ON 3-DIMENSIONAL \(\psi\)-RECURRENT \((LCS)_n\)-MANIFOLDS

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Abstract: The object of this paper is to study 3-dimensional \(\psi\)-recurrent \((LCS)_n\)-manifold and prove that it is a manifold of constant curvature and finally we prove that a 3-dimensional \((LCS)_n\)-manifold is locally \(\psi\)-concircularly symmetric if and only if the scalar curvature \(r\) is constant.

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1. Introduction

In (2003), A. A. Shaikh [1] introduced the notion of Lorentzian concircular structure manifolds (briefly \((LCS)_n\)-manifolds) with an example, which generalizes the notion of Lorentzian para-Sasakian manifolds introduced by Matsumoto. The notion of local symmetry of a Riemannian manifold has been

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studied by many authors in several ways to a different extent. As a weaker version of local symmetry, in (1977), Takahashi [12] introduced the notion of locally, $\varphi$-symmetric Sasakian manifold and obtained their several interesting results.

The importance of concircular curvature tensor is very well known in the differential geometry of certain $F$-structure such as complex, almost complex, Kahler, almost Kahler, contact and almost contact structure etc.

A transformation of an n-dimensional Riemannian manifold $M^n$, which transform every geodesic circle of $M^n$ in to a geodesic circle, is called a concircular transformation. A concircular transformation is always a conformal transformation. Here geodesic circle means a curve in $M^n$ whose first curvature is constant and second curvature is identically zero. Thus the geometry of concircular transformations, that is, the concircular geometry, is a generalization of inversive geometry in the sense that the change of metric is more general than that induced by a circle preserving diffeomorphism. An interesting invariant of a concircular transformation is the Concircular curvature tensor.

The Concircular curvature tensor $C$ of type $(1,3)$ on a Riemannian manifold is defined as

$$C(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y].$$  \hspace{1cm} (1.1)

**Definition 1.** An $(LCS)_n$- manifold is said to be an $\eta$-Einstein manifold if its Ricci tensor $S$ is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),$$  \hspace{1cm} (1.2)

where $a,b$ are associated functions on the manifold.

The object of the present paper is to study the 3-dimensional $(LCS)_n$-manifold. After preliminaries in Section 3, some basic results on 3-dimensional $(LCS)_n$- manifold are given. In Section 4, we prove that a 3-dimensional $\psi$-recurrent $(LCS)_n$- manifold is of constant curvature. And finally in Section 5, we prove that a 3-dimensional $(LCS)_n$- manifold is locally $\psi$-concircularly symmetric if and only if the scalar curvature $r$ is constant.

**2. Preliminaries**

An n-dimensional Lorentzian manifold $M^n$ is a smooth connected para compact Hausdroff manifold with a Lorentzian metric $g$ of type $(0,2)$ such that for each point $p \in M$, the tensor $g_p : T_p M \times T_p M \rightarrow R$ is a non-degenerate inner product
of signature \((-, +, \ldots, +)\), where \(T_pM\) denotes the tangent space of \(M\) at \(p\) and \(\mathbb{R}\) is the real number space. A non-zero vector \(v \in T_pM\) is said to be timelike (resp. non-spacelike, null, spacelike) if it satisfies \(g_p(v, v) < 0\) (resp. \(< 0, = 0, > 0\)) ([1], [4]).

**Definition 2.** In a Lorentzian manifold \((M^n, g)\), a vector field \(P\) is defined as
\[
g(X, P) = A(X),
\]
for any \(X \in \chi(M)\) is said to be a concircular vector field if
\[
(\nabla_X A)(Y) = \alpha \{g(X, Y) + \omega(X)A(Y)\},
\]
where \(\alpha\) is a non-zero scalar and \(\omega\) is a closed 1-form.

Let \(M^n\) be a Lorentzian manifold admitting a unit timelike concircular vector field \(\xi\), called the generator of manifold. Then we have
\[
g(\xi, \xi) = -1.
\]
Since \(\xi\) is a unit concircular vector field, it follows that there exists a non-zero 1-form \(\eta\) such that for
\[
g(X, \xi) = \eta(X),
\]
the equation of the following form holds
\[
(\nabla_X \eta)(Y) = \alpha \{g(X, Y) + \eta(X)\eta(Y)\}, (\alpha \neq 0)
\]
for all vector fields \(X, Y\); where \(\nabla\) denotes the operator of covariant differentiation with respect to the Lorentzian metric \(g\) and \(\alpha\) is a non-zero scalar function satisfies
\[
\nabla_X \alpha = (X \alpha) = d\alpha(X) = \rho\eta(X),
\]
\(\rho\) being a certain scalar function given by \(\rho = -(\xi \alpha)\). If we put
\[
\psi X = \frac{1}{\alpha} \nabla_X \xi.
\]
Then from (2.4) and (2.6), we get
\[
\psi^2 X = X + \eta(X)\xi,
\]
from which it follows that \(\psi\) is symmetric \((1, 1)\) tensor and called the structure tensor of the manifold. Thus the Lorentzian manifold \(M^n\) together with the unit timelike concircular vector field \(\xi\), its associated 1-form \(\eta\) and \((1, 1)\) tensor field
ψ is said to be Lorentzian concircular structure manifold (briefly \((LCS)_n\)-manifold). Especially, if we take \(\alpha=1\), then we can obtain the LP-Sasakian structure of Matsumoto ([4]). In an \((LCS)_n\)-manifold, the following relations holds ([1]):

