International Journal of Pure and Applied Mathematics
Volume 117 No. 1 2017, 59-67
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v117i1.6

ON 3-DIMENSIONAL ψ-RECURRENT $(LCS)_n$-MANIFOLDS

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Abstract: The object of this paper is to study 3-dimensional ψ-recurrent $(LCS)_n$-manifold and prove that it is a manifold of constant curvature and finally we prove that a 3-dimensional $(LCS)_n$-manifold is locally ψ-concircularly symmetric if and only if the scalar curvature r is constant.

AMS Subject Classification: 53C10, 53C15, 53C25

Key Words: 3-dimensional $(LCS)_n$-manifold, η-Einstein manifold, φ-recurrent manifold, φ-concircularly symmetric

1. Introduction

In (2003), A. A. Shaikh [1] introduced the notion of Lorentzian concircular structure manifolds (briefly $(LCS)_n$- manifolds) with an example, which generalizes the notion of Lorentzian para-Sasakian manifolds introduced by Matsumoto. The notion of local symmetry of a Riemannian manifold has been

Received: 2017-05-10
Revised: 2017-06-14
Published: November 28, 2017

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studied by many authors in several ways to a different extent. As a weaker version of local symmetry, in (1977), Takahashi [12] introduced the notion of locally, φ-symmetric Sasakian manifold and obtained their several interesting results.

The importance of concircular curvature tensor is very well known in the differential geometry of certain F-structure such as complex, almost complex, Kahler, almost Kahler, contact and almost contact structure etc.

A transformation of an n-dimensional Riemannian manifold M^n, which transform every geodesic circle of M^n in to a geodesic circle, is called a concircular transformation. A concircular transformation is always a conformal transformation. Here geodesic circle means a curve in M^n whose first curvature is constant and second curvature is identically zero. Thus the geometry of concircular transformations, that is, the concircular geometry, is a generalization of inversive geometry in the sence that the change of metric is more general than that induced by a circle preserving diffeomorphism. An interesting invariant of a concircular transformation is the Concircular curavture tensor.

The Concircular curvature tensor C of type $(1,3)$ on a Riemannian manifold is defined as

$$C(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]. \qquad (1.1)$$

Definition 1. *An $(LCS)_n$- manifold is said to be an η-Einstien manifold if its Ricci tensor S is of the form*

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \qquad (1.2)$$

where a,b are associated functions on the manifold.

The object of the present paper is to study the 3-dimensional $(LCS)_n$-manifold. After preliminaries in Section 3, some basic results on 3-dimensional $(LCS)_n$- manifold are given. In Section 4, we prove that a 3-dimensional ψ-recurrent $(LCS)_n$- manifold is of constant curvature. And finally in Section 5, we prove that a 3-dimensional $(LCS)_n$- manifold is locally ψ-concircularly symmetric if and only if the scalar curvature r is constant.

2. Preliminaries

An n-dimensional Lorentzian manifold M^n is a smooth connected para compact Hausdroff manifold with a Lorentzian metric g of type $(0,2)$ such that for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \to R$ is a non-degenerate inner product of signature $(-, +, \ldots\ldots\ldots, +)$, where T_pM denotes the tangent space of M at p and R is the real number space. A non-zero vector $v\epsilon T_pM$ is said to be timelike (resp. non-spacelike, null, spacelike) if it satisfies $g_p(v, v) < 0 (resp. < 0, = 0, > 0)$([1], [4]).

Definition 2. *In a Lorentzian manifold (M^n, g), a vector field P is defined as*

$$g(X, P) = A(X), \tag{2.1}$$

for any $X\epsilon\chi(M)$ is said to be a concircular vector field if

$$(\nabla_X A)(Y) = \alpha\{g(X, Y) + \omega(X)A(Y)\}, \tag{2.2}$$

where α is a non - zero scalar and ω is a closed 1-form.

