

**UPPER AND LOWER  
( $I, J$ )-CONTINUOUS MULTIFUNCTIONS**

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**Abstract:** The purpose of the present paper is to introduce, study and characterize upper and lower nearly  $(I, J)$ -continuous multifunctions. Also, we investigate its relation with another class of continuous multifunctions.

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**Key Words:** nearly  $(I, J)$ -continuous multifunctions,  $I$ -open set,  $I$ -closed set, lower nearly  $(I, J)$ -continuous multifunctions, upper almost nearly  $(I, J)$ -continuous multifunctions

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## 1. Introduction

It is well known today, that the notion of multifunction is playing a very impor-

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tant role in general topology, upper and lower continuity have been extensively studied on multifunctions  $F : (X, \tau) \rightarrow (Y, \sigma)$ . Currently using the notion of topological ideal, different types of upper and lower continuity in multifunction  $F : (X, \tau, I) \rightarrow (Y, \sigma)$  have been studied and characterized [2], [8], [9], [14], [17]. The concept of ideal topological spaces has been introduced and studied by Kuratowski [12] and the local function of a subset  $A$  of a topological space  $(X, \tau)$  was introduced by Vaidyanathaswamy [16] as follows: Given a topological space  $(X, \tau)$  with an ideal  $I$  on  $X$  and if  $P(X)$  is the set of all subsets of  $X$ , a set operator  $(\cdot)^* : P(X) \rightarrow P(X)$ , called the local function of  $A$  with respect to  $\tau$  and  $I$ , is defined as follows: for  $A \subseteq X$ ,  $A^*(\tau, I) = \{x \in X / U \cap A \notin I \text{ for every } U \in \tau_x\}$ , where  $\tau_x = \{U \in \tau : x \in U\}$ . A Kuratowski closure operator  $cl^*(\cdot)$  for a topology  $\tau^*(\tau, I)$  called the  $*$ -topology, finer than  $\tau$  is defined by  $cl^*(A) = A \cup A^*(\tau, I)$ . We will denote  $A^*(\tau, I)$  by  $A^*$ . In 1990, Jankovic and Hamlett [10], introduced the notion of  $I$ -open set in a topological space  $(X, \tau)$  with an ideal  $I$  on  $X$ . In 1992, Abd El-Monsef et al. [1] further investigated  $I$ -open sets and  $I$ -continuous functions. In 2007, Akdag [2], introduce the concept of  $I$ -continuous multifunctions in a topological space with and ideal on it. Given a multifunction  $F : (X, \tau) \rightarrow (Y, \sigma)$ , and two ideals  $I, J$  associate, now with the topological spaces  $(X, \tau, I)$  and  $(Y, \sigma, J)$ , consider the multifunction  $F : (X, \tau, I) \rightarrow (Y, \sigma, J)$ . We want to study some type of upper and lower continuity of  $F$ . In this paper, we introduce and study a new class of multifunction called a nearly  $(I, J)$ -continuous multifunctions in topological spaces. Investigate its relation with another class of continuous multifunctions. Also we introduce another types of  $(I, J)$ -continuous multifunctions.

## 2. Preliminaries

Throughout this paper,  $(X, \tau)$  and  $(Y, \sigma)$  (or simply  $X$  and  $Y$ ) always mean topological spaces in which no separation axioms are assumed, unless explicitly stated and if  $I$  is and ideal on  $X$ ,  $(X, \tau, I)$  mean an ideal topological space. For a subset  $A$  of  $(X, \tau)$ ,  $Cl(A)$  and  $\text{int}(A)$  denote the closure of  $A$  with respect to  $\tau$  and the interior of  $A$  with respect to  $\tau$ , respectively. A subset  $A$  is said to be regular open [15] (resp. semiopen [11], preopen[13], semi preopen [4]) if  $A = \text{int}(Cl(A))$  (resp.  $A \subseteq Cl(\text{int}(A))$ ,  $A \subseteq \text{int}(Cl(A))$ ,  $A \subseteq Cl(\text{int}(Cl(A)))$ ). The complement of regular open (resp. semiopen, semi-preopen) set is called regular closed (resp. semiclosed, semi-preclosed) set. A subset  $S$  of  $(X, \tau, I)$  is an  $I$ -open[10], if  $S \subseteq \text{int}(S^*)$ . The complement of an  $I$ -open set is called  $I$ -closed set. The  $I$ -closure and the  $I$ -interior, can be defined in the same way

