International Journal of Pure and Applied Mathematics
Volume 117 No. 1 2017, 45-57
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v117i1.5

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NECESSARY AND SUFFICIENT CONDITIONS FOR SOLUTION OF THE FOURTH ORDER CAUCHY DIFFERENCE EQUATION ON SYMMETRIC GROUPS

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Abstract: Let f: G $\to$ H be a function, where (G,.) is a group and (H,+) is an abelian group. In this paper, the following Fourth Order Cauchy difference of f : $C^{(4)}f(x_1,x_2,x_3,x_4,x_5)$ $= f(C_5(\prod_{i=1}^{5} x_i)) - f(C_4(\prod_{i=1}^{5} x_i)) + f(C_3(\prod_{i=1}^{5} x_i)) - f(C_2(\prod_{i=1}^{5} x_i)) + f(C_1(\prod_{i=1}^{5} x_i)) \forall x_1,x_2,x_3,x_4,x_5 \in G$ is studied. Where $f(C_r(\prod_{i=1}^{n} x_i))$ is defined as function of combination r at a time from n objects. Then sufficient and necessary conditions on symmetric groups are obtained.

AMS Subject Classification: 39xx
Key Words: Cauchy difference equation, symmetric groups

1. Introduction

It is well known from [1] that Jenson's functional equation

$$f(x+y) + f(x-y) = 2f(x) \tag{1.1}$$

with additional condition $f(0) = 0$, is equivalent to Cauchy's equation

Received: 2017-05-03
Revised: 2017-06-13
Published: November 26, 2017

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$$f(x+y) = f(x) + f(y)$$

on the real line. Let (G, .) be a group, (H, +) be an abelian group. Let e $\in$ G and 0 $\in$ H denote identity elements. The study of (1.1) was extended groups for f maps G into H in [2], where the general solution for a free group H with two generators and $G = GL_n(z), n \geq 3$ (see[3]). Since the functional equations involve Cauchy difference, which made it become much more interesting [4–7]. For a function f: G→H, its cauchy difference $C^{(m)}f$, is defined by

$$C^{(0)}f = f, \tag{1.2}$$
$$C^{(1)}f(x_1, x_2) = f(x_1x_2) - f(x_1) - f(x_2) \tag{1.3}$$

$$C^{(m+1)}f(x_1, x_2, \ldots, x_{m+2}) = C^{(m)}f(x_1, x_2, x_3 \ldots, x_{m+2}) - C^{(m)}f(x_1, x_3, \ldots, x_{m+2}) - C^{(m)}f(x_2, x_3, \ldots, x_{m+2}). \tag{1.4}$$

The first order cauchy difference $C^{(1)}f$ will be abbreviated as Cf. In [9],by using the reduction formulas and relations, as given in [2,3], the general solution of third order Cauchy difference equation was provided on symmetric groups.

In this paper, we consider the following functional equation:

$$f(C_5(\prod_{i=1}^{5} x_i)) - f(C_4(\prod_{i=1}^{5} x_i)) + f(C_3(\prod_{i=1}^{5} x_i)) - f(C_2(\prod_{i=1}^{5} x_i)) + f(C_1(\prod_{i=1}^{5} x_i)) = 0 \qquad \forall x_1, x_2, x_3, x_4, x_5 \in G. \tag{1.5}$$

It follows from (1.4) that (1.5) is equivalent to the vanishing fourth order cauchy difference equation

$$C^{(4)}f = 0$$

The purpose of this paper is to determine the solutions of equation (1.5). The solution of equation (1.5) will be denoted by

$$KerC^{(4)}(G, H) = \{f : G \to H | f \text{ satisfies } (1.5)\} \tag{1.6}$$

Remark 1. 1. $KerC^{(4)}(G, H)$ is an abelian group under the pointwise addition of functions;

2. $Hom(G, H) \subseteq KerC^{(4)}(G, H)$

2. Properties of Solutions

Lemma 1. *Suppose that $f \in KerC^{(4)}(G,H)$. Then*

$$\begin{aligned} f(e) &= 0, & (2.1)\\ Cf(x,y) &= 0, \quad \text{when } x=e \text{ or } y=e & (2.2)\\ C^{(2)}f(x,y,z) &= 0, \quad \text{when } x=e \text{ or } y=e \text{ or } z=e & (2.3)\\ C^{(3)}f(x,y,z,u) &= 0, \text{when } x=e \text{ or } y=e \text{ or } z=e \text{ or } u=e & (2.4)\\ C^{(3)}f & \quad \text{is a homomorphism w.r.t. each variable} & (2.5)\\ f(x^n) &= nf(x)+nC_2 \quad Cf(x,x)+nC_3 \quad C^{(2)}f(x,x,x) & \\ & \quad +nC_4 \quad C^{(3)}f(x,x,x,x) & (2.6) \end{aligned}$$

for all x,y,z,u∈ G and n∈ Z.