Let M^n be a Lorentzian manifold admitting a unit timelike concircular vector field ξ, called the generator of manifold. Then we have

$$g(\xi, \xi) = -1. \tag{2.3}$$

Since ξ is a unit concircular vector field, it follows that there exists a non-zero 1-form η such that for

$$g(X, \xi) = \eta(X), \tag{2.4}$$

the equation of the following form holds

$$(\nabla_X \eta)(Y) = \alpha\{g(X, Y) + \eta(X)\eta(Y)\}, (\alpha \neq 0) \tag{2.5}$$

for all vector fields X, Y; where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and α is a non-zero scalar function satisfies

$$\nabla_X \alpha = (X\alpha) = d\alpha(X) = \rho\eta(X), \tag{2.6}$$

ρ being a certain scalar function given by $\rho = -(\xi\alpha)$. If we put

$$\psi X = \frac{1}{\alpha}\nabla_X \xi. \tag{2.7}$$

Then from (2.4) and (2.6), we get

$$\psi^2 X = X + \eta(X)\xi, \tag{2.8}$$

from which it follows that ψ is symmetric $(1, 1)$ tensor and called the structure tensor of the manifold. Thus the Lorentzian manifold M^n together with the unit timelike concircular vector field ξ, its associated 1-form η and $(1, 1)$ tensor field

ψ is said to be Lorentzian concircular structure manifold (briefly ($(LCS)_n$-manifold). Especially, if we take α=1, then we can obtain the LP-Sasakian structure of Matsumoto ([4]). In an $(LCS)_n$- manifold, the following relations holds ([1]):

$$(a)\ \eta(\xi) = -1, (b)\ \psi\xi = 0, (c)\ \eta(\psi X) = 0, (d) g(\psi X, \psi Y) = g(X, Y) + \eta(X)\eta(Y), \quad (2.9)$$

$$\eta(R(X, Y)Z) = (\alpha^2 - \rho)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)], \quad (2.10)$$

$$S(X, \xi) = (n - 1)(\alpha^2 - \rho)\eta(X), \quad (2.11)$$

$$R(X, Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y], \quad (2.12)$$

$$R(\xi, X)Y = (\alpha^2 - \rho)[g(X, Y)\xi - \eta(Y)X], \quad (2.13)$$

$$R(\xi, X)\xi = (\alpha^2 - \rho)[\eta(X)\xi + X], \quad (2.14)$$

$$(\nabla_X \psi)(Y) = \alpha\{g(X, Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\}, \quad (2.15)$$

$$(X\rho) = d\rho(X) = \beta\eta(X). \quad (2.16)$$

In a 3-dimensional Riemannian manifold, we have

$$\begin{aligned} R(X, Y)Z \quad = \quad & g(Y, Z)QX - g(X, Z)QY \\ & + S(Y, Z)X - S(X, Z)Y \\ & - \frac{r}{2}[(g(Y, Z)X - g(X, Z)Y)], \end{aligned} \quad (2.17)$$

where Q is a Ricci tensor, i.e., $g(QX, Y) = S(X, Y)$ and r is a scalar curvature of the manifold.

3. Basic Results

Theorem 1. *In a 3-dimensional $(LCS)_n$- manifold, the Ricci operator is given by*

$$QX = [\frac{r}{2} - (\alpha^2 - \rho)]X + [\frac{r}{2} - 3(\alpha^2 - \rho)]\eta(X)\xi. \quad (3.1)$$

Proof. Taking $Z = \xi$ in (2.17) and using (2.8), (2.11)and(2.12), we get (3.1). □

Corollary 1. *In a 3-dimensional $(LCS)_n$- manifold, the Ricci tensor and curvature tensor are given by*

$$S(X,Y) = [\frac{r}{2} - (\alpha^2 - \rho)]g(X,Y) + [\frac{r}{2} - 3(\alpha^2 - \rho)]\eta(X)\eta(Y), \tag{3.2}$$

and

$$\begin{aligned} R(X,Y)Z &= [\frac{r}{2} - 2(\alpha^2 - \rho)][g(Y,Z)X - g(X,Z)Y] \\ &\quad +[\frac{r}{2} - 3(\alpha^2 - \rho)][(g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ &\quad +\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y)]. \end{aligned} \tag{3.3}$$

Proof. Equation (3.2) follows from (3.1). Using (3.1) and (3.2) in (2.17), we get (3.3). □

Remark 1. *Equation (3.2) shows that, a 3-dimensional $(LCS)_n$- manifold is an η-Einstein manifold.*

Lemma 1. *A 3-dimensional $(LCS)_n$- manifold is a manifold of constant curvature if and only if the scalar curvature $r = 6(\alpha^2 - \rho)$.*

Proof. From equation (3.3), the lemma follows. □

4. 3-Dimensional ψ-Recurrent $(LCS)_n$-Manifold

Definition 3. *An $(LCS)_n$-manifold is said to be ψ-recurrent $(LCS)_n$-manifold if there exists a non-zero 1-form A such that*

$$\psi^2((D_W R)(X,Y)Z) = A(W)R(X,Y)Z, \tag{4.1}$$

for arbitrary vector fields X, Y, Z, W.

