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INTEGRAL OPERATOR IN AMALGAM

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Abstract: Boundedness of the sum of two Hardy-type operators with not necessarily non-negative coefficients has been discussed between amalgams $\ell^q(X_u) - \ell^b(L^r, v)$ for the case $1 < r, q, b < \infty$ where X_u is weighted Banach function space.

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1. Introduction

Consider the operator

$$(Sf)(x) = \phi_1(x) \int_{-\infty}^{x} \psi_1(t) f(t) dt + \phi_2(x) \int_{x}^{\infty} \psi_2(t) f(t) dt$$

where $\phi_i, \psi_i; i = 1, 2$ are non-zero measurable functions not necessarily non-negative and f is positive measurable function. Boundedness of S between weighted Lebesgue spaces in one dimension has been considered in [8 (Remark 2.4), 15], between weighted Lebesgue spaces in two dimension in [9], between suitable weighted amalgam spaces $\ell^q(L^p, u) - \ell^{\bar{q}}(L^{\bar{p}}, v)$ in [7] where u, v are weight function, between Banach function space (BFS) in [10]. Boundedness conditions for the sum of two Hardy-type operators gives better value of best constant for Hardy inequality than the same for single Hardy-type operator.

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Motivated by this, in this paper, we have considered the boundedness of S between $\ell^q(X_u)$ and $\ell^b(L^r, v)$ for certain ranges of indices , where X_u is weighted BFS. Boundedness of the Hardy operator $(Tf)(x) = \int_{-\infty}^{x} f(t)dt$ and it's adjoint operator $(T^*f)(x) = \int_x^\infty f(t)dt$ between $\ell^q(X_u)$ and $\ell^b(L^r, v)$ has been considered in [6], while $\ell^q(L^p, u) - \ell^{\bar{q}}(L^{\bar{p}}, v)$ boundedness of T and T^* has been proved in [3]. The notion of BFS was introduced by Luxemburg [12]. For detailed studies of BFS we used to refer [1]. Amalgams were introduced by Wiener [13, 14]. A good treatment of such spaces is available in [4, 5]. We use to refer [8] for the study of Hardy inequality.

Throughout the paper, the symbol u and v denotes weight function, that is, measurable functions, positive almost everywhere in the interval $(-\infty, \infty)$, u_n and v_n denotes sequence weight, χ_n denotes characteristic function over $[n, n+1]$ and $\chi_{[\alpha,\beta]}$ denotes characteristic function over $[\alpha, \beta], p' = p/(p-1)$ denotes conjugate to p and the same is true for other indices. For a BFS X, X' denotes it's adjoint space. $\mathbb{Z}$ represents the set of integers. We conclude this section by giving some results which will be further used as follows:

$\ell^p(v_n) - \ell^q(u_n)$ boundedness of the operator $H^*(a_n) = \sum_{k=n}^{\infty} a_k$, is stated below:

Proposition 1.1. Let $1 < p, q < \infty$, $u_n \geq 0$, $v_n > 0$, $n \in \mathbb{Z}$ and $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$. Then there exists a constant $C > 0$ such that

$$\left(\sum_{n\in\mathbb{Z}} \left(\sum_{k=n}^{\infty} a_k\right)^q u_n\right)^{1/q} \leq C \left(\sum_{n\in\mathbb{Z}} a_n^p v_n\right)^{1/p}$$

holds for all non-negative sequence $\{a_k\} \in \ell^q(v_n)$, if and only if

(a) for $1 < p \leq q < \infty$

$$\underset{m\in\mathbb{Z}}{sup} \left(\sum_{n=-\infty}^{m} u_n\right)^{1/q} \left(\sum_{n=m}^{\infty} v_n^{1-p'}\right)^{1/p'} < \infty,$$

(b) for $1 < q < b < \infty$

$$\left\{\sum_{m\in\mathbb{Z}} \left(\sum_{n=-\infty}^{m} u_k\right)^{r/q} \left(\sum_{n=m}^{\infty} v_n^{1-p'}\right)^{r/q'} u_m\right\}^{1/r} < \infty.$$

Boundedness of the operator $(H_1^*f)(x) = \psi(x) \int_x^\infty \phi(t)f(t)dt$, between BFS X and Y where ϕ, ψ are weight function is stated below:

Proposition 1.2. Let X, Y are BFS satisfying the ℓ-condition [2]. Then the inequality $||H_1^* f||_Y \leq C||f||_X$ holds for measurable function f and a constant C if and only if $\sup_{0<t<\infty} ||\psi\chi_{[0,t]}||_Y ||\phi\chi_{[t,\infty]}||_{X'} < \infty$.