Proof. Putting $x_1 = e$ in (1.5) we get (2.1).

$$\begin{aligned} f(x_2x_3x_4x_5) &- f(x_2x_3x_4) - f(x_2x_3x_5) - f(x_2x_4x_5) - f(x_3x_4x_5)\\ &- f(x_2x_3x_4x_5) + f(x_2x_3) + f(x_2x_4) + f(x_2x_5) + f(x_3x_4)\\ &+ f(x_3x_5) + f(x_4x_5) + f(x_2x_3x_4) + f(x_2x_3x_5) + f(x_2x_4x_5)\\ &+ f(x_3x_4x_5) - f(x_2) - f(x_3) - f(x_4) - f(x_5) - f(x_2x_3)\\ &- f(x_2x_4) - f(x_2x_5) - f(x_3x_4) - f(x_3x_5) - f(x_4x_5)\\ &+ f(e) + f(x_2) + f(x_3) + f(x_4) + f(x_5) = 0, \end{aligned}$$

therefore f(e)=0.

Then from (2.1) we obtain (2.2)-(2.4)

$$\begin{aligned} Cf(x,e) &= f(xe) - f(x) - f(e)\\ &= f(x) - f(x)\\ &= 0. \end{aligned}$$

Similarly we can obtain

$$\begin{aligned} Cf(e,y) &= 0,\\ C^{(2)}f(e,y,z) &= f(eyz) - f(ey) - f(ez) - f(yz) + f(e)\\ &\qquad + f(y) + f(z)\\ &= 0. \end{aligned}$$

Similarly we can obtain

$$C^{(2)}f(x,e,z) = 0,$$

$$\begin{aligned} C^{(2)}f(x,y,e) &= 0,\\ C^{(3)}f(e,y,z,u) &= 0,\\ C^{(3)}f(x,e,z,u) &= 0,\\ C^{(3)}f(x,y,e,u) &= 0,\\ C^{(3)}f(x,y,z,e) &= 0. \end{aligned}$$

Furthermore, by the definition of $C^{(3)}f$, we have

$$\begin{aligned} C^{(3)}f(x,yw,z,u) \;=\; & f(xywzu) - f(xywz) - f(xywu) - f(xzu)\\ & -f(ywzu) + f(xyw) + f(xz) + f(xu)\\ & +f(ywz) + f(ywu) + f(zu) - f(x)\\ & -f(yw) - f(z) - f(u) \end{aligned}$$

and

$$\begin{aligned} & C^{(3)}f(x,y,z,u) + C^{(3)}f(x,w,z,u)\\ &\quad = f(xyzw) - f(xyz) - f(xzu) - f(xyu) - f(yzu) + f(xy)\\ &\quad +f(xz) + f(xu) + f(yz) + f(yu) + f(zu) - f(x) - f(y)\\ &\quad -f(z) - f(u) + f(xwzu) - f(xwz) - f(xwu) - f(xzu)\\ &\quad -f(wzu) + f(xw) + f(xz) + f(xu) + f(wz) + f(wu)\\ &\quad +f(zu) - f(x) - f(w) - f(z) - f(u) \end{aligned}$$

One can easily check that

$$\begin{aligned} & C^{(3)}f(x,yw,z,u) - C^{(3)}f(x,y,z,u) - C^{(3)}f(x,w,z,u)\\ &\quad = C^{(4)}f(x,y,w,z,u) = 0\\ & C^{(3)}f(x,yw,z,u) - C^{(3)}f(x,y,z,u) - C^{(3)}f(x,w,z,u)\\ &\quad = f(xywzu) - f(xywz) - f(xywu) - f(xzu)\\ &\qquad - f(ywzu) + f(xyw) + f(xz) + f(xu) + f(ywz)\\ &\qquad + f(ywu) + f(zu) - f(x) - f(yw) - f(z) - f(u)\\ &\qquad - f(xyzw) + f(xyz) + f(xzu) + f(xyu) + f(yzu)\\ &\qquad - f(xy) - f(xz) - f(xu) - f(yz) - f(yu) - f(zu)\\ &\qquad + f(x) + f(y) + f(z) + f(u) - f(xwzu) + f(xwz)\\ &\qquad + f(xwu) + f(xzu) + f(wzu) - f(xw) - f(xz) - f(xu)\\ &\qquad - f(wz) - f(wu) - f(zu) + f(x) + f(w) + f(z) + f(u) \end{aligned}$$

