GENERALIZED COMMON COUPLED FIXED POINT THEOREM OF NONLINEAR CONTRACTIVE MAPPING IN CONE METRIC SPACE

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Abstract: The existence and uniqueness of common coupled fixed point theorem has been proved under different contractive condition in cone metric space. The result is verified with the help of suitable example.

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1. Introduction and Mathematical Preliminaries

Banach’s fixed point theorem for contraction mapping is one of the pivotal result of functional analysis. Many generalization of Banach’s contraction mapping have been introduced in order to obtain more general fixed point results in metric spaces and their generalizations. Some of these generalizations are mentioned below.

In 1968, Kannan [15], [16] in his result shows that contractive mapping which does not imply continuity has fixed point.

In 1997, the concept of weak contraction which is a generalization of Banach’s contraction principle was introduced by Alber and Guerre-Delabriere.

In 2007, Huang and Zhang [12] introduced the concept of cone metric space, where they generalized metric space by replacing the set of real numbers with

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an ordering Banach space. Thus the cone naturally induces a partial order in Banach space. Some of the works are noted in [2], [7], [8], [9], etc.

The concept of coupled fixed point was given by Bhaskar and Lakshmikantham in [19]. After that several authors proved various coupled and common coupled fixed point theorem in partial ordered metric space, G-metric space, b-metric space, Fuzzy metric space, cone metric space etc. Some of the works are noted in [3], [13], [17], [21], etc.

The aim of this paper is to established the existence and uniqueness of common coupled fixed point theorem satisfying some generalized contractive condition in cone metric space. In this paper we do not impose the normality condition of the cone. The only assumption is that cone $P$ has nonempty interior.

Some definitions and results.

**Definition 1.** [12] Let $E$ always be a real Banach space and $P$ a subset of $E$. $P$ is called a cone if and only if:

(i) $P$ is nonempty, closed, and $P \neq \{0\}$;

(ii) $a, b \in \mathbb{R}, \quad a, b \geq 0, \quad x, y \in P \implies ax + by \in P$;

(iii) $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, a partial ordering $\leq$ with respect to $P$ is naturally defined by $x \leq y$ if and only if $x - y \in P$ for $x, y \in E$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int} P$, where $\text{int} P$ denote the interior of $P$.

The cone $P$ is said to be normal if there exist a real number $K > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \implies \|x\| \leq k\|y\|.$$  

The least positive number $K$ satisfying the above statement is called the normal constant of $P$.

The cone $P$ is called regular if every increasing sequence which is bounded from above is convergent; that is, if $\{x_n\}$ is a sequence such that

$$x_1 \leq x_2 \leq \ldots \leq x_n \leq \ldots \leq y,$$

for some $y \in E$, then there is $x \in E$ such that $\|x_n - x\| \to 0$ as $n \to \infty$. Equivalently, the cone $P$ is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone.
In the following we always suppose that E is a real Banach space with cone P in E with intP ≠ ∅ and ≤ is partial ordering with respect to P.

**Definition 2.** [12] Let X be a nonempty set. Let the mapping \( d : X \times X \rightarrow E \) satisfies:

(i) \( 0 \leq d(x, y) \), for all \( x, y \in X \) and \( d(x, y) = 0 \) if and only if \( x = y \),

(ii) \( d(x, y) = d(y, x) \), for all \( x, y \in X \),

(iii) \( d(x, y) \leq d(x, z) + d(z, y) \), for all \( x, y, z \in X \).

Then \( d \) is called a cone metric on \( X \) and \((X, d)\) is called a cone metric space.

**Definition 3.** [2] Let \((X, d)\) be a cone metric space, \(\{x_n\}\) a sequence in \(X\) and \(x \in X\).

(i) If for every \( c \in E \) with \( 0 \ll c \), there exists \( n_0 \in \mathbb{N} \) such that for all \( n > n_0 \), \( d(x_n, x) \ll c \), then \( \{x_n\} \) is said to be convergent and \( \{x_n\} \) converges to \( x \), and \( x \) is the limit of \( \{x_n\} \). This limit is denoted by \( \lim_n \{x_n\} = x \) or \( x_n \rightarrow x \) as \( n \rightarrow \infty \).