as  $Cl(A)$  and  $\text{int}(A)$ , respectively, will be denoted by  $I Cl(A)$  and  $I \text{int}(A)$ , respectively. The family of all  $I$ -open (resp.  $I$ -closed, regular open, regular closed, semiopen, semi closed, preopen, semi-preclosed) subsets of a  $(X, \tau, I)$ , denoted by  $IO(X)$  (resp.  $IC(X)$ ,  $RO(X)$ ,  $RC(X)$ ,  $SO(X)$ ,  $SC(X)$ ,  $PO(X)$ ,  $SPO(X)$ ,  $SPC(X)$ ). We set  $IO(X, x) = \{A : A \in IO(X) \text{ and } x \in A\}$ . It is well known that in a topological space  $(X, \tau, I)$ ,  $X^* \subseteq X$  but if the ideal is codense, that is  $\tau \cap I = \emptyset$ , then  $X^* \subseteq X$ .

By a multifunction  $F : X \rightarrow Y$ , we mean a point-to-set correspondence from  $X$  into  $Y$ , also we always assume that  $F(x) \neq \emptyset$  for all  $x \in X$ . For a multifunction  $F : X \rightarrow Y$ , the upper and lower inverse of any subset  $A$  of  $Y$  by  $F^+(A)$  and  $F^-(A)$ , respectively, that is  $F^+(A) = \{x \in X : F(x) \subseteq A\}$  and  $F^-(A) = \{x \in X : F(x) \cap A \neq \emptyset\}$ . In particular,  $F^+(y) = \{x \in X : y \in F(x)\}$  for each point  $y \in Y$ .

**Definition 1.** [2] A multifunction  $F : (X, \tau, I) \rightarrow (Y, \sigma)$  is said to be

1. upper  $I$ -continuous if for each  $x \in X$  and each open set  $V$  of  $Y$  such that  $x \in F^+(V)$ , there exists an  $I$ -open set  $U$  containing  $x$  such that  $U \subseteq F^+(V)$ .
2. lower  $I$ -continuous if for each  $x \in X$  and each open set  $V$  of  $Y$  such that  $x \in F^-(V)$ , there exists an  $I$ -open set  $U$  containing  $x$  such that  $U \subseteq F^-(V)$ .
3.  $I$ -continuous if it is both upper  $I$ -continuous and lower  $I$ -continuous.

**Definition 2.** [6] A multifunction  $F : (X, \tau) \rightarrow (Y, \sigma)$  is said to be

1. upper semi continuous at a point  $x \in X$  if for each open set  $V$  of  $Y$  with  $F(x) \in V$ , there exists an open set  $U$  containing  $x$  such that  $F(U) \subseteq V$ .
2. lower semi continuous at a point  $x \in X$  if for each open set  $V$  of  $Y$  with  $F(x) \cap V \neq \emptyset$ , there exists an open set  $U$  containing  $x$  such that  $F(a) \cap V \neq \emptyset$  for all  $a \in U$ .

**Definition 3.** A subset  $A$  of a topological space  $(X, \tau)$  is said to be  $N$ -closed [6] if every cover of  $A$  by regular open sets of  $X$  has a finite subcover.

**Definition 4.** [8] A multifunction  $F : (X, \tau) \rightarrow (Y, \sigma)$  is said to be:

1. upper nearly continuous at a point  $x \in X$  if for each open set  $V$  containing  $F(x)$  and having  $N$ -closed complement, there exists an open set  $U$  containing  $x$  such that  $F(U) \subset V$ .

2. lower nearly continuous at a point  $x \in X$  if for each open set  $V$  of  $Y$  meeting  $F(x)$  and having  $N$ -closed complement, there exists an open set  $U$  of  $X$  containing  $x$  such that  $F(u) \cap V \neq \emptyset$  for each  $u \in U$ .
3. upper (resp. lower) nearly continuous on  $X$  if it has this property at every point of  $X$ .

**Definition 5.** [9] A multifunction  $F : (X, \tau) \rightarrow (Y, \sigma)$  is said to be:

1. upper almost nearly continuous at a point  $x \in X$  if for each open set  $V$  containing  $F(x)$  and having  $N$ -closed complement, there exists an open set  $U$  containing  $x$  such that  $F(U) \subset \text{int}(Cl(V))$ .
2. lower almost nearly continuous at a point  $x \in X$  if for each open set  $V$  of  $Y$  meeting  $F(x)$  and having  $N$ -closed complement, there exists an open set  $U$  of  $X$  containing  $x$  such that  $F(u) \cap \text{int}(Cl(V)) \neq \emptyset$  for each  $u \in U$ .
3. upper (resp. lower) almost nearly continuous on  $X$  if it has this property at every point of  $X$ .