The following result states the adjoint case of [6, Theorem 2]:

Proposition 1.3. Let $1 < b, q, r < \infty$, u, v are weight functions and X_u is a weighted BFS. Suppose X_u and the Lebesgue space L^r satisfy ℓ-condition. Then the inequality

$$||T^* f||_{\ell^q(X_u)} \leq C||f||_{r,v,b}$$

holds for all $f \in \ell^b(L^r, u)$ if and only if for $b \leq q$

$$A_1 = \sup_{m\in\mathbb{Z}} \sup_{m<t<m+1} ||u\chi_{[m,t]}||_X \left(\int_t^{m+1} v^{1-b'}\right)^{1/b'} < \infty$$

$$A_2 = \sup_{m\in\mathbb{Z}} \left(\sum_{n=-\infty}^{m} ||u\chi_n||_X^q\right)^{1/q} \left(\sum_{n=m}^{\infty} \left(\int_{n-1}^{n} v^{1-r'}\right)^{\frac{b'}{r'}}\right)^{1/b'} < \infty.$$

2. Boundedness of S between Amalgam

In this section we will consider $\ell^q(X_u) - \ell^b(L^r, v)$ boundedness of S for the case of $1 < r, q, b < \infty$. First consider the following notations:

$$C_1 = \sup_{m\in\mathbb{Z}} \sup_{m<t<m+1} ||u\phi_1\chi_{[t,m+1]}||_X \left(\int_m^t (v|\psi_1|^{-r})^{1-r'}\right)^{1/r'}$$

$$C_2 = \sup_{m\in\mathbb{Z}} \sup_{m<t<m+1} ||u\phi_2\chi_{[m,t]}||_X \left(\int_t^{m+1} (v|\psi_2|^{-r})^{1-r'}\right)^{1/r'}$$

$$C_3 = \sup_{m\in\mathbb{Z}} \left(\sum_{n=m}^{\infty} ||u\phi_1\chi_n||_X^q\right)^{1/q} \left(\sum_{n=-\infty}^{m} \left(\int_{n-1}^{n} (v|\psi_1|^{-r})^{1-r'}\right)^{\frac{b'}{r'}}\right)^{1/b'}$$

$$C_4 = \sup_{m\in\mathbb{Z}} \left(\sum_{n=-\infty}^{m} ||u\phi_2\chi_n||_X^q\right)^{1/q} \left(\sum_{n=m}^{\infty} \left(\int_{n-1}^{n} (v|\psi_2|^{-r})^{1-r'}\right)^{\frac{b'}{r'}}\right)^{1/b'}$$

where $\frac{1}{s} = \frac{1}{q} - \frac{1}{b}$.

Theorem 2.1. Let $1 < r, q, b < \infty$, u, v are weight functions, X_u is weighted BFS. Suppose X_u and L^r satisfy ℓ-condition. Then there exist a constant $C > 0$ for which the inequality

$$||Sf||_{\ell^q(X_u)} \leq C||f||_{r,v,b} \tag{1}$$

holds for all $f \in \ell^b(L^r, v)$ if and only if for $b \leq q$, $\max(C_1, C_2, C_3, C_4) < \infty$.

Proof. Denote

$$(S_1 f)(x) = \phi_1(x) \int_{-\infty}^{x} \psi_1(t) f(t) dt; \quad (S_2 f)(x) = \phi_2(x) \int_{x}^{\infty} \psi_2(t) f(t) dt.$$

Then $S = S_1 + S_2$, consequently,

$$||Sf||_{\ell^q(X_u)} \leq ||S_1 f||_{\ell^q(X_u)} + ||S_2 f||_{\ell^q(X_u)}. \tag{2}$$

Sufficiency. By an application of [6, Theorem 2(a)], the following can be proved.

Lemma A. Suppose $1 < r, q, b < \infty$, u, v are weight functions, X_u is weighted BFS. Suppose X_u and L^r satisfy ℓ-condition. Then there exist a constant $C > 0$ for which the inequality

$$||S_1 f||_{\ell^q(X_u)} \leq C\, ||f||_{r,v,b} \tag{3}$$

holds for all $f \in \ell^b(L^r, v)$ if and only if for $b \leq q$, $\max(C_1, C_3) < \infty$.

In [6], Lemma A has been proved for $\phi_1 \equiv \psi_1 \equiv 1$ and $f \geq 0$. The proof of Lemma A is analogous to that proof. We omit the detail. When X_u is weighted Lebesgue space, $\phi_1 \equiv \psi_1 \equiv 1$ and $f \geq 0$, Lemma A has been proved in [3].

