number of ones on the edges differ by at most one and the number of zeros and the number of ones on vertices also differ by at most one.

A graph G which admits cordial labeling is called a cordial graph.

We adopt the following notations in this paper.

(i) $e_{f^*}(0)$ is the number of edges of the graph G having label 0 under f^*.

(ii) $e_{f^*}(1)$ is the number of edges of the graph G having label 1 under f^*.

3. Main Results

Definition 3. Let $G = (V, E)$ be a graph. An injective function $f : V \to N$ is said to be a Zumkeller cordial labeling of the graph G if there exists an induced function $f^* : E \to \{0, 1\}$ defined by $f^*(xy) = f(x)f(y)$ satisfies the following conditions

(i) For every $xy \in E$

$$f^*(xy) = \begin{cases} 1, & \text{if } f(x)f(y) \text{ is a Zumkeller number;} \\ 0, & \text{otherwise.} \end{cases}$$

(ii) $|e_{f^*}(0) - e_{f*}(1)| \leq 1$.

Definition 4. A graph $G = (V, E)$ which admits a Zumkeller cordial labeling is called a Zumkeller cordial graph.

Example 5. A Zumkeller cordial graph is shown in Figure 1.

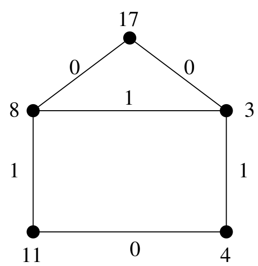

Figure 1: Zumkeller cordial graph

Theorem 6. *The helm graph H_n admits a Zumkeller cordial labeling when $n \equiv 0$ (mod 2).*

Proof. Let $V = v_0 \cup \{v_i \mid 1 \leq i \leq n\} \cup \{u_j \mid 1 \leq j \leq n\}$ be the vertex set where v_0 is the centre vertex, $\{v_i \mid 1 \leq i \leq n\}$ is the set of vertices on the cycle C_n and $\{u_j \mid 1 \leq j \leq n\}$ is the set of pendent vertices of H_n.

Let

$$E = \{e_{0i} = v_0 v_i \mid 1 \le i \le n\} \cup \{e_{ii+1} = v_i v_{i+1} \mid 1 \le i \le n-1\} \\ \cup \{e_{ij} = v_i u_j \mid 1 \le i \le n; 1 \le j \le n\}$$

be the edge set of H_n.

Define an injective function $f : V \to N$ as follows

$$f(v_0) = 5, \quad f(v_1) = 6, \quad f(v_2) = 3,$$

for $i \equiv 1 \pmod 2$ and $i \ge 3$

$$f(v_i) = 2^{\frac{i+1}{2}}, \quad f(v_{i+1}) = p_{i+1},$$

where p_{i+1} is the smallest prime such that $p_{i+1} \ge 2^{\frac{i+3}{2}} + 1$,

$f(u_1) = 12$ for $j \equiv 0 \pmod 2$

$f(u_j) = 2^{\frac{n+j}{2}}$

$f(u_{j+1}) = p_{j+1}$ where p_{j+1} is a prime such that $p_{j+1} \ge f(v_i) + 1$ for $i \equiv 1 \pmod 2$.

Define an induced function $f^* : E \to \{0, 1\}$ as

$$f^*(e_{ii+1}) = f(v_i v_{i+1}) = \begin{cases} 1, & \text{if } f(v_i)f(v_{i+1}) \text{ is a Zumkeller number;} \\ 0, & \text{otherwise.} \end{cases}$$

Now we identify the edges which are belonging to E_1 and those are belonging to E_2 where E_1 is the set of edges whose labels are Zumkeller numbers and $E_2 = E - E_1$.

Case 1: Consider the edges on the cycle C_n. From the vertex function defined above the cycle C_n has Zumkeller numbers on the alternate edges starting from the first edge.

Case 2: For the pendent edges $f^*(v_i u_j)$ of H_n, we have the following:

(i) $f^*(v_1 u_1) = f(v_1)f(u_1) = 6 \times 12 = 72$ which is not a Zumkeller number and hence the edge $v_1 u_1 \in E_2$.

(ii) $f^*(v_2 u_2) = 3 \times 2^{\frac{n+2}{2}}$ is a Zumkeller number by the property 1 of Zumkeller numbers and hence $v_2 u_2 \in E_1$

(iii) for $i \equiv 1 \pmod 2$, $j \equiv 1 \pmod 2$ and $i, j \ge 3$

$$f^*(v_i u_j) = f(v_i)f(u_j) = 2^{\frac{i+1}{2}} \times p_j$$

is not a Zumkeller number since $p_j \geq f(v_i)+1$.

Hence the edges $v_iu_j \in E_2$.

(iv) for $i \equiv 0 \pmod 2$, $j \equiv 0 \pmod 2$ and $i, j \geq 4$

$$f^*(v_iu_j) = p_i \times 2^{\frac{n+j}{2}}$$

is a Zumkeller number by the property 1 of Zumkeller numbers. Hence the edges $v_iu_j \in E_1$.