\[(a) \eta(\xi) = -1, (b) \psi\xi = 0, (c) \eta(\psi X) = 0, (d) g(\psi X, \psi Y) = g(X, Y) + \eta(X)\eta(Y), \quad (2.9)\]

\[\eta(R(X, Y)Z) = (\alpha^2 - \rho)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)], \quad (2.10)\]

\[S(X, \xi) = (n - 1)(\alpha^2 - \rho)\eta(X), \quad (2.11)\]

\[R(X, Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y], \quad (2.12)\]

\[R(\xi, X)Y = (\alpha^2 - \rho)[g(X, Y)\xi - \eta(Y)X], \quad (2.13)\]

\[R(\xi, X)\xi = (\alpha^2 - \rho)[\eta(X)\xi + X], \quad (2.14)\]

\[(\nabla_X \psi)(Y) = \alpha\{g(X, Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\}, \quad (2.15)\]

\[(X\rho) = d\rho(X) = \beta\eta(X). \quad (2.16)\]

In a 3-dimensional Riemannian manifold, we have

\[R(X, Y)Z = g(Y, Z)QX - g(X, Z)QY + S(Y, Z)X - S(X, Z)Y - \frac{r}{2}[(g(Y, Z)X - g(X, Z)Y)], \quad (2.17)\]

where \(Q\) is a Ricci tensor, i.e., \(g(QX, Y) = S(X, Y)\) and \(r\) is a scalar curvature of the manifold.

3. Basic Results

**Theorem 1.** In a 3-dimensional \((LCS)_n\)-manifold, the Ricci operator is given by

\[QX = \left[\frac{r}{2} - (\alpha^2 - \rho)\right]X + \left[\frac{r}{2} - 3(\alpha^2 - \rho)\right]\eta(X)\xi. \quad (3.1)\]

**Proof.** Taking \(Z = \xi\) in (2.17) and using (2.8), (2.11) and (2.12), we get (3.1). \(\square\)

**Corollary 1.** In a 3-dimensional \((LCS)_n\)-manifold, the Ricci tensor and curvature tensor are given by
\[ S(X,Y) = \left[ \frac{r}{2} - (\alpha^2 - \rho) \right] g(X,Y) + \left[ \frac{r}{2} - 3(\alpha^2 - \rho) \right] \eta(X)\eta(Y), \quad (3.2) \]

and

\[ R(X,Y)Z = \left[ \frac{r}{2} - 2(\alpha^2 - \rho) \right] [g(Y,Z)X - g(X,Z)Y] \]
\[ + \left[ \frac{r}{2} - 3(\alpha^2 - \rho) \right] [g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi] \]
\[ + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y]. \quad (3.3) \]

**Proof.** Equation (3.2) follows from (3.1). Using (3.1) and (3.2) in (2.17), we get (3.3).

**Remark 1.** Equation (3.2) shows that, a 3-dimensional \((LCS)_n\)-manifold is an \(\eta\)-Einstein manifold.

**Lemma 1.** A 3-dimensional \((LCS)_n\)-manifold is a manifold of constant curvature if and only if the scalar curvature \(r = 6(\alpha^2 - \rho)\).

**Proof.** From equation (3.3), the lemma follows.

### 4. 3-Dimensional \(\psi\)-Recurrent \((LCS)_n\)-Manifold

**Definition 3.** An \((LCS)_n\)-manifold is said to be \(\psi\)-recurrent \((LCS)_n\)-manifold if there exists a non-zero 1-form \(A\) such that

\[ \psi^2((D_WR)(X,Y)Z) = A(W)R(X,Y)Z, \quad (4.1) \]

for arbitrary vector fields \(X, Y, Z, W\).

Taking covariant differentiation on both the sides of equation (3.3), we get

\[ (D_WR)(X,Y)Z = \left[ \frac{dr(W)}{2} - 2(2\alpha\rho - \beta)\eta(W) \right] \]
\[ + \frac{dr(W)}{2} - 3(2\alpha\rho - \beta)\eta(W)] \]
\[ [g(Y,Z)X - g(X,Z)Y] \]
\[ + \left[ \frac{r}{2} - 3(\alpha^2 - \rho) \right][g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi] \]
\[ + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y] \]
\[ + \left[ \frac{r}{2} - 3(\alpha^2 - \rho) \right][g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi] \]
\[ + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y]. \quad (4.2) \]
\[-g(X,Z)\eta(Y)](D_W\xi) + \left[\frac{r}{2} - 3(\alpha^2 - \rho)\right][g(Y,Z)(D_W\eta)(X) - g(X,Z)((D_W\eta)Y)]\xi \\
+ \left[\frac{r}{2} - 3(\alpha^2 - \rho)\right][((D_W\eta)Y)\eta(Z)X + \eta(Y)(D_W\eta)(Z)X - (D_W\eta)(X)\eta(Z)Y - \eta(X)(D_W\eta)(Z)Y]].

