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NUMERICAL SOLUTION OF FREDHOLM INTEGRAL EQUATIONS OF THE FIRST KIND USING LEGENDRE WAVELET COLLOCATION METHOD

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Abstract: We present an algorithm based on regularization method and wavelet collocation method to find approximate solutions of Fredholm integral equations of the first kind. First kind Fredholm integral equations are often ill-posed problems. Fredholm integral equation of the first kind is converted into approximate well posed Fredholm integral equation of the second kind by applying regularization method and then solution to the resulting equation is obtained by using collocation method with Legendre wavelets.

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Key Words: first kind Fredholm integral equation, regularization method, Legendre wavelet, collocation method

1. Introduction

Integral equations appear in many different scientific applications. Many inverse problems can be easily converted into first kind Fredholm integral equations. These problems arise in signal processing, goephysics, medical imaging, spectroscopy, backward heat equation and physical problems such as angular variation of scattered light and measurement of spectral distribution [2] and [12].

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The linear Fredholm integral equation of the first kind are given as

$$g(x) = \int_a^b K(x,t)f(t)dt, \qquad x \in [a,b], \tag{1.1}$$

where $K(x,t)$ is the kernel of integral equation, $g(x)$ is the observed data. We want to determine the function $f(x)$ for given $K(x,t)$ and $g(x)$. In terms of operator equation, eq.(1.1) can be written as

$$Tf = g,$$

where operator $T : X \to Y$ defined by

$$(Tf)(x) = \int_a^b K(x,t)f(t)dt, \quad x \in [a,b], \tag{1.2}$$

where X and Y are normed linear spaces. In this paper, we have taken $X = Y = L^2[0,1]$.

In first kind Fredholm integral equation, the unknown function $f(x)$ appears only under integral sign. However, if the function $f(x)$ appears both inside and outside the integral sign then the equation is called Fredholm integral equation of the second kind which is defined as

$$g(x) = f(x) + \int_a^b K(x,t)f(t)dt \qquad x \in [a,b]. \tag{1.3}$$

Fredholm integral equation of the second kind is a well posed problem whereas Fredholm integral equation of the first kind is an ill-posed problem that is it may have no solution, or if a solution exists, it is not unique and it may not depend continuously on the data function $g(x)$[11]. As it is ill posed problem, we need regularization method to solve this problem.

Regularization method is used to convert an ill-posed problem into a well posed problem. There are different techniques used for regularizing first kind Fredholm integral equations. In this paper, we have used regularization method which converts first kind integral equation into second kind integral equation. The regularization method combined with different techniques like homotopy perturbation method [3], direct method, successive approximation and Adomian decomposition method [4] and mean value method [1] have been employed to solve first kind Fredholm integral equations. Wavelets are used in different numerical problems as they provide fast algorithms. Many authors proposed algorithms based on the wavelet basis to solve integral equations. Legendre wavelets have been successfully applied in solving linear integral equations of the second kind [7]. K. Maleknejad et al.[9] applied wavelet Galerkin with wavelet preconditioner to solve first kind Fredholm integral equations. K. Maleknejad et al.[10] used wavelet Galerkin Discretization with Legendre basis and applied CG method to solve Fredholm integral equations of the first kind. Haar wavelet collocation method has been used for solving Fredholm integral equations of the first kind by K. Maleknejad et al.[8]. He used wavelet basis and collocation method to discretize integral equation and then applied Conjugate gradient (CG) method to solve the resulting equation. In this paper, we applied regularization method combined with collocation method using Legendre wavelet for solving Fredholm integral equations of the first kind.

The paper is organized as follows. Section 2, provides a brief description to Legendre wavelet. Section 3 presents the numerical method used to solve the eq.(1.1). In Section 4, the proposed method is used to approximate solution of some numerical problems and conclusion is drawn in Section 5.

2. Legendre Wavelet

Wavelets can be summarized as a family of functions constructed from translation and dilation of a single function called mother wavelet. For dilation parameter a and translation parameter b, the family of continuous wavelets are defined as

$$\psi_{a,b}(t) = \psi\left(\frac{t-b}{a}\right), \qquad a, b \in R,\ a \neq 0.$$

The family of discrete wavelets can be obtained by restricting the values of a and b to discrete values. Putting $a = 2^{-j},\ b = 2^{-j}k;\ j,\ k$ are arbitrary integers, we get a family of discrete wavelets $\psi_{j,k}(t)$ defined as

$$\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k).$$

The wavelets $\psi_{j,k}(t)$ form an orthonormal basis in the Hilbert space $L^2(R)$ and $\psi(t)$ is called mother wavelet [5] and [6].

