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MIXED QUADRATURE RULE FOR DOUBLE INTEGRALS

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Abstract: A mixed quadrature rule of precision five for double integrals which is a linear combination of Simpson's $\frac{3}{8}th$ rule and Gauss-Legendre-2 point rule, where each constituent rule is of degree of precision three in two variables is formulated. The rule is numerically tested taking some suitable texts and the error bound is determined.

AMS Subject Classification: 65D30, 65D31
Key Words: mixed quadrature rule, Maclaurin's theorem, Error bound, Degree of precision

1. Introduction

A mixed quadrature rule of higher degree of precision has been formed by different researchers [2], [3], [4]. These rules were meant for single integral. In the same vein here we have developed a mixed quadrature rule of degree of precision-5 for double integrals taking the convex combination of Simpson's $\frac{3}{8}th$ and Gauss-Legendre-2 point rule each of degree of precision 3. This paper has been designed as follows. Section 2 contains formulation of quadrature of con- stituent rules and the corresponding errors in two variables. Section 3 has developed for construction of mixed quadrature rules. The error analysis has been done in Section 4. In Section 5 the rule is nu merically verified by taking suitable examples. The conclusions are drawn in Section 6.

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2. Construction of Quadrature Rule

For approximate evaluation of real definite integral

$$I(f) = \int_{-1}^{1} f(x,y)dxdy \tag{1}$$

The Simpson's $\frac{3}{8}th$ rule is

$$\begin{aligned} I(f) \simeq & R_{S\frac{3}{8}}(f) = \frac{1}{16}[\{f(-1,-1) + f(-1,1) + f(1,-1) + f(1,1)\} \\ & + 3\{f(-1,-\frac{1}{3}) + f(-1,\frac{1}{3}) + f(1,-\frac{1}{3}) + f(1,\frac{1}{3}) \\ & + f(-\frac{1}{3},-1) + f(-\frac{1}{3},1) + f(\frac{1}{3},-1) + f(\frac{1}{3},1)\} \\ & + 3\{f(-1,-\frac{1}{3}) + f(-1,\frac{1}{3}) + f(1,-\frac{1}{3}) + f(1,\frac{1}{3})\} \\ & + 9\{f(-\frac{1}{3},-\frac{1}{3}) + f(-\frac{1}{3},\frac{1}{3}) + f(\frac{1}{3},-\frac{1}{3}) + f(\frac{1}{3},\frac{1}{3})\}. \end{aligned} \tag{2}$$

The Gauss-Legendre's two point rule is

$$\begin{aligned} I(f) \simeq R_{GL2}(f) = f(-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}) + f(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}) + f(\frac{1}{\sqrt{3}}, \\ -\frac{1}{\sqrt{3}}) + f(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}). \end{aligned} \tag{3}$$

Here each rule, i.e. equation (2)and (3) has a precision 3. Hence

$$I(f) = R_{s\frac{3}{8}}(f) + E_{s\frac{3}{8}}(f), \tag{4}$$

$$I(f) = R_{s\frac{3}{8}}(f) + E_{s\frac{3}{8}}(f), \tag{5}$$

where $E_{S\frac{3}{8}}(f)$ and $E_{GL2}(f)$ are error in approximating the integrals $I(f)$ by equation (2) and equation (3) respectively.

Assuming $f(x,y)$ to be sufficiently differentiable in $-1 \leq x \leq 1$, $-1 \leq y \leq 1$, we can rewrite equation (1) using Maclaurin's expansion