Taking $X, Y, Z, W$ orthogonal to $\xi$ and using (2.9), we get

\[(D_WR)(X,Y)Z = \frac{dr(W)}{2}[g(Y,Z)X - g(X,Z)Y] \tag{4.3}\]

Applying $\psi^2$ on both the sides of (4.3) and using (2.9), we get

\[\psi^2(D_WR)(X,Y)Z = \frac{dr(W)}{2}[g(Y,Z)X - g(X,Z)Y]. \tag{4.4}\]

By (4.1), equation (4.4) reduces to

\[A(W)R(X,Y)Z = \frac{dr(W)}{2}[g(Y,Z)X - g(X,Z)Y]. \tag{4.5}\]

Putting $W = \{e_i\}$, where $\{e_i\}, i = 1, 2, 3,$ is an orthonormal basis of the tangent space at any point of the manifold and taking summation over $i, 1 \leq i \leq 3,$ we get

\[R(X,Y)Z = \lambda[g(Y,Z)X - g(X,Z)Y], \tag{4.6}\]

where $\lambda = \frac{dr(e_i)}{2A(e_i)}$ is a scalar, since $A$ is a non zero 1-form. Then by Schur’s theorem, $\lambda$ will be a constant on the manifold. Therefore $M^3$ is of constant curvature $\lambda$. Hence, we can state the following theorem:

**Theorem 2.** A 3-dimensional $\psi$-recurrent $(LCS)_n$-manifold is of constant curvature.
5. 3-Dimensional Locally $\psi$-Concircularly Symmetric $(LCS)_n$-Manifold

**Definition 4.** An $(LCS)_n$-manifold is said to be locally $\psi$-concircularly symmetric if the Concircular curvature tensor $C$ satisfies

$$\psi^2((D_W C)(X,Y)Z) = 0, \quad (5.1)$$

for arbitrary vector fields $X, Y, Z, W$ orthonormal to $\xi$.

Using (3.3) in (1.1) in a 3-dimensional $(LCS)_n$-manifold, the Concircular curvature tensor $C$ is given by

$$C(X,Y)Z = \left[\frac{r^3}{3} - 2(\alpha^2 - \rho)\right][g(Y, Z)X - g(X, Z)Y] \quad (5.2)$$

$$+ \left[\frac{r^2}{2} - 3(\alpha^2 - \rho)\right][g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi]$$

$$+ \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y].$$

Taking covariant differentiation on both sides of the equation (5.2), we get

$$(D_W C)(X,Y)Z = \left[\frac{dr(W)}{3} - 2(2\alpha\rho - \beta)\eta(W)\right]$$

$$[g(Y, Z)X - g(X, Z)Y]$$

$$+ \left[\frac{dr(W)}{2} - 3(2\alpha\rho - \beta)\eta(W)\right]$$

$$[g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)]$$

$$+ \left[\frac{r^2}{2} - 3(\alpha^2 - \rho)\right][g(Y, Z)(D_W \eta)(X)\xi$$

$$- g(X, Z)(D_W \eta)(Y)\xi + g(Y, Z)\eta(X)(D_W \xi)$$

$$- g(X, Z)\eta(Y)(D_W \xi) + (D_W \eta)(Y)\eta(Z)X$$

$$+ \eta(Y)(D_W \eta)(Z)X - (D_W \eta)(X)\eta(Z)Y$$

$$- \eta(X)(D_W \eta)(Z)Y].$$

Now assume that $X, Y$ and $Z$ are horizontal vector fields, then equation (5.3) becomes

$$(D_W C)(X,Y)Z = \left[\frac{dr(W)}{3} - 2(2\alpha\rho - \beta)\eta(W)\right]$$

$$[g(Y, Z)X - g(X, Z)Y]$$

$$+ \left[\frac{r^2}{2} - 3(\alpha^2 - \rho)\right][g(Y, Z)(D_W \eta)(X)\xi$$

$$- g(X, Z)(D_W \eta)(Y)\xi + g(Y, Z)\eta(X)(D_W \xi)$$

$$- g(X, Z)\eta(Y)(D_W \xi) + (D_W \eta)(Y)\eta(Z)X$$

$$+ \eta(Y)(D_W \eta)(Z)X - (D_W \eta)(X)\eta(Z)Y$$

$$- \eta(X)(D_W \eta)(Z)Y].$$
Applying $\psi^2$ on both the sides of (5.4) and using (2.9), we get

$$\psi^2(D_W C)(X,Y)Z = \frac{dr(W)}{3}[g(Y,Z)X - g(X,Z)Y + g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi].$$

Again taking $X, Y, Z, W$ orthogonal to $\xi$, we get

$$\psi^2(D_W C)(X,Y)Z = \frac{dr(W)}{3}[g(Y,Z)X - g(X,Z)Y].$$

Hence, we can state the following theorem:

**Theorem 3.** A 3-dimensional $(LCS)_n$-manifold is locally $\psi$-concircularly symmetric if and only if the scalar curvature $r$ is constant.

**References**