(ii) If for every \( c \in E \) with \( 0 \ll c \), there exists \( n_0 \in \mathbb{N} \) such that for all \( n, m > n_0 \), \( d(x_n, x_m) \ll c \), then \( \{x_n\} \) is called a Cauchy sequence in \( X \).

(iii) If every Cauchy sequence in \( X \) is convergent in \( X \), then \( X \) is called a complete cone metric space.

(iv) If \( P \) is a normal cone, then \( \{x_n\} \) converges to \( x \) if and only if \( d(x_n, x) \rightarrow 0 \) as \( n \rightarrow \infty \) and \( \{x_n\} \) is a Cauchy sequence if and only if \( d(x_n, x_m) \rightarrow 0 \) as \( n, m \rightarrow \infty \).

**Lemma 4.** [2] If \( P \) is a normal cone in \( E \), then

(i) If \( 0 \leq x \leq y \) and \( a \geq 0 \), where \( a \) is real number, then \( 0 \leq ax \leq ay \).

(ii) If \( 0 \leq x_n \leq y_n \), for \( n \in \mathbb{N} \) and \( \lim_n \{x_n\} = x, \lim_n \{y_n\} = y \), then \( 0 \leq x \leq y \).

**Lemma 5.** [2] If \( E \) be a real Banach space with cone \( P \) in \( E \), then for \( a, b, c \in E \),

(i) If \( a \leq b \) and \( b \ll c \), then \( a \ll c \).

(ii) If \( a \ll b \) and \( b \ll c \), then \( a \ll c \).

**Definition 6.** [19] An element \((x, y) \in X \times X\) is called a coupled fixed point of the mapping \( F : X \times X \rightarrow X \) if \( x = f(x, y) \) and \( y = f(y, x) \).

**Definition 7.** An element \((x, y) \in X \times X\) is called a common coupled fixed point of the mapping \( F : X \times X \rightarrow X \) and \( g : X \rightarrow X \) if \( x = g(x) = F(x, y) \).
2. Main Results

**Theorem 8.** Let \((X, d)\) be a complete cone metric space with a cone \(P\) having nonempty interior. Let \(F, G : X \times X \rightarrow X\) satisfying

\[
d(F(x, y), G(u, v)) \leq l_1 \max \{d(x, u), d(x, F(x, y))\} + l_2 \max \{d(y, v), d(y, F(y, x))\} + l_3 \max \{d(u, F(x, y)), d(u, G(u, v))\}
\]

for all \(x, y, u, v \in X\), and \(l_1, l_2, l_3\) are non-negative real numbers such that \(l_1 + l_2 + l_3 < 1\).

Then \(F(x, y)\) and \(G(x, y)\) have a unique common coupled fixed point in \(X \times X\).

**Proof.** Let \(x_0\) and \(y_0\) be two arbitrary elements of \(X\).

Let

\[
x_{2k+1} = F(x_{2k}, y_{2k}), \quad y_{2k+1} = F(y_{2k}, x_{2k})
\]

and

\[
x_{2k+2} = G(x_{2k+1}, y_{2k+1}), \quad y_{2k+2} = G(y_{2k+1}, x_{2k+1})
\]

for \(k = 0, 1, 2, 3, \ldots\)