$$\begin{aligned}
&= f(xywzu) - f(xywz) - f(xywu) - f(ywzu) - f(xyzu)\\
&\quad - f(xwzu) + f(xyw) + f(ywz) + f(ywu) + f(xyz) + f(xyu)\\
&\quad + f(yzu) + f(xwz) + f(xwu) + f(xzu) + f(wzu) - f(yw)\\
&\quad - f(xy) - f(yz) - f(yu) - f(xw) - f(xz) - f(xu) - f(wz)\\
&\quad - f(wu) - f(zu) + f(y) + f(x) + f(w) + f(z) + f(u)\\
&= C^{(4)}f(x,y,w,z,u)\\
&= 0 \quad \text{by (1.5)}.
\end{aligned}$$

Hence, the above relations imply the $C^{(3)}f(x,.,z,u)$ is a homomorphism. Similarly, the fact is also true for $C^{(3)}f(.,y,z,u)$, $C^{(3)}f(x,y,.,u)$ and $C^{(3)}f(x,y,z,.)$. This proves (2.5). □

We now consider (2.6). Actually, it is trivial for n = 0,1,2,3 by (2.1) and by the definition of Cf. Suppose that (2.6) holds for all natural numbers smaller than n≥5, then

$$\begin{aligned}
f(x^n) =& f(x^{n-3}xxx)\\
=& f(x^{n-3}xx) + f(x^{n-3}xx) + f(x^{n-3}xx) + f(xxx) - f(x^{n-3}x)\\
&- f(x^{n-3}x) - f(x^{n-3}x) - f(xx) - f(xx) - f(xx)\\
&+ f(x^{n-3}) + f(x) + f(x) + f(x) + C^{(3)}f(x^{n-3},x,x,x)\\
=& f(x^{n-1}) + f(x^{n-1}) + f(x^{n-1}) + f(x^3) - f(x^{n-2}) - f(x^{n-2})\\
&- f(x^{n-2}) - f(x^2) - f(x^2) + f(x^{n-3}) + f(x) + f(x) + f(x)\\
&+ C^{(3)}f(x^{n-3},x,x,x)\\
=& 3f(x^{n-1}) + f(x^3) - 3f(x^{n-2}) - 3f(x^2) + f(x^{n-3}) + 3f(x)\\
&+ C^{(3)}f(x^{n-3},x,x,x)\\
=& 3\Big[(n-1)f(x) + (n-1)C_2Cf(x,x) + (n-1)C_3C^{(2)}f(x,x,x)\\
&+(n-1)C_4C^{(3)}f(x,x,x,x)\Big]\\
&+ \left[3f(x) + 3C_2Cf(x,x) + 3C_3C^2f(x,x,x)\right]\\
&- 3\Big[(n-2)f(x) + (n-2)C_2Cf(x,x) + (n-2)C_3C^{(2)}f(x,x,x)\\
&+(n-2)C_4C^{(3)}f(x,x,x,x)\Big] - 3\left[2f(x) + 2C_2Cf(x,x)\right]\\
&+ \Big[(n-3)f(x) + (n-3)C_2Cf(x,x) + (n-3)C_3C^{(2)}f(x,x,x)\\
&+(n-3)C_4C^{(3)}f(x,x,x,x)\Big] + \Big[3f(x) + (n-3)C^{(3)}f(x,x,x,x)\Big]
\end{aligned}$$

$$\begin{aligned}&=nf(x)+nC_2Cf(x,x)+nC_3C^{(2)}f(x,x,x)\\&\quad+nC_4C^{(3)}f(x,x,x,x)\end{aligned}$$

where the definition of $C^{(3)}f$ and (2.5) are used in the second equation. This gives (2.6) fora all n≥0. On the other hand, for any fixed integer n>0, by (1.4) and (2.1), we have