The following can be proved from Proposition 1.3, by applying suitable substitutions:

Lemma B. Suppose $1 < r, q, b < \infty$, u, v are weight functions, X_u is weighted BFS. Suppose X_u and L^r satisfy ℓ-condition. Then there exist a constant $C > 0$ for which the inequality

$$||S_2 f||_{\ell^q(X_u)} \leq C||f||_{r,v,b} \tag{4}$$

holds for all $f \in \ell^b(L^r, v)$ if and only if for $b \leq q$, $\max(C_2, C_4) < \infty$.

Sufficiency now follows from Lemma A, Lemma B and the inequality (2.2).

Necessity. for non-negative functions, ϕ_i, ψ_i. Suppose that the inequality (2.1) holds and that $f \geq 0$. Then

$$||S_i f||_{\ell^q(X_u)} \leq ||Sf||_{\ell^q(X_u)}; \quad i = 1, 2$$

and consequently (2.3) and (2.4) holds. This implies that $\max(C_1, C_3) < \infty$ (for $b \le q, i = 1$), $\max(C_2, C_4) < \infty$ (for $b \le q, i = 2$) . Necessity is now proved.

Necessity for general functions ϕ_i, ψ_i. Suppose again that the inequality (2.1) holds. Define, for $\varepsilon > 0$ and a measurable function $g > 0$, a new weight function v_ε as

$$v_\varepsilon(x) = \max\left\{v(x),\ \frac{|\psi_1(x)|^r}{(g(x))^r}\varepsilon\right\} \tag{5}$$

Since $v \le v_\varepsilon$, we get $||f||_{r,v,b} \le ||f||_{r,v_\varepsilon,b}$, consequently the inequality (2.1) is equivalent to

$$||Sf||_{\ell^q(X_u)} \le C||f||_{r,v_\varepsilon,b}. \tag{6}$$

Define

$$f(x) = g(x)|\psi_1(x)|^{-1}\mathrm{sgn}(\psi_1(x))\chi_{[\alpha,\beta]}(x) \tag{7}$$

for $m \in \mathbb{Z}$ be fixed and $m < \alpha < \beta < m+1$. For v_ε and f defined above

$$\begin{aligned} ||f||_{r,v_\varepsilon,b} &= \left(\int_\alpha^\beta g^r|\psi_1|^{-r}v_\varepsilon\right)^{1/r} \le \left(\int_m^{m+1} g^r|\psi_1|^{-r}v_\varepsilon\right)^{1/r} \\ &\le \left(\int_m^{m+1} g^r|\psi_1|^{-r}|\psi_1|^r g^{-r}\varepsilon\right)^{1/r} = (\varepsilon)^{1/r} < \infty \end{aligned}$$

and

$$\begin{aligned} ||Sf||_{\ell^q(X_u)} &= \left(\sum_{n\in\mathbb{Z}} ||Sf.u\chi_n||_X^q\right)^{1/q} = ||\,|Sf|u.\chi_m||_X \ge ||\,|Sf|u\chi_{[\beta,m+1]}||_X \\ &= ||\,|\phi_1 \int_{-\infty}^x \psi_1 f|\,u\chi_{[\beta,m+1]}||_X \ge ||\,|\int_m^x g\,|(u|\phi_1|)\ \chi_{[\beta,m+1]}||_X \end{aligned}$$

since $\int_x^\infty \psi_2 f = 0$ when $x > \beta$ of f defined as (2.7). Consequently, the inequality (2.6) yields

$$||\,|\int_m^x g\,|.(u|\phi_1|)\ \chi_{[\beta,m+1]}||_X \left(\int_m^{m+1} g^r|\psi_1|^{-r}v_\varepsilon\right)^{-1/r} \le C.$$

Since C is independent of β and ε, we have when $\beta \to m$ and $\varepsilon \to 0$ (via a subsequence)

$$||\,|\int_m^x g\,|(u|\phi_1|)\ \chi_{[m,m+1]}||_X \le C\left(\int_m^{m+1} g^r|\psi_1|^{-r}v\right)^{1/r}.$$

By an application of [11, Theorem 4] (also see [2]), this holds implies $C_1 < \infty$. Next, we replace v_ε and f defined as in (2.5) and (2.7) by, respectively,

$$v_\varepsilon(x) = \max\left\{v(x),\ \frac{|\psi_2(x)|^r}{(g(x))^r}\varepsilon\right\}$$

and

$$f(x) = g(x)|\psi_2(x)|^{-1}\mathrm{sgn}(\psi_2(x))\chi_{[\alpha,\beta]}(x)$$

where $m \in \mathbb{Z}$ is fixed and $m < \alpha < \beta < m+1$. The necessity of $C_2 < \infty$ can be achieved by making parallel arguments in the inequality (2.6) for this new v_ε and f and using Proposition 1.2. For $\varepsilon > 0$, we define

$$\tilde{v}_\varepsilon(x) = \max\left\{v(x),\ |\psi_1(x)|^r\varepsilon\right\}. \tag{8}$$