Then

\[
d(x_{2k+1}, x_{2k+2}) = d(F(x_{2k}, y_{2k}), G(x_{2k+1}, y_{2k+1}))
\]

\[
\leq l_1 \max \{d(x_{2k}, x_{2k+1}), d(x_{2k}, F(x_{2k}, y_{2k}))\} + l_2 \max \{d(y_{2k}, y_{2k+1}), d(y_{2k}, F(y_{2k}, x_{2k}))\}
\]

\[
+ l_3 \max \{d(x_{2k+1}, F(x_{2k}, y_{2k})), d(x_{2k+1}, G(x_{2k+1}, y_{2k+1}))\}
\]

\[
= l_1 \max \{d(x_{2k}, x_{2k+1}), d(x_{2k}, x_{2k+1})\} + l_2 \max \{d(y_{2k}, y_{2k+1}), d(y_{2k}, y_{2k+1})\}
\]

\[
+ l_3 \max \{d(x_{2k+1}, x_{2k+1}), d(x_{2k+1}, x_{2k+2})\}
\]

\[
= l_1 d(x_{2k}, x_{2k+1}) + l_2 d(y_{2k}, y_{2k+1}) + l_3 d(x_{2k+1}, x_{2k+2})
\]

\[\Rightarrow (1 - l_3)d(x_{2k+1}, x_{2k+2}) \leq l_1 d(x_{2k}, x_{2k+1}) + l_2 d(y_{2k}, y_{2k+1})\]

i.e

\[
d(x_{2k+1}, x_{2k+2}) \leq \frac{l_1}{1 - l_3} d(x_{2k}, x_{2k+1}) + \frac{l_2}{1 - l_3} d(y_{2k}, y_{2k+1})
\]
Similarly, we have

\[ d(y_{2k+1}, y_{2k+2}) = d(F(y_{2k}, x_{2k}), G(y_{2k+1}, x_{2k+1})) \]

\[ \leq l_1 \max[d(y_{2k}, y_{2k+1}), d(y_{2k}, F(y_{2k}, x_{2k})] \]

\[ + l_2 \max[d(x_{2k}, x_{2k+1}), d(x_{2k}, F(x_{2k}, y_{2k}))] \]

\[ + l_3 \max[d(y_{2k+1}, F(y_{2k}, x_{2k})), d(y_{2k+1}, G(y_{2k+1}, x_{2k+1}))] \]

\[ = l_1 \max[d(y_{2k}, y_{2k+1}), d(y_{2k}, y_{2k+1})] \]

\[ + l_2 \max[d(x_{2k}, x_{2k+1}), d(x_{2k}, y_{2k+1})] \]

\[ + l_3 \max[d(y_{2k+1}, y_{2k+1}), d(y_{2k+1}, y_{2k+2})] \]

\[ = l_1 d(y_{2k}, y_{2k+1}) + l_2 d(x_{2k}, x_{2k+1}) + l_3 d(y_{2k+1}, y_{2k+2}) \quad (4) \]

\[ \implies (1 - l_3)d(y_{2k+1}, y_{2k+2}) \leq l_1 d(y_{2k}, y_{2k+1}) + l_2 d(x_{2k}, x_{2k+1}) \]

i.e

\[ d(y_{2k+1}, y_{2k+2}) \leq \frac{l_1}{1 - l_3} d(y_{2k}, y_{2k+1}) + \frac{l_2}{1 - l_3} d(x_{2k}, x_{2k+1}) \quad (5) \]

Adding (3) and (5) we get

\[ d(x_{2k+1}, x_{2k+2}) + d(y_{2k+1}, y_{2k+2}) \leq \frac{l_1 + l_2}{1 - l_3} d(x_{2k}, x_{2k+1}) + \frac{l_2 + l_2}{1 - l_3} d(y_{2k}, y_{2k+1}) \]

\[ \leq \frac{l_1 + l_2}{1 - l_3} [d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})] \]

\[ = p[d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})] \quad (6) \]

where

\[ 0 < p = \frac{l_1 + l_2}{1 - l_3} < 1 \]

Hence, from (4) we have

\[ d(x_{2k+2}, x_{2k+3}) + d(y_{2k+2}, y_{2k+3}) \leq p[d(x_{2k+1}, x_{2k+2}) + d(y_{2k+1}, y_{2k+2})] \]