We have used Legendre wavelets for obtaining numerical solution of first kind Fredholm integral equations. First, we will give an introduction to Legendre wavelets. Legendre wavelets $\psi_{n,m}(t) = \psi(k, \hat{n}, m, t)$ have four arguments : $\hat{n} = 2n - 1,\ n = 1, 2, 3, ..., 2^{k-1},$ where k can assume any positive integer, $m = 0, 1, 2, ..., M - 1$; M is a positive integer and represents the order of Legendre polynomials and t represents normalized time. Legendre wavelets $\psi_{n,m}(t)$

on interval [0,1] are defined as

$$\psi_{n,m}(t) = \begin{cases} \sqrt{(m+\frac{1}{2})}2^{k/2}P_m(2^k t - \hat{n}), & \text{for } \frac{\hat{n}-1}{2^k} \le t \le \frac{\hat{n}+1}{2^k}, \\ 0, & \text{otherwise} \end{cases} \tag{2.1}$$

where $P_m(t)$ are Legendre polynomials of order m satisfying following recursive formula

$$P_0(t) = 1,$$

$$P_1(t) = t,$$

$$P_{m+1}(t) = \left(\frac{2m+1}{m+1}\right) tP_m(t) - \left(\frac{m}{m+1}\right) P_{m-1}(t), \quad \text{for} \quad m = 1, 2, 3....,$$

The Legendre wavelets $\psi_{n,m}(t)$ form an orthonormal basis in $L^2[0,1]$[7] and [10]. So, any function $f(t)$ defined over [0,1] can be written as sum of these Legendre wavelets as

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(t), \tag{2.2}$$

where $\psi_{n,m}(t)$ are Legendre wavelets and $c_{n,m}$ are Legendre wavelet coefficients given by

$$c_{n,m} = \langle f(t), \psi_{n,m}(t)\rangle = \int_0^1 f(t)\psi_{n,m}(t)dt. \tag{2.3}$$

After truncation, eq.(2.2) can be written as

$$f(t) \simeq \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(t), \tag{2.4}$$

In matrix notation, we can write

$$f(t) = C\psi^T(t),$$

where C and $\psi(t)$ are $1 \times 2^{k-1}M$ matrices and are given by

$$C = [c_{1,0}, c_{1,1}, ..., c_{1,M-1}, c_{2,0}, ..., c_{2,M-1}, ..., c_{2^{k-1},0}......c_{2^{k-1},M-1}], \tag{2.5}$$

$$\begin{aligned} \psi(t) \quad = \quad & [\psi_{1,0}(t), \psi_{1,1}(t), ..., \psi_{1,M-1}(t), \psi_{2,0}(t), ..., \psi_{2,M-1}(t), ... \\ & ..., \psi_{2^{k-1},0}(t), ..., \psi_{2^{k-1},M-1}(t)]. \end{aligned} \tag{2.6}$$

The error bound for truncation of Legendre wavelet series is given by following theorem.

Theorem 2.1. *Let $f(x) \in C^m[0,1]$ and $C\Psi^T$ is the approximate solution of $f(x)$ using Legendre wavelet. Then the error bound obtained is as follows :*

$$\|f - C\Psi^T\| \leq \frac{1}{m!2^{mk}} sup_{x\in[0,1]}|f^m(x)|.$$

Proof. See [13]. □

3. Numerical Method

In this section, we present our algorithm which consists of three steps. In first step, we convert ill-posed first kind Fredholm integral equations into well posed second kind Fredholm integral equations using regularization method. The regularization method is very reliable in handling Fredholm integral equations of first kind [1], [2], [3] and [4].

Consider Fredholm integral equation of the first kind of form

$$g(x) = \int_0^1 K(x,t)f(t)dt, \quad x \in [0,1]. \tag{3.1}$$

Using regularization method, eq.(3.1) transforms into the second kind Fredholm intgeral equation

$$\alpha f_\alpha(x) = g(x) - \int_0^1 K(x,t)f_\alpha(t)dt, \tag{3.2}$$

where α is a small positive parameter known as regularization parameter. This is a well posed integral equation. Now we will solve eq.(3.2) instead of eq.(3.1). Also from [2], $f_\alpha(x)$ converges to $f(x)$ as $\alpha \to 0$, i.e.,