For the similar reason discussed as earlier, the inequality (2.1) is equivalent to

$$||Sf||_{\ell^q(X_u)} \le C||f||_{r,\tilde{v}_\varepsilon,b}. \tag{9}$$

For a non-negative sequence $\{a_n\} \in \ell^b$ and fixed $\alpha \in \mathbb{Z}$, define

$$\tilde{f}(x) = \begin{cases} \sum\limits_{n\in\mathbb{Z}} a_n\left(\frac{\tilde{v}_\varepsilon(x)}{|\psi_1(x)|}\right)^{1-r'}\mathrm{sgn}(\psi_1(x))\chi_{[n,n+1]}(x), & x \le \alpha \\ 0, & x > \alpha. \end{cases} \tag{10}$$

For f defined as above

$$||\tilde{f}||_{r,\tilde{v}_\varepsilon,b} = \left\{\sum_{n=-\infty}^{\alpha} A_n^b V_n\right\}^{1/b} \le \left\{\sum_{n\in\mathbb{Z}} A_n^b V_n\right\}^{1/b},$$

where

$$A_n = a_{n-1}\int_{n-1}^{n}(\tilde{v}_\varepsilon|\psi_1|^{-r})^{1-r'} \text{ and } V_n = \left(\int_{n-1}^{n}(\tilde{v}_\varepsilon|\psi_1|^{-r})^{1-r'}\right)^{-b/r'}.$$

Since f defined as (2.10) vanishes for $x > \alpha$, we have for $\alpha < x < \alpha + 1$

$$(S\tilde{f})(x) = \phi_1(x)\left\{\sum_{n=-\infty}^{\alpha}\int_{n-1}^{n}\psi_1(t)\tilde{f}(t)dt\right\} = \phi_1(x)\sum_{n=-\infty}^{\alpha}A_n$$

which implies

$$||S\tilde{f}||_{X_u} = \left(\sum_{n=-\infty}^{\alpha}A_n\right)||u\phi_1\chi_\alpha||_X.$$

Consequently,

$$||S\tilde{f}||_{\ell^q(X_u)} = \left\{ \sum_{\alpha\in\mathbb{Z}} \left(\sum_{n=-\infty}^{\alpha} A_n \right)^q ||u\phi_1\chi_\alpha||_X^q \right\}^{1/q},$$

the inequality (2.9) yields

$$\left\{ \sum_{\alpha\in\mathbb{Z}} \left(\sum_{n=-\infty}^{\alpha} A_n \right)^q ||u\phi_1\chi_\alpha||_X^q \right\}^{1/q} \leq C \left(\sum_{n\in\mathbb{Z}} A_n^b V_n \right)^{1/b}, \tag{11}$$

where the RHS of the above inequality is dominated by

$$C(\varepsilon^{1-r'})^{1/r} \left(\sum_{n\in\mathbb{Z}} a_{n-1}^b \right)^{1/b} < \infty$$

since $\{a_n\} \in \ell^b$. The inequality (2.11) holds implies by [3, Corollary 1.3],

$$\tilde{C}_3 = \sup_{m\in\mathbb{Z}} \left(\sum_{n=m}^{\infty} ||u\phi_1\chi_n||_X^q \right)^{1/q} \left\{ \sum_{n=-\infty}^{m} V_n^{1-b'} \right\}^{1/b'} < \infty \text{ holds for } b \leq q .$$

In inequality (2.11), C is independent of ε. Since $C \sim \tilde{C}_3$ for $b \leq q$, we have, by taking $\varepsilon \to 0$ (via a subsequence) $\tilde{C}_3 = C_3$ and therefore the necessity of $C_3 < \infty$ for $b \leq q$ follows. Next, we replace $\tilde{v}_\varepsilon$ and $\tilde{f}$ defined as in (2.8) and (2.10) by, respectively,

$$\tilde{v}_\varepsilon(x) = \max\{v(x),\ |\psi_2(x)|^r\varepsilon\} \text{ and}$$

$$\tilde{f}(x) = \begin{cases} \sum_{n\in\mathbb{Z}} a_n \left(\frac{v_\varepsilon(x)}{|\psi_2(x)|} \right)^{1-r} \operatorname{sgn}(\psi_2(x))\chi_{[n,n+1]}(x), & x \geq \alpha \\ 0, & x < \alpha \end{cases}$$

where $\alpha \in \mathbb{Z}$ is fixed. The necessity of $C_4 < \infty$, now, can be achieved by making parallel argument in the inequality (2.9) for this new $\tilde{v}_\varepsilon$ and $\tilde{f}$ and using Proposition 1.1. □

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