Therefore

\[ d(x_n, x_{n+1}) + d(y_n, y_{n+1}) \leq p[d(x_{n-1}, x_n) + d(x_{n-1}, x_n)] \]

\[ \leq p^2[d(x_{n-2}, x_{n-1}) + d(x_{n-2}, x_{n-1})] \]

\[ \vdots \]

\[ \vdotswithin{\vdots} \]

\[ \vdotswithin{\vdots} \vdotswithin{\vdots} \]
Now if we take
\[ d(x_n, x_{n+1}) + d(y_n, y_{n+1}) = \beta_n \]
then,
\[ \beta_n \leq p \beta_{n-1} \leq \ldots \leq p^n \beta_0 \]
For \( m > n \) we have

\[
\begin{align*}
&d(x_n, x_m) + d(y_n, y_m) \leq \beta_{m-1} + \beta_{m-2} + \ldots + \beta_n \\
&\leq (p^{m-1} + p^{m-2} + \ldots + p^n) \beta_0 \\
&= p^n(1 + p + \ldots + p^{n-1}) \beta_0 \\
&= \frac{p^n(1 - p^{m-n})}{1 - p} \beta_0 \\
&= \frac{p^n - p^m}{1 - p} \beta_0 \\
&= \frac{p^n}{1 - p} \beta_0
\end{align*}
\]
Consequently,
\[
\begin{align*}
&d(x_n, x_m) \leq \frac{p^n}{1 - p} \beta_0 \\
&d(y_n, y_m) \leq \frac{p^n}{1 - p} \beta_0
\end{align*}
\]
Let 0 \( \ll c \) be given. Choose a natural number \( N \) such that \( \frac{p^n}{1 - p} \beta_0 \ll c \) for \( n > N \). Thus \( d(x_n, x_m) \ll c \) and \( d(y_n, y_m) \ll c \) for \( m > n \). Therefore both the sequences \( \{x_n\} \) and \( \{y_n\} \) are Cauchy sequences.

From the compactness of \( X \), there exist \( x, y \in X \) such that
\[
\lim_{n \to \infty} x_n = x \quad \text{and} \quad \lim_{n \to \infty} y_n = y
\]
Now we prove that \( F(x, y) = x \) and \( F(y, x) = y \).
Substituting \( x_{2k+1} = F(x, y) \) and \( y_{2k+1} = F(y, x) \) in (6), we obtain
\[
d(F(x, y), x_{2k+2}) + d(F(y, x), y_{2k+2}) \leq p[d(x_{2k}, F(x, y)) + d(y_{2k}, F(y, x))]
\]
Letting $k \to \infty$ we obtain

$$d(F(x, y), x) + d(F(y, x), y) \leq p[d(x, F(x, y)) + d(y, F(x, y))]$$

$$\implies (1 - p)[d(F(x, y), x) + d(F(y, x), y)] \leq 0$$

Since $0 < p < 1$ so $1 - p > 0$ therefore

$$d(F(x, y), x) + d(F(y, x), y) \leq 0$$

Which is a contradiction.

Therefore,

$$d(F(x, y), x) + d(F(y, x), y) = 0$$

$$\implies d(F(x, y), x) = 0 \text{ and } d(F(y, x), y) = 0$$

So $F(x, y) = x$ and $F(y, x) = y$

Similarly on substituting $x_{2k+1} = G(x, y)$ and $y_{2k+1} = G(y, x)$ in (6), and letting $k \to \infty$ we obtain

$$G(x, y) = x \text{ and } G(y, x) = y$$

So we have proved that $(x, y)$ is a common coupled fixed point of $F$ and $G$.