$$\begin{aligned}C^{(3)}f(x^n,x^{-n},x^n,x^n)=&f(x^{2n})-f(x^n)-f(x^n)-f(x^{3n})-f(x^n)\\&+f(e)+f(x^{2n})+f(x^{2n})+f(e)+f(e)\\&+f(x^{2n})-f(x^n)-f(x^{-n})-f(x^n)\\&-f(x^n)\\=&4f(x^{2n})-6f(x^n)-f(x^{3n})-f(x^{-n})\\\Rightarrow&f(x^{-n})=4f(x^{2n})-6f(x^n)-f(x^{3n})\\&-C^{(3)}f(x^n,x^{-n},x^n,x^n)\\=&4\,[2nf(x)+2nC_2Cf(x,x)+2nC_3\\&C^{(2)}f(x,x,x)+2nC_4C^{(3)}f(x,x,x,x)\Big]\\&-6\,[nf(x)+nC_2Cf(x,x)+nC_3\\&C^{(2)}f(x,x,x)+nC_4C^{(3)}f(x,x,x,x)\Big]\\&-[3nf(x)+3nC_2Cf(x,x)+3nC_3\\&C^{(2)}f(x,x,x)+3nC_4C^{(3)}f(x,x,x,x)\Big]\\&+n^4C^{(3)}f(x,x,x,x)\\=&[-nf(x)+-nC_2Cf(x,x)+-nC_3\\&C^{(2)}f(x,x,x)+-nC_4C^{(3)}f(x,x,x,x)\Big]\end{aligned}$$

from (2.5) and the above claim for n>0. This confirms (2.6) for n<0.

Remark 2. For any function f:G→ H, the following statements are pairwise equivalent:

(i) The function $f\in KerC^{(4)}(G,H)$;

(ii) $C^{(3)}f(.,y,z,u)$ is a homomorphism;

(iii) $C^{(3)}f(x,.,z,u)$ is a homomorphism;

(iv) $C^{(3)}f(x,y,.,u)$ is a homomorphism;

(v) $C^{(3)}f(x,y,z,.)$ is a homomorphism;

Before presenting Proposition 1, we first introduce the following useful lemma, which was given in [8]

Lemma 2. (Lemma 2.4 in [8]) *The following identity is valid for any function f:G→H and $l \in N$;*

$$f(x_1x_2\dots x_l) = \sum_{m\le l}\sum_{1\le i_1<i_2<\dots<i_m\le l} C^{(m-1)}f(x_{i_1},x_{i_2},\dots,x_{i_m}) \tag{2.7}$$

Proposition 1. *Suppose that $f \in KerC^{(4)}(G,H)$. Then*

$$\begin{aligned} &f(x_1^{n_1}x_2^{n_2}\dots x_l^{n_l})\\ &= \sum_{1\le i\le l}\Big[n_if(x_i) + n_iC_2 \quad Cf(x_i,x_i) + n_iC_3 \quad C^{(2)}f(x_i,x_i,x_i)\\ &\quad + n_iC_4 \quad C^{(3)}f(x_i,x_i,x_i,x_i)\Big] + \sum_{1\le i_1<i_2\le l} Cf(x_{i_1}^{n_{i_1}},x_{i_2}^{n_{i_2}})\\ &\quad + \sum_{1\le i_1<i_2<i_3\le l} n_{i_1}n_{i_2}n_{i_3}C^{(2)}f(x_{i_1},x_{i_2},x_{i_3})\\ &\quad + \sum_{1\le i_1<i_2<i_3<i_4\le l} n_{i_1}n_{i_2}n_{i_3}n_{i_4}C^{(3)}f(x_{i_1},x_{i_2},x_{i_3},x_{i_4}) \end{aligned} \tag{2.8}$$

for $n_i \in Z$ and all $x_i \in G$, $i = 1,2,\dots l$ such that $x_j \neq x_{j+1}$, $j = 1,2,\dots,l-1$

Proof. Replacing x_i in (2.7) by $x_i^{n_i}$, we have

$$f(x_1^{n_1}x_2^{n_2}\dots x_l^{n_l}) = \sum_{m\le l}\sum_{1\le i_1<i_2<\dots<i_m\le l} C^{(m-1)}f(x_{i_1}^{n_{i_1}},x_{i_2}^{n_{i_2}},\dots,x_{i_m}^{n_{i_m}})$$

The vanishing of $C^{(m-1)}f$ for m≥5 yields

$$\begin{aligned} f(x_1^{n_1}x_2^{n_2}\dots x_l^{n_l}) &= \sum_{1\le i\le l} f(x_i^{n_i}) + \sum_{1\le i_1<i_2\le l} Cf(x_{i_1}^{n_{i_1}},x_{i_2}^{n_{i_2}})\\ &+ \sum_{1\le i_1<i_2<i_3\le l} C^{(2)}f(x_{i_1}^{n_{i_1}},x_{i_2}^{n_{i_2}},x_{i_3}^{n_{i_3}})\\ &+ \sum_{1\le i_1<i_2<i_3<i_4\le l} C^{(3)}f(x_{i_1}^{n_{i_1}},x_{i_2}^{n_{i_2}},x_{i_3}^{n_{i_3}},x_{i_4}^{n_{i_4}}) \end{aligned}$$