$$f(x) = \lim_{\alpha\to 0} f_\alpha(x). \tag{3.3}$$

In the second step, Legendre wavelet collocation method is employed to handle the well posed problem given in eq.(3.2). This method will transform integral equation into a system of algebraic equations. For this purpose, first we expand $f_\alpha(x)$ as sum of series of Legendre wavelets, i.e., we can write

$$f_\alpha(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(x). \tag{3.4}$$

Putting these values in eq.(3.2), we get

$$\begin{aligned}\alpha\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{n,m}\psi_{n,m}(x) &= g(x)-\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{n,m}\int_0^1 K(x,t)\psi_{n,m}(t)dt\\ \alpha\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{n,m}\psi_{n,m}(x) &= g(x)-\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{n,m}L_{n,m}(x), \qquad (3.5)\end{aligned}$$

where $L_{n,m}(x)=\int_0^1 K(x,t)\psi_{n,m}(t)dt$.
Collocation points are defined as

$$x_i=\frac{2i-1}{2p},\quad i=1,2,.....,p,$$

where $p=2^{k-1}M$.

Satisfying eq.(3.5) at all collocation points x_i, we get the following linear system of equations,

$$\alpha\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{n,m}\psi_{n,m}(x_i)=g(x_i)-\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{n,m}L_{n,m}(x_i),\quad i=1,2,....,p.$$

Writing this system in matrix form, we have

$$\begin{aligned}\alpha C\Psi &= G-CL\\ C(\alpha\Psi+L) &= G\\ C &= G(\alpha\Psi+L)^{-1}. \qquad (3.6)\end{aligned}$$

The value of C is same as given in eq.(2.5). The function G, Ψ and L are given by

$$G=[g(x_1),g(x_2),.....,g(x_p)],$$

$$\Psi=\begin{bmatrix}\psi_{1,0}(x_1) & \psi_{1,0}(x_2) & \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \psi_{1,0}(x_p)\\ \psi_{1,1}(x_1) & \psi_{1,1}(x_2) & \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \psi_{1,1}(x_p)\\ \cdot & \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\\ \psi_{1,M-1}(x_1) & \psi_{1,M-1}(x_2) & \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \psi_{1,M-1}(x_p)\\ \psi_{2,0}(x_1) & \psi_{2,0}(x_2) & \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \psi_{2,0}(x_p)\\ \cdot & \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\\ \cdot & \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\\ \psi_{2^{k-1},M-1}(x_1) & \psi_{2^{k-1},M-1}(x_2) & \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \psi_{2^{k-1},M-1}(x_p)\end{bmatrix},$$

$$L = \begin{bmatrix} L_{1,0}(x_1) & L_{1,0}(x_2) & \cdots\cdots & L_{1,0}(x_p) \\ L_{1,1}(x_1) & L_{1,1}(x_2) & \cdots\cdots & L_{1,1}(x_p) \\ . & . & \cdots\cdots & . \\ L_{1,M-1}(x_1) & L_{1,M-1}(x_2) & \cdots\cdots & L_{1,M-1}(x_p) \\ L_{2,0}(x_1) & L_{2,0}(x_2) & \cdots\cdots & L_{2,0}(x_p) \\ . & . & \cdots\cdots & . \\ . & . & \cdots\cdots & . \\ L_{2^{k-1},M-1}(x_1) & L_{2^{k-1},M-1}(x_2) & \cdots\cdots & L_{2^{k-1},M-1}(x_p) \end{bmatrix}.$$

Using the value of coefficients $C = \{c_{n,m}\}$ as given in eq.(3.6), we now find value of $f_\alpha(x)$ using eq.(3.4).

In third step, we find the value of $f(x)$ from $f_\alpha(x)$ using eq.(3.3). We start with $\alpha = 0.1$ and decrease value of α until the numerical solution converges to the exact solution of integral equation. To check accuracy of the proposed method, we use root mean square error (rms) which is given by

$$\text{rms} = \frac{1}{N}\sum_{i=1}^{N} |f(x_i) - f_{ex}(x_i)|,$$

where $f(x_i)$ and $f_{ex}(x_i)$ are numerical and exact solution and N is the number of points.

4. Numerical Examples

Some numerical examples are presented to demonstrate efficiency of the proposed method. All examples are solved using $k = 2,\ M = 3$, i.e., $p = 6$. The results are also compared with their exact values and shown through figures also. All computations are done with help of Matlab.

Example 4.1. Consider the following first kind Fredholm integral equation (see [8]):

$$-\frac{e^{\sin x}(0.54\sin x + 0.84) - \sin x}{\cos x^2 - 2} = \int_0^1 e^{t\sin x} f(t)dt.$$

The exact solution is $f(x) = \cos x$.