Therefore we conclude that the alternate pendent edges have Zumkeller numbers.

Case 3: For the spokes $f^*(v_0v_i)$ of H_n, we have the following:

(i) $f^*(v_0v_1) = f(v_0)f(v_1) = 5 \times 6 = 30$ is a Zumkeller number and therefore the edge $v_0v_1 \in E_1$;

(ii) $f^*(v_0v_2) = f(v_0)f(v_2) = 5 \times 3 = 15$ is not a Zumkeller number and therefore the edge $v_0v_2 \in E_2$;

(iii) for $i \equiv 1 \pmod 2$ and $i \geq 3$, $f^*(v_0v_i) = 5 \times 2^{\frac{i+1}{2}}$ is a Zumkeller number and $f^*(v_0v_{i+1}) = 5 \times p_{i+1}$ is not a Zumkeller number.

Hence the edges $v_0v_i \in E_1$ and $v_0v_{i+1} \in E_2$.

Therefore in this we can conclude that the alternate edges of the spokes have Zumkeller numbers.

Since the cycle C_n has n edges where n is even, $n/2$ edges belongs to each E_1 and E_2.

Similarly among the pendent edges and spokes $n/2$ edges belongs to each E_1 and E_2.

That is $e_{f^*}(1) = 3n/2$ and $e_{f^*}(0) = 3n/2$.

Therefore the condition $|e_{f^*}(0) - e_{f^*}(1)| \leq 1$ is satisfied. Hence the helm H_n admits Zumkeller cordial labeling, when $n \equiv 0 (\text{mod } 2)$. □

A Zumkeller cordial labeling of the helm graph H_6 is given in the Example 7.

Example 7. See Figure 2.

Corollary 8. *The wheel W_n admits a Zumkeller cordial labeling, when $n \equiv 0(\text{mod } 2)$.*

Proof. By removing the pendent edges from the helm graph H_n, the proof follows. □

Corollary 9. *The crown graph $C_n \odot K_1$ admits a Zumkeller cordial labeling, when $n \equiv 0 \ (\text{mod } 2)$.*

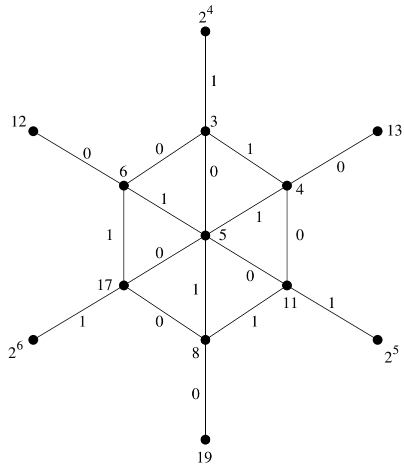

Figure 2: Zumkeller cordial helm H_6

Proof. The proof follows by removing the centre vertex from the helm graph H_n. □

The Example 10 shows the Zumkeller cordial wheel W_6 and Zumkeller cordial crown $C_6 \odot K_1$.

Example 10. See Figure 3.

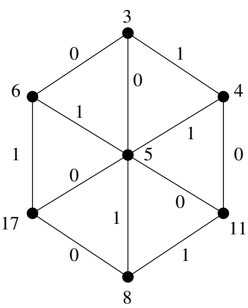

Figure 3: Zumkeller cordial wheel W_6

Theorem 11. *The helm graph H_n admits a Zumkeller cordial labeling when $n \equiv 1$ (mod 2).*

Proof. We define the vertex set and the edge set as in Theorem 1. Define an injective function $f : V \to N$ such that $f(v_0) = 10$, $f(v_1) = 3$, for $i \equiv 0 \pmod 2$ and $i \geq 2$,

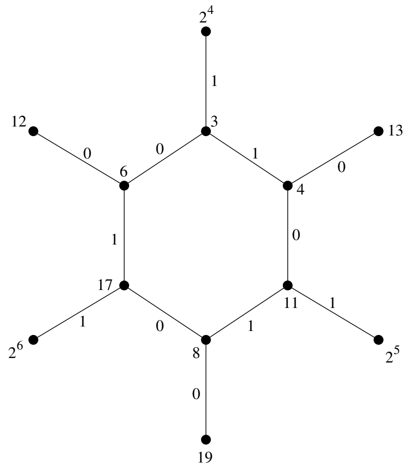

Figure 4: Zumkeller cordial crown $C_6 \odot K_1$

$f(v_i) = 2^{\frac{i+2}{2}}$,

$f(v_{i+1}) = p_{i+1}$, where p_{i+1} is the smallest prime such that $p_{i+1} \geq 2^{\frac{i+4}{2}} + 1$ for $j \equiv 1 \pmod 2$)

$f(u_j) = 2^{\frac{n+j+2}{2}}$,

$f(u_{j+1}) = p_{j+1}$, where p_{j+1} is a prime such that $p_{j+1} \geq f(v_i) + 1$ for $i \equiv 0 \pmod 2$ and $i = j + 1$.