$$\begin{aligned} I(f) = \int_{-1}^{1}\int_{-1}^{1} & [f_{0,0}(0,0) + \{xf_{1,0}(0,0) + yf_{0,1}(0,0)\} \\ & + \frac{1}{2!}\{x^2 f_{2,0}(0,0) + 2xyf_{1,1}(0,0) + y^2 f_{0,2}(0,0)\} \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{3!}\{x^3 f_{3,0}(0,0) + 3x^2 y f_{2,1}(0,0) \\
&+3xy^2 f_{1,2}(0,0) + y^3 f_{0,3}(0,0)\} \\
&+\frac{1}{4!}\{x^4 f_{4,0}(0,0) + 4x^3 y f_{3,1}(0,0) + 6x^2y^2 f_{2,2}(0,0) \\
&+4xy^3 f_{1,3}(0,0) + y^4 f_{0,4}(0,0)\} \\
&+\frac{1}{5!}\{x^5 f_{5,0}(0,0) + 5x^4 y f_{4,1}(0,0) \\
&\quad + 5x^4 y f_{4,1}(0,0) + 10x^3y^2 f_{3,2}(0,0) \\
&\quad + 10x^2y^3 f_{2,3}(0,0) + 5xy^4 f_{1,4}(0,0) + y^5 f_{0,5}(0,0)\} \\
&\quad + \frac{1}{6!}\{x^6 f_{6,0}(0,0) + 6x^5 y f_{5,1}(0,0) \\
&\quad + 15x^4y^2 f_{4,2}(0,0) + 20x^3y^3 f_{3,3}(0,0) \\
&\quad + 15x^2y^4 f_{2,4}(0,0) + 6xy^5 f_{1,5}(0,0) + y^6 f_{0,6}(0,0)\}]dxdy. \qquad (6)
\end{aligned}
$$

Integrating equation (6) we receive

$$
\begin{aligned}
I(f) =& 4f_{0,0}(0,0) + \frac{2}{3}\{f_{2,0}(0,0) \\
&+ f_{0,2}(0,0)\} + \frac{1}{30}\{f_{4,0}(0,0) + f_{0,4}(0,0)\} \\
&+ \frac{1}{9} f_{2,2}(0,0) + \frac{1}{180}\{f_{4,2}(0,0) + f_{2,4}(0,0)\} \\
&+ \frac{4}{7!}\{f_{6,0}(0,0) + f_{0,6}(0,0)\}. \qquad (7)
\end{aligned}
$$

Again using Maclaurin's expansion:

$$
\begin{aligned}
f(-1,-1) =& f_{0,0}(0,0) - \{f_{1,0}(0,0) + f_{0,1}(0,0)\} \\
&+ \frac{1}{2!}\{f_{2,0}(0,0) + 2f_{1,1}(0,0) + f_{0,2}(0,0)\} \\
&- \frac{1}{3!}\{f_{3,0}(0,0) \\
&- 3f_{2,1}(0,0) - 3f_{1,2}(0,0) - f_{0,3}(0,0)\} \\
&+ \frac{1}{4!}\{f_{4,0}(0,0) + 4f_{3,1}(0,0) \\
&+ 6f_{2,2}(0,0) + 4f_{1,3}(0,0) + f_{0,4}(0,0)\} \\
&- \frac{1}{5!}\{f_{5,0}(0,0) + 5f_{4,1}(0,0) + 10f_{3,2}(0,0) \\
&+ 10f_{2,3}(0,0) + 5f_{1,4}(0,0) + f_{0,5}(0,0)\}
\end{aligned}
$$

$$+\frac{1}{6!}\{f_{6,0}(0,0)+6f_{5,1}(0,0)+15f_{4,2}(0,0)$$
$$+20f_{3,3}(0,0)+15f_{2,4}(0,0)$$
$$+6f_{1,5}(0,0)+f_{0,6}(0,0)\}+... \tag{8}$$

Using equation (8) and $f(0,-1), f(1,-1), f(-1,0), f(1,0), f(0,1), f(-1,1)$, $f(1,1)$, $f(-1,-\frac{1}{3}), f(-1,\frac{1}{3}), f(1,-\frac{1}{3}), f(1,\frac{1}{3}), f(-\frac{1}{3},-1), f(-\frac{1}{3},1,), f(\frac{1}{3},-1,)$, $f(\frac{1}{3},1)$, $f(-\frac{1}{3},-\frac{1}{3}), f(-\frac{1}{3},\frac{1}{3}), f(\frac{1}{3},-\frac{1}{3})$, and $f(\frac{1}{3},\frac{1}{3})$ could be evaluated and using them in equation (2)

$$\begin{aligned}R_{S\frac{3}{8}} =&4f_{0,0}(0,0)+\frac{2}{3}\{f_{2,0}(0,0)+f_{0,2}(0,0)\}\\&+\frac{1}{18}\{f_{4,0}(0,0)+f_{0,4}(0,0)\}+\frac{1}{9}\{f_{2,2}(0,0)\}\\&+\frac{7}{27\times 36}\{f_{4,2}(0,0)+f_{2,4}(0,0)\}\\&+\frac{3904}{243\times 6!}\{f_{6,0}(0,0)+f_{0,6}(0,0)\}.\end{aligned} \tag{9}$$