In order to prove the uniqueness, let $(x', y') \in X \times X$ be another common coupled fixed point of $F$ and $G$. Then

$$d(x, x') = d(F(x, y), G(x', y'))$$

$$\leq l_1 \max[d(x, x'), d(x, F(x, y))] + l_2 \max[d(y, y'), d(y, F(x, y))]$$

$$+ l_3 \max[d(x', F(x, y)), d(x', G(x', y'))]$$

$$= l_1 \max[d(x, x'), d(x, x)] + l_2 \max[d(y, y'), d(y, y)]$$

$$+ l_3 \max[d(x', x), d(x', x')]$$

$$= l_1 d(x, x') + l_2 d(y, y') + l_3 d(x', x)$$

$$\implies (1 - l_1 - l_3) d(x, x') \leq l_2 d(y, y')$$

$$\implies d(x, x') \leq \frac{l_2}{1 - l_1 - l_3} d(y, y')$$

(9)

Similarly, we can prove that

$$d(y, y') \leq \frac{l_2}{1 - l_1 - l_3} d(x, x')$$

(10)
Adding (9) and (10) we get
\[ d(x, x') + d(y, y') \leq \frac{l_2}{1 - l_1 - l_3}[d(x, x') + d(y, y')] \]
\[ \Rightarrow (1 - \frac{l_2}{1 - l_1 - l_3})[d(x, x') + d(y, y')] \leq 0 \]
\[ \Rightarrow d(x, x') + d(y, y') = 0 \]
i.e. \( x = x' \) and \( y = y' \)
The proof is completed. \( \square \)

**Corollary 9.** Since \( \max[d(x, u), d(x, F(x, y))] \leq [d(x, u) + d(x, F(x, y))] \)
and \( \max[d(y, v), d(y, F(y, x))] \leq [d(y, v) + d(y, F(y, x))] \)
and \( \max[d(u, F(x, y)), d(u, G(u, v))] \leq [d(u, F(x, y)) + d(u, G(u, v))] \).
So we have from theorem 8,
Let \( (X, d) \) be a complete cone metric space with a cone \( P \) having nonempty interior. Let \( F, G : X \times X \to X \) satisfying
\[ d(F(x, y), G(u, v)) \leq l_1[d(x, u) + d(x, F(x, y))] + l_2[d(y, v) + d(y, F(y, x))] + l_3[d(u, F(x, y)) + d(u, G(u, v))] \]
(11)
for all \( x, y, u, v \in X \), and \( l_1, l_2, l_3 \) are non-negative real numbers such that \( l_1 + l_2 + l_3 < 1 \).
Then \( F \) and \( G \) have a unique common coupled fixed point in \( X \times X \).

**Corollary 10.** Let \( (X, d) \) be a complete cone metric space with a cone \( P \) having nonempty interior. Let \( F, G : X \times X \to X \) satisfying
\[ d(F(x, y), G(u, v)) \leq l_1d(x, u) + l_2d(y, v) + l_3d(u, G(u, v)) \]
(12)
for all \( x, y, u, v \in X \), and \( l_1, l_2, l_3 \) are non-negative real numbers such that \( l_1 + l_2 + l_3 < 1 \).
Then \( F \) and \( G \) have a unique common coupled fixed point in \( X \times X \).

**3. Example**

Let \( E = R \) and \( P = \{x \in E : x \geq 0\} \). Let \( X = [0, 1] \) and define a mapping \( d : X \times X \to X \) by \( d(x, y) = \frac{1}{5}|x - y| \) for all \( x, y \in X \). Then \( (X, d) \) is a complete cone metric space. Define \( F, G : X \times X \to X \) by \( F(x, y) = \frac{x+y}{2} \) and \( G(x, y) = \frac{x+y}{3} \) for all \( x, y \in X \).
Then for all \( x, y, u, v \in X \) we have,

\[
d(F(x, y), G(u, v)) = d\left(\frac{xy}{3}, \frac{uv}{3}\right) = \frac{1}{3} |xy - 2uv|
\]

\[
\max[d(x, u), d(x, F(x, y))] = \max[\frac{1}{5} |x - u|, d(x, \frac{xy}{2})] = \frac{1}{5} |x - \frac{xy}{2}|
\]

\[
\max[d(y, v), d(y, F(y, x))] = \max[\frac{1}{5} |y - v|, d(y, \frac{xy}{2})] = \frac{1}{5} |y - \frac{xy}{2}|
\]

\[
\max[d(u, F(x, y)), d(u, G(u, v))] = \max[d(u, \frac{xy}{3}), d(u, \frac{uv}{3})] = \max[\frac{1}{5} |u - \frac{xy}{2}|, \frac{1}{5} |u - \frac{uv}{3}|]
\]