Taking covariant differentiation on both the sides of equation (3.3), we get

$$\begin{aligned} (D_W R)(X,Y)Z &= [\frac{dr(W)}{2} - 2(2\alpha\rho - \beta)\eta(W)] \\ &\quad [g(Y,Z)X - g(X,Z)Y] \\ &\quad +[\frac{dr(W)}{2} - 3(2\alpha\rho - \beta)\eta(W)] \\ &\quad [g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ &\quad +\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y] \\ &\quad +[\frac{r}{2} - 3(\alpha^2 - \rho)][g(Y,Z)\eta(X) \end{aligned} \tag{4.2}$$

$$
\begin{aligned}
&-g(X,Z)\eta(Y)](D_W\xi)\\
&+[\frac{r}{2}-3(\alpha^2-\rho)][g(Y,Z)(D_W\eta)(X)\\
&-g(X,Z)((D_W\eta)Y)]\xi\\
&+[\frac{r}{2}-3(\alpha^2-\rho)]\,[(D_W\eta)(Y)\eta(Z)X\\
&+\eta(Y)(D_W\eta)(Z)X-(D_W\eta)(X)\eta(Z)Y\\
&-\eta(X)(D_W\eta)(Z)Y)].
\end{aligned}
$$

Taking X, Y, Z, W orthogonal to ξ and using (2.9), we get

$$
\begin{aligned}
(D_W R)(X,Y)Z \;=\; & \frac{dr(W)}{2}[g(Y,Z)X-g(X,Z)Y] \\
& +[\frac{r}{2}-3(\alpha^2-\rho)] \\
& [g(Y,Z)(D_W\eta)(X)-g(X,Z)(D_W\eta)(Y)]\xi.
\end{aligned} \tag{4.3}
$$

Applying ψ^2 on both the sides of (4.3) and using (2.9), we get

$$\psi^2(D_W R)(X,Y)Z=\frac{dr(W)}{2}[g(Y,Z)X-g(X,Z)Y]. \tag{4.4}$$

By (4.1), equation (4.4) reduces to

$$A(W)R(X,Y)Z=\frac{dr(W)}{2}[g(Y,Z)X-g(X,Z)Y]. \tag{4.5}$$

Putting $W=\{e_i\}$, where $\{e_i\}$, $i=1,2,3$, is an orthonormal basis of the tangent space at any point of the manifold and taking summation over $i, 1\le i\le 3$, we get

$$R(X,Y)Z=\lambda[g(Y,Z)X-g(X,Z)Y], \tag{4.6}$$

where $\lambda=\frac{dr(e_i)}{2A(e_i)}$ is a scalar, since A is a non zero 1-form. Then by Schur's theorem, λ will be a constant on the manifold. Therefore M^3 is of constant curvature λ. Hence, we can state the following theorem:

Theorem 2. *A 3-dimensional ψ-recurrent $(LCS)_n$-manifold is of constant curvature.*

5. 3-Dimensional Locally ψ-Concircularly Symmetric $(LCS)_n$-Manifold

Definition 4. *An $(LCS)_n$-manifold is said to be locally ψ-concircularly symmetric if the Concircular curvature tensor C satisfies*

$$\psi^2((D_W C)(X,Y)Z) = 0, \tag{5.1}$$

for arbitrary vector fields X, Y, Z, W orthonormal to ξ.

Using (3.3) in (1.1) in a 3-dimensional $(LCS)_n$- manifold, the Concircular curvature tensor C is given by

$$\begin{aligned} C(X,Y)Z &= [\frac{r}{3} - 2(\alpha^2 - \rho)][g(Y,Z)X - g(X,Z)Y] \\ &\quad +[\frac{r}{2} - 3(\alpha^2 - \rho)]\,[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ &\quad +\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y]\,. \end{aligned} \tag{5.2}$$