Error in Simpson's $\frac{3}{8}th$ rule: Substituting equation (6), (9) in equation (4) error associated with Simpson's $\frac{3}{8}th$ rule $E_{S\frac{3}{8}}(f)=I(f)-R_{S\frac{3}{8}}(f)$

$$\begin{aligned}E_{S\frac{3}{8}}(f)=&-\frac{4}{405}\{f_{4,0}(0,0)+f_{0,4}(0,0)\}-\frac{2}{1215}\{f_{4,2}(0,0)+f_{2,4}(0,0)\}\\&-\frac{26356}{243\times 7!}\{f_{6,0}(0,0)+f_{0,6}(0,0)\}.\end{aligned} \tag{10}$$

Therefore, we obtain the error.

As the error contains at least fourth derivative of the integral function, it vanishes for all polynomials of degree ≤ 3 in x and y. Thus the formula becomes exact for all polynomials of degree up to 3 i.e. degree of precision of the formula is 3.

2.1. Gauss Legendre-2 Point Rule

$$\begin{aligned}f(-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}) =&f(0,0)-\frac{1}{\sqrt{3}}\{f_{1,0}(0,0)+f_{0,1}(0,0)\}\\&+\frac{1}{2!}\{(-\frac{1}{\sqrt{3}})^2 f_{2,0}(0,0)\end{aligned}$$

$$
\begin{aligned}
&+ 2(-\frac{1}{\sqrt{3}})(-\frac{1}{\sqrt{3}})f_{1,1}(0,0)\\
&+ (-\frac{1}{\sqrt{3}})^2 f_{0,2}(0,0)\}\\
&- \frac{1}{3!}\{(-\frac{1}{\sqrt{3}})^3 f_{3,0}(0,0) + 3(-\frac{1}{\sqrt{3}})^3 f_{2,1}(0,0)\\
&+ 3(-\frac{1}{\sqrt{3}})^3 f_{1,2}(0,0) + (-\frac{1}{\sqrt{3}})^3 f_{0,3}(0,0)\}\\
&+ \frac{1}{4!}\{(-\frac{1}{\sqrt{3}})^4 f_{4,0}(0,0) + 4(-\frac{1}{\sqrt{3}})^4 f_{3,1}(0,0)\\
&+ 6(-\frac{1}{\sqrt{3}})^4 f_{2,2}(0,0) + 4(-\frac{1}{\sqrt{3}})^4 f_{1,3}(0,0)\\
&+ (-\frac{1}{\sqrt{3}})^4 f_{0,4}(0,0)\}\\
&- \frac{1}{5!}\{(-\frac{1}{\sqrt{3}})^5 f_{5,0}(0,0) + 5(-\frac{1}{\sqrt{3}})^5 f_{4,1}(0,0)\\
&+ 10(-\frac{1}{\sqrt{3}})^5 f_{3,2}(0,0) + 10(-\frac{1}{\sqrt{3}})^5 f_{2,3}(0,0)\\
&+ 5(-\frac{1}{\sqrt{3}})^5 f_{1,4}(0,0) + (-\frac{1}{\sqrt{3}})^5 f_{0,5}(0,0)\}\\
&+ \frac{1}{6!}\{(-\frac{1}{\sqrt{3}})^6 f_{6,0}(0,0) + 6(-\frac{1}{\sqrt{3}})^6 f_{5,1}(0,0)\\
&+ 15(-\frac{1}{\sqrt{3}})^6 f_{4,2}(0,0) + 20(-\frac{1}{\sqrt{3}})^6 f_{3,3}(0,0)\\
&+ 15(-\frac{1}{\sqrt{3}})^6 f_{2,4}(0,0) + 6(-\frac{1}{\sqrt{3}})^6 f_{1,5}(0,0)\\
&+ (-\frac{1}{\sqrt{3}})^6 f_{0,6}(0,0)\} + \ldots \qquad (11)
\end{aligned}
$$

It is possible to calculate $f(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}), f(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}), f(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}})$ using the same arguments as in equation (10).