Therefore, by (2.6) and (2.5), we have

$$f(x_i^{n_i}) = n_if(x_i) + n_iC_2 \quad Cf(x_i,x_i) + n_iC_3 \quad C^{(2)}f(x_i,x_i,x_i)$$

$$+n_iC_4 \quad C^{(3)}f(x_i,x_i,x_i,x_i)$$

$$\begin{aligned} C^{(2)}f(x_{i_1}^{n_{i_1}},x_{i_2}^{n_{i_2}},x_{i_3}^{n_{i_3}}) &= n_{i_1}n_{i_2}n_{i_3}C^{(2)}f(x_{i_1},x_{i_2},x_{i_3}) \\ C^{(3)}f(x_{i_1}^{n_{i_1}},x_{i_2}^{n_{i_2}},x_{i_3}^{n_{i_3}},x_{i_4}^{n_{i_4}}) &= n_{i_1}n_{i_2}n_{i_3}n_{i_4}C^{(3)}f(x_{i_1},x_{i_2},x_{i_3},x_{i_4}) \end{aligned}$$

which is (2.8). This completes proof. □

Remark 3. In particular, if $l = 1$, then Proposition 1 holds.

3. Solution on Symmetric Group S_n

The symmetric group on a finite set X is the group whose elements are all bijective functions from X to X and whose group operation is that of function composition. If $X = 1, 2, \ldots n$, then it is called the symmetric group of degree n and denoted S_n.

In this section, we consider (1.5) for $G = S_n$.

Lemma 3. *If $f \in KerC^{(4)}(S_n, H)$, then*

$$C^{(3)}f(x_1x_2\ldots x_m,y,z,u) = C^{(3)}f(x_{\pi(1)}x_{\pi(2)}\ldots x_{\pi(m)},y,z,u), \tag{3.1}$$

$$C^{(3)}f(x,y_1y_2\ldots y_m,z,u) = C^{(3)}f(x,y_{\pi(1)}y_{\pi(2)}\ldots y_{\pi(m)},z,u), \tag{3.2}$$

$$C^{(3)}f(x,y,z_1z_2\ldots z_m,u) = C^{(3)}f(x,y,z_{\pi(1)}z_{\pi(2)}\ldots z_{\pi(m)},u), \tag{3.3}$$

$$C^{(3)}f(x,y,z,u_1u_2\ldots u_m) = C^{(3)}f(x,y,z,u_{\pi(1)}u_{\pi(2)}\ldots u_{\pi(m)}), \tag{3.4}$$

for all $x, y, z, u, x_i, y_i, z_i, u_i \in S_n, i = 1, 2, 3, \ldots m$, and all rearrangements π.

Proof. Note that $C^{(3)}f(.,y,z,u)$ is a homomorphism and H is an abelian group, which yields

$$\begin{aligned} &C^{(3)}f(x_1x_2\ldots x_m,y,z,u) \\ &\quad = C^{(3)}f(x_1,y,z,u) + C^{(3)}f(x_2,y,z,u) + \ldots + C^{(3)}f(x_m,y,z,u) \\ &\quad = C^{(3)}f(x_{\pi(1)},y,z,u) + C^{(3)}f(x_{\pi(2)},y,z,u) + \ldots + \\ &\qquad C^{(3)}f(x_{\pi(m)},y,z,u) \\ &\quad = C^{(3)}f(x_{\pi(1)}x_{\pi(2)}\ldots x_{\pi(m)},y,z,u) \end{aligned}$$

This proves (3.1). By a similar procedure, we can also verify (3.2)-(3.4). □

Lemma 4. *Let τ be an arbitrary 2-cycle $\in S_n$, $f \in KerC^{(4)}(S_n, H)$, then*

$$\begin{aligned} f(\tau^2) &= 0 & (3.5)\\ Cf(\tau,\tau) &= -2f(\tau) & (3.6)\\ C^2f(\tau,\tau,\tau) &= 4f(\tau) & (3.7)\\ C^3f(\tau,\tau,\tau,\tau) &= -8f(\tau) & (3.8)\\ C^4f(\tau,\tau,\tau,\tau,\tau) &= 16f(\tau), 16f(\tau) = 0 & (3.9) \end{aligned}$$