Taking covariant differentiation on both sides of the equation (5.2), we get

$$\begin{aligned} (D_W C)(X,Y)Z &= [\frac{dr(W)}{3} - 2(2\alpha\rho - \beta)\eta(W)] \\ &\quad [g(Y,Z)X - g(X,Z)Y] \\ &\quad +[\frac{dr(W)}{2} - 3(2\alpha\rho - \beta)\eta(W)] \\ &\quad [g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y) \\ &\quad +\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y] \\ &\quad +[\frac{r}{2} - 3(\alpha^2 - \rho)][g(Y,Z)(D_W\eta)(X)\xi \\ &\quad -g(X,Z)(D_W\eta)(Y)\xi + g(Y,Z)\eta(X)(D_W\xi) \\ &\quad -g(X,Z)\eta(Y)(D_W\xi) + (D_W\eta)(Y)\eta(Z)X \\ &\quad +\eta(Y)(D_W\eta)(Z)X - (D_W\eta)(X)\eta(Z)Y \\ &\quad -\eta(X)(D_W\eta)(Z)Y]. \end{aligned} \tag{5.3}$$

Now assume that X, Y and Z are horizontal vector fields, then equation (5.3) becomes

$$\begin{aligned} (D_W C)(X,Y)Z &= \frac{dr(W)}{3}[g(Y,Z)X - g(X,Z)Y] \\ &\quad +[\frac{r}{2} - 3(\alpha^2 - \rho)][g(Y,Z)(D_W\eta)(X)\xi \end{aligned} \tag{5.4}$$

$$-g(X,Z)(D_W\eta)(Y)\xi].$$

Applying ψ^2 on both the sides of (5.4) and using (2.9), we get

$$\psi^2(D_WC)(X,Y)Z = \frac{dr(W)}{3}[g(Y,Z)X - g(X,Z)Y + g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi]. \quad (5.5)$$

Again taking X, Y, Z, W orthogonal to ξ, we get

$$\psi^2(D_WC)(X,Y)Z = \frac{dr(W)}{3}[g(Y,Z)X - g(X,Z)Y]. \quad (5.6)$$

Hence, we can state the following theorem:

Theorem 3. *A 3-dimensional $(LCS)_n$-manifold is locally ψ-concircularly symmetric if and only if the scalar curvature r is constant.*

References

[1] A. A. Shaikh, On Lorentzian almost para-contact manifolds with a structure of the concircular type, *Kyungpook Math. J.*, **43** (2003), 305-314.

[2] A. A. Shaikh, K. K. Baishya, On ψ-symmetric L P-Sasakian manifolds, *Yokohama Math. J.*, **52** (2006), 97-112.

[3] B. O'neill, Semi-Riemannian Geometry with Applications to Relativity", Academic Press, Inc., New York (1983).K. Matsumoto, On Lorentzian almost para-contact manifolds, *Bull. Yama-gata Univ. Nat. Sci.*, **12** (1989), 151-156.

[4] A. A. Shaikh, T. Basu, and S. Eyasmin, On the Existence of ψ-Recurrent $(LCS)_n$ Manifolds, *Extracta Mathematica*, **23**, No. 1 (2008), 71-83.

[5] C. Özgür and U. C. De, On the quasi-conformal curvature tensor of a Kenmotsu manifold, *Mathematica Pannonica*, **17**, No. 2 (2006), 221-228.

[6] D. E. Blair, *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Math.No. 509. Springer, 1976.

[7] J. B. Jun, U. C. De and G. Pathak, On Kenmotsu manifolds, *J. Korean Math. Soc.*, **42** (2005), 435-445.

[8] K. Kenmotsu. A class of almost contact Riemannian manifolds, *Tohoku Math. J.*, **24** (1972), 93-103.

[9] K. Yano, Concircular geometry, I, *Proc. Imp. Acad.*, Tokyo, **16** (1940), 195-200.

[10] S. Tanno, Isometric Immersions of Sasakian manifold in spheres, *Kodai Math. Sem. Rep.*, **21** (1969), 448-458.

[11] T. Takahashi, Sasakian ψ-symmetric spaces, *Tohoku Math. J.*, **29** (1977), 91-113.

[12] U. C. De and G. Pathak, On 3-dimensional Kenmotsu Manifolds, *Indian J. Pure Appl. Math.*, **35**, No. 2 (2004), 159-165.

[13] U. C. De, A. A. Shaikh, and S. Biswas, On ψ-Recurrent Sasakian manifolds, *NoviSad J. Math.*, **33**, No. 2 (2003),43-48.

[14] U. C. De, A. Yildiz and A. F. Yaliniz On ψ-Recurrent Kenmotsu manifolds, *Turk. J. Math.*, **33** (2009), 17-25.

[15] A. A. Shaikh and U. C. De, On 3-dimensional LP-Sasakian manifolds, *Soochow J. of Mathematics*, **26**, No. 1 (2000), 359-368.