Substituting the value of $f(-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}), f(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}), f(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}), f(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}})$ in equation (3), we recceive

$$
\begin{aligned}
R_{GL2}(f) =&4f(0,0) + \frac{2}{3}\{f_{2,0}(0,0) + f_{0,2}(0,0)\}\\
&+ \frac{1}{9\times 3!}\{f_{4,0}(0,0) + f_{0,4}(0,0)\}
\end{aligned}
$$

$$+\frac{20}{9\times 6!}\{f_{4,2}(0,0)+f_{2,4}(0,0)\}$$
$$+\frac{1}{9}\{f_{2,2}(0,0)\}$$
$$+\frac{4}{27\times 6!}\{f_{6,0}(0,0)+f_{0,6}(0,0)\}+... \tag{12}$$

2.2. Error in Gauss-Legendre's Two Point Rule

The error associated with the Gauss-Legendre 2-point rule is obtained substituting equation(6)and equation (11) in equation (5), i.e.

$$\begin{aligned}E_{GL2}(f) =& I(f)-R_{GL2}(f)\\ &=\frac{2}{135}\{f_{4,0}(0,0)+f_{0,4}(0,0)\}+\frac{1}{405}\{f_{4,2}(0,0)+f_{2,4}(0,0)\}\\ &+\frac{1}{1701}\{f_{6,0}(0,0)+f_{0,6}(0,0)\}.\end{aligned} \tag{13}$$

In this case also, the error contains at least fourth derivative of the integral function. Thus the degree of the precision is 3.

3. Mixed Quadrature Rule

Referring [5], [6], and [7] Now multiplying equation (9) by 2 and equation (12) by 3 and then adding them, we get

$$I(f)=\frac{1}{5}[2R_{S\frac{3}{8}}(f)+3R_{GL2}(f)]+\frac{1}{5}[2E_{S\frac{3}{8}}(f)+3E_{GL2}(f)] \tag{6}$$

Where

$$R_{mix}(f)=R_{S\frac{3}{8}GL2}(f)+E_{S\frac{3}{8}GL2}(f) \tag{7}$$

Where

$$R_{S\frac{3}{8}GL2}=\frac{1}{5}\{2R_{S\frac{3}{8}}(f)+3R_{GL2}(f)\} \tag{8}$$

Where $R_{S\frac{3}{8}GL2}(f)$ and $E_{S\frac{3}{8}GL2}(f)$
are mixed quadrature rule and its error obtained by Simpson's $\frac{3}{8}th$
and Gauss-Legendre's 2-point rule respectively.
The truncation error generated by this approximation is given by

$$\begin{aligned}R_{S\frac{3}{8}GL2} =& \frac{1}{5}\{2E_{S\frac{3}{8}}(f)+3R_{GL2}(f)\}\\ &=-\frac{7}{97200}\{f_{6,0}(0,0)+f_{0,6}(0,0)\}\end{aligned} \tag{9}$$

The rule equation 17 may be called as mixed quadrature rule integrate exactly all polynomial of degree ≤ 5 in x and y.

4. Error Analysis

Theorem 1. *Let $f(x, y)$ be sufficiently differentiable function in the closed interval $[-1, 1]$. The bounds of truncatinal error $E_{S\frac{3}{8}GL2}(f)$ associated with the rule $R_{S\frac{3}{8}GL2}(f)$ is given by*

$$|E_{S\frac{3}{8}GL2}| = \frac{7}{972}|f_{6,0}(0,0) + f_{0,6}(0,0)|...$$

The proof obviously follows from the equation (17).