Now the following cases arise

**Case - i.**
Let \( x = 0 \) or \( y = 0 \) and \( u = 0 \) or \( v = 0 \). Then

\[
d(F(x, y), G(u, v)) = 0
\]

Hence for all non-negative real numbers \( l_1, l_2, l_3 \) with \( l_1 + l_2 + l_3 < 1 \). We have,

\[
d(F(x, y), F(u, v)) \leq l_1 \max[d(x, u), d(x, F(x, y))] + l_2 \max[d(y, v), d(y, F(y, x))] + l_3 \max[d(u, F(x, y)), d(u, G(u, v))]
\]

**Case - ii.**
Let non of \( x, y, u, v \) is 0. Let \( x = u = y = v \) then

\[
d(F(x, y), G(u, v)) = \frac{x^2}{30}
\]

\[
\max[d(x, u), d(x, F(x, y))] = \frac{1}{5} |x - \frac{x^2}{2}|
\]

\[
\max[d(y, v), d(y, F(y, x))] = \frac{1}{5} |y - \frac{x^2}{2}|
\]

\[
\max[d(u, F(x, y)), d(u, G(u, v))] = \frac{1}{5} |x - \frac{x^2}{2}|
\]

Let \( l_1 = \frac{1}{5}, l_2 = l_3 = \frac{2}{5} \) then \( l_1 + l_2 + l_3 = \frac{5}{5} < 1 \) then

\[
l_1 \max[d(x, u), d(x, F(x, y))] + l_2 \max[d(y, v), d(y, F(y, x))] + l_3 \max[d(u, F(x, y)), d(u, G(u, v))] = \frac{1}{5} |x - \frac{x^2}{2}|
\]

As \( x \in (0, 1) \) so \( \frac{x^2}{30} < \frac{1}{5} |x - \frac{x^2}{2}| \). Therefore,

\[
d(F(x, y), F(u, v)) \leq l_1 \max[d(x, u), d(x, F(x, y))] + l_2 \max[d(y, v), d(y, F(y, x))] + l_3 \max[d(u, F(x, y)), d(u, G(u, v))]
\]

**Case - iii.**
Let non of \( x, y, u, v \) is 0. Let \( x = \frac{1}{2}, y = \frac{1}{3}, u = \frac{1}{4}, v = \frac{1}{5} \)

\[
d(F(x, y), G(u, v)) = \frac{1}{75}
\]

\[
\max[d(x, u), d(x, F(x, y))] = \frac{1}{20}
\]

\[
\max[d(y, v), d(y, F(y, x))] = \frac{7}{75}
\]
\[
\max[d(u, F(x, y)), d(u, G(u, v))] = \frac{1}{50}
\]
Let \(l_1 = l_2 = l_3 = \frac{2}{11}\) then \(l_1 + l_2 + l_3 = \frac{6}{11} < 1\) then
\[
\begin{align*}
l_1\max[d(x, u), d(x, F(x, y))] + l_2\max[d(y, v), d(y, F(y, x))] \\
+ l_3\max[d(u, F(x, y)), d(u, G(u, v))] = \frac{1}{50}
\end{align*}
\]
Therefore,
\[
d(F(x, y), F(u, v)) \leq l_1\max[d(x, u), d(x, F(x, y))] + l_2\max[d(y, v), d(y, F(y, x))] \\
+ l_3\max[d(u, F(x, y)), d(u, G(u, v))]
\]
Hence all the condition of theorem 2.1 are satisfied.
Here it is seen that \((0, 0)\) is the unique common coupled fixed point of \(F\) and \(G\).

**References**