An induced function $f^* : E \to \{0, 1\}$ is defined as in Theorem 1.

We follow the same procedure to prove the condition $|e_{f^*}(0) - e_{f^*}(1)| \leq 1$.

Hence the proof is complete. □

Example 12. A Zumkeller cordial labeling of the helm graph H_7 given in Figure 5.

Corollary 13. *The crown graph $C_n \odot K_1$ admits a Zumkeller cordial labeling when $n \equiv 1$ (mod 2).*

Corollary 14. *The wheel graph W_n admits a Zumkeller cordial labeling when $n \equiv 1$ (mod 2).*

Theorem 15. *The flower graph with cycle C_n for $n \equiv 1$ (mod 2) admits a Zumkeller cordial labeling.*

Proof. We define the vertex set and the edge set as in Theorem 1. Define an injective function $f : V \to N$ such that

$f(v_0) = 5$, $f(v_1) = 3$

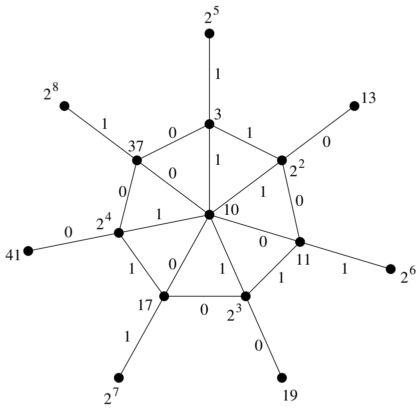

Figure 5: Zumkeller cordial labeling of helm H_7

for $i \equiv 0 \pmod 2$ and $i \geq 2$
$f(v_i) = 2^{\frac{i+2}{2}}$,
$f(v_{i+1}) = p_{i+1}$ where p_{i+1} is the smallest prime such that

$$p_{i+1} \geq 2^{\frac{i+2}{2}} + 1,$$

for $j \equiv 1 \pmod 2$,

$$f(u_j) = 2^{\frac{n+j+2}{2}},$$

$$f(u_{j+1}) = p_{j+1},$$

where p_{j+1} is a prime such that $p_{j+1} > f(v_i)+1$ for $i \equiv 0 \pmod 2$ and $i = j+1$.

An induced function $f^* : E \to \{0, 1\}$ is defined as

$$\begin{aligned} f^*(e_i e_{i+1}) &= f(v_i v_{i+1}) \\ &= \begin{cases} 1, & \text{if } f(v_i)f(v_{i+1}) \text{ is a Zumkeller number;} \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Now we identify the edges which belong to E_1 and those belong to E_2, where E_1 is the set of edges whose labels are Zumkeller numbers and $E_2 = E - E_1$.

From the vertex function defined above, the numbers of the alternate edges starting from the first edge are Zumkeller numbers among the cycle C_n, among the pendent edges $f^*(v_i u_j)$ and among the spokes $f^*(v_0 u_j)$.

In the cycle C_n the edge $f^*(v_n v_1)$ is not receiving a Zumkeller number.

Among the spokes $f^*(v_0 v_i)$ the alternate edges starting from the first edge are not Zumkeller numbers.

Since n is odd, it is clear that the cycle C_n and the spokes $f^*(v_0v_i)$ have $\frac{n-1}{2}$ edges which belong to E_1 and $\frac{n+1}{2}$ edges belong to E_2.

The spokes $f^*(v_0u_j)$ and the pendent edges $f^*(v_iu_j)$ have $\frac{n+1}{2}$ edges which belong to E_1 and $\frac{n-1}{2}$ edges belong to E_2.

That is $e_{f^*}(1) = 2n$ and $e_{f*}(0) = 2n$.

Therefore the condition $|e_{f^*}(0) - e_{f^*}(1)| \leq 1$ is satisfied.

Hence the odd flower admits Zumkeller cordial labeling. □

The Zumkeller cordial labeling of the flower graph is given in the following example.

Example 16. See Figure 6.

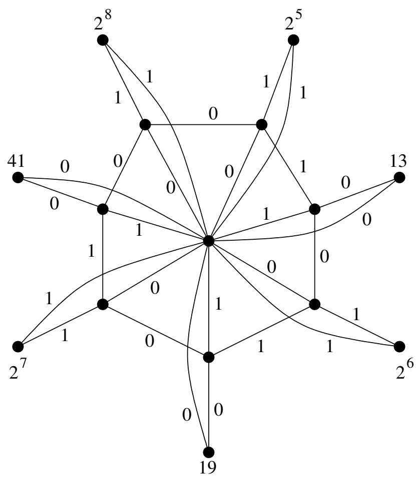

Figure 6: Zumkeller cordial labeling of flower graph

4. Conclusion

In this paper, we have proved the existence of Zumkeller cordial labeling for graphs such as helm, wheel, crown graph and flower graph.

5. Open Question

Check whether an even flower graph admits Zumkeller cordial labeling or not.

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