Theorem 2. *The bounds for the truncational error is*

$$|E_{S\frac{3}{8}GL2}(f)| \leq \frac{4M}{675}|\eta_2 - \eta_1|,$$

where $\eta_1, \eta_2[-1, 1]$ and

$$M = Max|f_{4,0}(0,0) + f_{0,4}(0,0)|, -1 \leq x \leq 1, -1 \leq y \leq 1.$$

Proof. We have

$$\begin{aligned} E_{S\frac{3}{8}}(f) &= -\frac{4}{405}|f_{4,0}(n_1) + f_{0,4}(n_1)|, \\ E_{GL2}(f) &= \frac{2}{135}|f_{4,0}(n_2) + f_{0,4}(n_2)|, \end{aligned}$$

where $\eta_1, \eta_2[-1, 1]$. Hence

$$\begin{aligned} E_{S\frac{3}{8}GL2} &= \frac{1}{5}|2E_{S\frac{3}{8}}(f) + 3E_{GL2}(f)| \\ &= \frac{4}{675}\{f_{4,0}(\eta_2, 0) + f_{0,4}(0, \eta_2) - f_{4,0}(\eta_1, 0) - f_{0,4}(0, \eta_1)\} \\ &= \frac{4}{675}\{\int_{\eta_1}^{\eta_2} f_{5,0}(x, 0)dx + \int_{\eta_1}^{\eta_2} f_{0,5}(0, y)dy\} \\ &= \frac{4}{675}\int_{\eta_1}^{\eta_2}\int_{\eta_1}^{\eta_2}[f_{5,0}(x, *) + f_{0,5}(*, y)]. \end{aligned}$$

Hence

$$|E_{S\frac{3}{8}GL2}(f)| \leq \frac{4M|\eta_1 - \eta_2|}{675},$$

$R_{S\frac{3}{8}}(f)$	$R_{GL2}(f)$	$R_{S\frac{3}{8}GL2}(f)$	$E_{S\frac{3}{8}GL2}(f)$
5.614586191129401	5.488065843621398	5.538673982624600	0.014282600457338
2.589849108367626	2.024691358024692	2.250754458161866	0.019769316757732
0.611821774881802	0.609491314229328	0.610423498490318	0.002836871491600

Table 1: Numerical Verification

where

$$M = max|f_{4,0}(x,*) + f_{0,4}(*,y)|, \quad -1 \le x \le 1, -1 \le y \le 1.$$

Which gives only truncation error bound on η_1, η_2 are known points in $[-1,1]$.

Corollary 3. (see [1]) *The error bound for the truncational error is* $|E_{S\frac{3}{8}GL2}(f)| \le \frac{8M}{675}$, *where* $|\eta_1 - \eta_2| \le 2$.

5. Numerical Verification

The approximate value of integrals:

$$I_1 = \int_{-1}^{1}\int_{-1}^{1} e^{x+y}dxdy = 5.524391382167262,$$
$$I_2 = \int_{-1}^{1}\int_{-1}^{1} e^{-(x^2+y^2)}dxdy = 2.230985141404134,$$
$$I_3 = \int_{0}^{1}\int_{0}^{1} \frac{sin^2(x+y)}{(x+y)}dxdy = 0.613260369981918.$$

are presented in Table 1.

6. Conclusion

From Table 1, we find that

$$|E_{S\frac{3}{8}}(f)| \le |E_{GL2}(f)|, \tag{10}$$

and

$$|E_{S\frac{3}{8}}GL2(f)| \le |E_{RS\frac{3}{8}}(f)|. \tag{11}$$

From equation (18) and equation (19):

$$|E_{S\frac{3}{8}}(f)| \le |E_{GL2}(f)| \le |E_{RS\frac{3}{8}}(f)|.$$

It is evident that the mixed quadrature rule $R_{S_{\frac{3}{8}}GL2}(f)$ of degree of precision 5 provided us better result than constituent rule $R_{S_{\frac{3}{8}}}(f)$ and $R_{GL2}(f)$.

Each of degree of precision three. Hence, the mixed quadrature rule is more efficient and numerically better convergent than exact result.

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