For $\alpha = 0.0001$, the numerical solution converges to exact solution and rms value obtained is

$$\text{rms} = 3.95338299e - 03.$$

The exact and approximated solutions are compared in table 1 and Figure 1.

Table 1: Exact and numerical solution for Example 4.1

t	Exact solution $f_{ex}(t)$	Approximate solution $f(t)$	Error $\lvert f_{ex}(t) - f(t)\rvert$
0	1.00000000	0.99308299	1.48847206e-03
0.1	0.99500416	0.99056829	2.02011478e-03
0.2	0.98006657	0.97967458	2.53185302e-03
0.3	0.95533648	0.96040188	1.95994098e-04
0.4	0.92106099	0.93275017	4.74036713e-03
0.5	0.87758256	0.87285458	6.58689337e-03
0.6	0.82533561	0.81712397	2.92061636e-03
0.7	0.76484218	0.75697473	6.56532489e-03
0.8	0.69670670	0.69240688	4.951662547e-03
0.9	0.62160996	0.62342041	1.239583497e-03

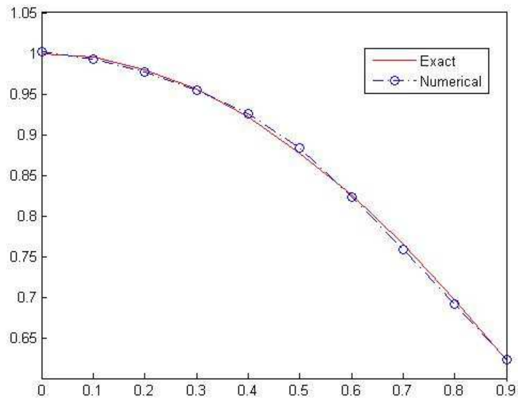

Figure 1: Comparison of exact and numerical solution for Example 4.1

Example 4.2. Consider the following first kind Fredholm integral equation (see [8]):

$$1 + \cos(x) - \cos(x + 1) = \int_0^1 e^t(\sin(x - t + 1) + 1)f(t)dt.$$

The exact solution is $f(t) = e^{-x}$.

Here the minimum value of rms is obtained for regularization parameter

$\alpha = 10^{-8}$ and its value is

$$\text{rms} = 6.07245313\text{e} - 03.$$

The exact and approximated solutions are compared in table 2 and Figure 2.

Table 2: Exact and numerical solution for Example 4.2

t	Exact solution $f_{ex}(t)$	Approximate solution $f(t)$	Error $\lvert f_{ex}(t) - f(t) \rvert$
0	1.00000000	0.98489923	1.51007686e-02
0.1	0.90483741	0.90312649	1.71091928e-03
0.2	0.81873075	0.82400119	5.27043699e-03
0.3	0.74081822	74752330	6.70508465e-03
0.4	0.67032004	0.67369284	3.37279851e-03
0.5	0.60653065	0.60039031	6.14034096e-03
0.6	0.54881163	0.54534975	3.46187840e-03
0.7	0.49658530	0.49484266	1.74263893e-03
0.8	0.44932896	0.44886904	4.5992385e-04
0.9	0.40656965	0.40742883	8.59224167e-04

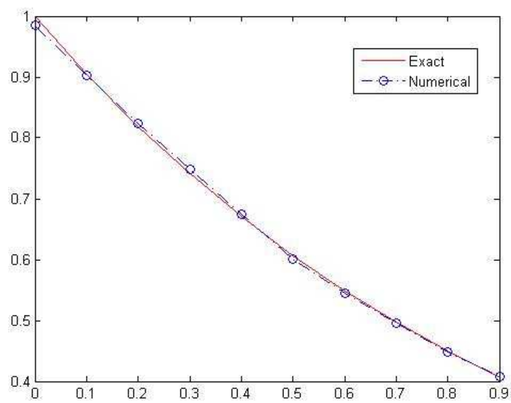

Figure 2: Comparison of exact and numerical solution for Example 4.2

5. Conclusion

In this paper, approximate solution to linear Fredholm integral equations of the first kind is derived by combining regularization method and Collocation method with Legendre wavelets. The reliability of the method is shown by applying the proposed method on two numerical examples. We have used six collocation points. A small number of collocation points are providing good results. Numerical results obtained show that the method is reliable and provides better results than the method used in [8] which uses Haar wavelet basis at scale $J = 4,\ 6$.

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