Proof. It suffices to prove (3.9). The proofs for the rest of the statements are straightforward. Using $\tau^2 = e, f(e) = 0$ and (1.4) we get

$$\begin{aligned} C^{(4)}f(\tau,\tau,\tau,\tau,\tau) &= f(\tau^5) - 5f(\tau^4) + 10f(\tau^3) - 10f(\tau^2) + 5f(\tau)\\ &= f(\tau) - 5f(e) + 10f(\tau) - 10f(e) + 5f(\tau)\\ &= 16f(\tau) \end{aligned}$$

which implies that $16f(\tau) = 0$ since $C^{(4)}f = 0$. This completes the proof. □

Lemma 5. *For any 2-cycle $\sigma, \tau, \nu, \omega$ and $f \in KerC^{(4)}(S_n, H)$, we have*

$$C^{(3)}f(\sigma,\tau,\nu,\omega) = C^{(3)}f\left((12),(12),(12),(12)\right) \tag{3.10}$$

Proof. For any 2-cycle σ, there exists $u \in S_n$ such that $\sigma = u(12)u^{-1}$. Hence, for any $x, y, z \in S_n$, by (3.4) we have

$$\begin{aligned} C^{(3)}f(x,y,z,\sigma) &= C^{(3)}f(x,y,z,u(12)u^{-1})\\ &= C^{(3)}f(x,y,z,(12)uu^{-1})\\ &= C^{(3)}f(x,y,z,(12)) & (3.11) \end{aligned}$$

similarly,

$$\begin{aligned} C^{(3)}f(x,y,\sigma,z) &= C^{(3)}f(x,y,(12),z) & (3.12)\\ C^{(3)}f(x,\sigma,y,z) &= C^{(3)}f(x,(12),y,z) & (3.13)\\ C^{(3)}f(\sigma,x,y,z) &= C^{(3)}f((12),x,y,z) & (3.14) \end{aligned}$$

In particular, (3.10) follows from (3.11)-(3.14). □

Lemma 6. *Assume that $Cf(\sigma,\tau) = Cf((12),(12))$ and $C^{(2)}f(\sigma,\tau,\nu) = C^{(2)}f((12),(12),(12))$ for every 2-cycle $\sigma,\tau,\nu \in S_n$. Then for any $x, y, z, \beta, \sigma_i \in S_n, i = 1, 2, \ldots n$, rearrangement π where σ_i, β are 2-cycles, we have*

$$f(\sigma_1\sigma_2\ldots\sigma_l) = f(\sigma_{\pi(1)}\sigma_{\pi(2)}\cdots\sigma_{\pi(l)}) \tag{3.15}$$

$$f(x\beta yz) = f(x(12)yz) \quad (3.16)$$
$$f(xy\beta z) = f(xy(12)z) \quad (3.17)$$
$$f(\beta) = f((12)) \quad (3.18)$$

for every $f \in KerC^{(4)}(S_n, H)$.

Proof. Firstly for any 2-cycle $\sigma_i \in S_n, i = 1, 2, \dots, l$ and rearrangement π, it follows from the assumption $Cf(\sigma, \tau) = Cf((12), (12))$ and $C^{(2)}f(\sigma, \tau, \nu) = C^{(2)}f((12), (12), (12))$, Proposition 1, and (3.10) that

$$\begin{aligned} f(\sigma_1\sigma_2\dots\sigma_l) &= \sum_{i=1}^{l} f(\sigma_i) + \sum_{1\le i<j\le l} Cf(\sigma_i, \sigma_j) \\ &+ \sum_{1\le i<j<k\le l} C^{(2)}f(\sigma_i, \sigma_j, \sigma_k) \\ &+ \sum_{1\le i<j<k<q\le l} C^{(3)}f(\sigma_i, \sigma_j, \sigma_k, \sigma_q) \\ &= \sum_{i=1}^{l} f(\sigma_i) + lC_2 \quad Cf((12), (12)) \\ &+ lC_3 \quad C^{(2)}f((12), (12), (12)) \\ &+ lC_4 \quad C^{(3)}f((12), (12), (12), (12)) \\ &= \sum_{i=1}^{l} f(\sigma_{\pi(i)}) + lC_2 \quad Cf((12), (12)) \\ &+ lC_3 \quad C^{(2)}f((12), (12), (12)) \\ &+ lC_4 \quad C^{(3)}f((12), (12), (12), (12)) \\ &= f(\sigma_{\pi(1)}\sigma_{\pi(2)}\dots\sigma_{\pi(l)}) \end{aligned}$$

which gives (3.15).

On the other hand, for every $x, y, z, \beta \in S_n$ there exists 2-cycles $\sigma_i, \tau_j, \nu_k, u \in S_n, i = 1, 2, \dots, p, j = 1, 2, \dots, q, k = 1, 2, \dots, r$, such that $x = \sigma_1\sigma_2\dots\sigma_p, y = \tau_1\tau_2\dots\tau_q, z = \nu_1\nu_2\dots\nu_r$ and $\beta = u(12)u^{-1}$. Note that $u = \delta_1\delta_2\dots\delta_s$ for some 2-cycle $\delta_1\delta_2\dots\delta_s \in S_n$ we have

$$\begin{aligned} f(x\beta yz) &= f(\sigma_1\sigma_2\dots\sigma_p\delta_1\delta_2\dots\delta_s(12)\delta_s^{-1}\delta_{s-1}^{-1}\dots\delta_1^{-1}\tau_1\tau_2\dots\tau_q\nu_1\nu_2\dots\nu_r) \\ &= f(\sigma_1\sigma_2\dots\sigma_p(12)\delta_1\delta_2\dots\delta_s\delta_s^{-1}\delta_{s-1}^{-1}\dots\delta_1^{-1}\tau_1\tau_2\dots\tau_q\nu_1\nu_2\dots\nu_r) \\ &= f(x(12)yz) \end{aligned}$$

by (3.15).
similarly, we can prove (3.17). In particular, taking $x = y = z = e$ in (3.16) or (3.17), we obtain (3.18). This completes the proof.
According to lemma 6, we give the following main result in this section. □

Theorem 1. *Assume that* $Cf(\sigma,\tau) = Cf((12),(12))$ *and* $C^{(2)}f(\sigma,\tau,\nu) = C^{(2)}f((12),(12),(12))$ *for every 2-cycle* $\sigma,\tau,\nu \in S_n$*. Then* $f \in KerC^{(4)}(S_n,H)$ *if and only if there is an* $h_0 \in H$ *such that* $16h_0 = 0$ *and*

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is even,} \\ h_0 & \text{if } x \text{ is odd.} \end{cases} \tag{3.19}$$

Proof. Necessity. Let $f \in KerC^{(4)}(S_n,H)$. Then for any $x \in S_n$, there exists 2-cycles $\alpha_i \in S_n, i = 1,2,\ldots,p$, such that $x = \alpha_1\alpha_2\ldots\alpha_p$. In view of (3.10),(3.18),(3.6)-(3.8) and Proposition 1, we get

$$\begin{aligned} f(x) &= f(\alpha_1\alpha_2\ldots\alpha_p) \\ &= \sum_{i=1}^{p} f(\alpha_i) + \sum_{1\le i<j\le p} Cf(\alpha_i,\alpha_j) + \sum_{1\le i<j<k\le p} C^{(2)}f(\alpha_i,\alpha_j,\alpha_k) \\ &\quad + \sum_{1\le i<j<k<l\le p} C^{(3)}f(\alpha_i,\alpha_j,\alpha_k,\alpha_l) \\ &= pf((12)) + pC_2 \quad Cf((12),(12)) + pC_3C^{(2)}f((12),(12),(12)) \\ &\quad +pC_4 \quad C^{(3)}f((12),(12),(12),(12)) \\ &= pf((12)) + \frac{p^2-p}{2}(-2f((12)) + \frac{p(p^2-3p+2)}{6}4f((12)) \\ &\quad + \frac{p(p^3-6p^2+11p-6)}{24}(-8f((12))) \\ &= \frac{-p^4+8p^3-20p^2+16p}{3}f((12)) \end{aligned} \tag{3.20}$$

Let $g(p) = \frac{-p^4+8p^3-20p^2+16p}{3}$, we claim that

$$g(p) \in \begin{cases} 16\text{N} & \text{if } p \text{ is even,} \\ 16\text{N+1} & \text{if } p \text{ is odd.} \end{cases} \tag{3.21}$$

We first prove the even case. Obviously, (3.21) is true for p=2 since $g(2) = 0$. For an inductive proof, Suppose that (3.21) also holds for p=2n, $n \in Z$. Then we compute that

$$g(p) = \frac{-(2n+2)^4 + 8(2n+2)^3 - 20(2n+2)^2 + 16(2n+2)}{3}$$

$$
\begin{aligned}
&= -\frac{1}{3}((2n)^4 + 4(2n)^3(2) + 6(2n)^2(2)^2 + 4(2n)(2)^3 + 16) \\
&\quad + \frac{8}{3}((2n)^3 + 3(2n)^2(2) + 3(2n)(4) + 8) \\
&\quad - \frac{20}{3}((2n)^2 + 2(2n)(2) + 4) + \frac{16}{3}(2n + 2) \\
&= (-\frac{1}{3}(2n)^4 + \frac{8}{3}(2n)^3 - \frac{20}{3}(2n)^2 + \frac{16}{3}(2n)) \\
&\quad - \frac{8}{3}(2n)^3 - 8(2n)^2 + 16(2n)^2 - \frac{32}{3}(2n) + 32(2n) - \frac{80}{3}(2n) \\
&\quad - \frac{16}{3} + \frac{64}{3} - \frac{80}{3} + \frac{32}{3} \\
&= g(2n) - \frac{64}{3}n^3 + 32n^2 - \frac{32}{3}n
\end{aligned}
$$

which yields $g(2n+2) \in 16N$. This confirms the even case of (3.21). When p is odd, (3.21) is true for p=1 because $g(1) = 1$. Suppose that (3.21) holds for $p = 2n - 1$, and then we get

$$
\begin{aligned}
g(2n+1) &= -\frac{1}{3}(2n+1)^4 + \frac{8}{3}(2n+1)^3 - \frac{20}{3}(2n+1)^2 + \frac{16}{3}(2n+1) \\
&= -\frac{1}{3}((2n-1)^4 + 4(2n-1)^3(2) + 6(2n-1)^2(2)^2 \\
&\quad + 4(2n-1)(2)^3 + 16) + \frac{8}{3}((2n-1)^3 + 3(2n-1)^2(2) \\
&\quad + 3(2n-1)(4) + 8) - \frac{20}{3}((2n-1)^2 + 2(2n-1)(2) + 4) \\
&\quad + \frac{16}{3}(2n - 1 + 2) \\
&= (-\frac{1}{3}(2n-1)^4 + \frac{8}{3}(2n-1)^3 - \frac{20}{3}(2n-1)^2 \\
&\quad + \frac{16}{3}(2n-1)) - \frac{8}{3}(2n-1)^3 + 8(2n-1)^2 \\
&\quad - \frac{16}{3}(2n-1)(-\frac{16}{3} + \frac{64}{3} - \frac{80}{3} + \frac{32}{3}) \\
&= g(2n-1) - \frac{64}{3}n^3 + 64n^2 - \frac{176}{3}n + 16
\end{aligned}
$$

This completes the proof of (3.21). According to (3.21),(3.20) becomes

$$
f(x) = \begin{cases} 0 & \text{if } x \text{ is even}, \\ \text{f((12))} & \text{if } x \text{ is odd}. \end{cases}
$$

This proves that f must have the form (3.19) with $h_0 = f((12))$.

Sufficiently. Let $f : S_n \to H$ be defined by (3.19), where h_0 is a constant with $16h_0 = 0$. In order to prove the identity of (1.5), by the symmetry of x_1, x_2, x_3, x_4, x_5 it suffices to verify the following four cases:

case (i): x_1 is odd, and x_2, x_3, x_4, x_5 are even;
case (ii): x_1, x_2 is odd, and x_3, x_4, x_5 are even;
case (iii): x_1, x_2, x_3 is odd, and x_4, x_5 are even;
case (iv): x_1, x_2, x_3, x_4 is odd, and x_5 are even;
case (iv): x_1, x_2, x_3, x_4, x_5 are odd.

In fact, for case (i) it is easy to see that

$$x_1, x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_1x_2x_3, x_1x_2x_4, x_1x_2x_5, x_1x_3x_4, x_1x_3x_5, x_1x_4x_5,$$
$$x_1x_2x_3x_4, x_1x_2x_3x_5, x_1x_2x_4x_5, x_1x_3x_4x_5, x_1x_2x_3x_4x_5$$

are odd, and

$$x_2, x_3, x_4, x_5, x_2x_3, x_2x_4, x_2x_5, x_3x_4, x_3x_5, x_4x_5, x_2x_3x_4, x_2x_3x_5, x_2x_4x_5,$$
$$x_3x_4x_5, x_2x_3x_4x_5$$

are even, which leads to the equality of (1.5).

The proofs of the other cases are similar. □

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