**Definition 6.** [5] A multifunction  $F : (X, \tau, I) \rightarrow (Y, \sigma)$  is said to be:

1. upper nearly  $I$ -continuous at a point  $x \in X$  if for each open set  $V$  containing  $F(x)$  and having  $N$ -closed complement, there exists an  $I$ -open set  $U$  containing  $x$  such that  $F(U) \subset V$ .
2. lower nearly  $I$ -continuous at a point  $x \in X$  if for each open set  $V$  of  $Y$  meeting  $F(x)$  and having  $N$ -closed complement, there exists an  $I$ -open set  $U$  of  $X$  containing  $x$  such that  $F(u) \cap V \neq \emptyset$  for each  $u \in U$ .
3. upper (resp. lower) nearly  $I$ -continuous on  $X$  if it has this property at every point of  $X$ .

**Definition 7.** [7] A multifunction  $F : (X, \tau, I) \rightarrow (Y, \sigma)$  is said to be:

1. upper almost nearly  $I$ -continuous at a point  $x \in X$  if for each open set  $V$  containing  $F(x)$  and having  $N$ -closed complement, there exists an  $I$ -open set  $U$  containing  $x$  such that  $F(U) \subset \text{int}(Cl(V))$ .
2. lower almost nearly  $I$ -continuous at a point  $x \in X$  if for each open set  $V$  of  $Y$  meeting  $F(x)$  and having  $N$ -closed complement, there exists an  $I$ -open set  $U$  of  $X$  containing  $x$  such that  $F(u) \cap \text{int}(Cl(V)) \neq \emptyset$  for each  $u \in U$ .
3. upper (resp. lower) almost nearly  $I$ -continuous on  $X$  if it has this property at every point of  $X$ .

### 3. Upper and Lower Nearly $I$ -Continuous Multifunctions

**Definition 8.** A multifunction  $F : (X, \tau, I) \rightarrow (Y, \sigma, J)$  is said to be:

1. upper nearly  $(I, J)$ -continuous at a point  $x \in X$  if for each  $J$ -open set  $V$  containing  $F(x)$  and having  $N$ -closed complement, there exists an  $I$ -open set  $U$  containing  $x$  such that  $F(U) \subset V$ .
2. lower nearly  $(I, J)$ -continuous at a point  $x \in X$  if for each  $J$ -open set  $V$  of  $Y$  meeting  $F(x)$  and having  $N$ -closed complement, there exists an  $I$ -open set  $U$  of  $X$  containing  $x$  such that  $F(u) \cap V \neq \emptyset$  for each  $u \in U$ .
3. upper (resp. lower) nearly  $(I, J)$ -continuous on  $X$  if it has this property at every point of  $X$ .

**Example 3.1.** Let  $X = Y = \{a, b, c\}$  with two topologies  $\tau = \{\emptyset, X, \{b\}\}$   $\sigma = \{\emptyset, Y, \{a\}\}$  and two ideals  $I = \{\emptyset, \{a\}\}$ ,  $J = \{\emptyset, \{b\}\}$ . Define a multifunction  $f : (X, \tau, I) \rightarrow (Y, \sigma, J)$  as follows:  $f(a) = \{a\}$ ,  $f(b) = \{c\}$  and  $f(c) = \{b\}$ . It is easy to see that:

The set of all  $I$ -open is  $\{\emptyset, X, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$ .

The set of all  $J$ -open is  $\{\emptyset, \{a, b\}, \{a, c\}\}$ .

In consequence,  $f$  is upper nearly  $(I, J)$ -continuous on  $X$ .

**Example 3.2.** Let  $X = Y = \{a, b, c\}$  with two topologies  $\tau = \{\emptyset, X, \{b, c\}\}$   $= \sigma$  and two ideals  $I = \{\emptyset, \{b\}\} = J$ . Define a multifunction  $f : (X, \tau, I) \rightarrow (Y, \sigma, J)$  as follows:  $f(a) = \{a\}$ ,  $f(b) = \{c\}$  and  $f(c) = \{b\}$ . It is easy to see that:

The set of all  $I$ -open is  $\{\emptyset, X, \{c\}, \{a, c\}, \{b, c\}\}$ .

In consequence,  $f$  is not upper nearly  $(I, J)$ -continuous.

Recall that if  $(X, \tau, I)$  is an ideal topological space and  $I$  is the empty ideal, then for each  $A \subseteq X$ ,  $A^* = cl(A)$ , that is to said, every  $I$ -open set is a preopen set, in consequence, if  $f : (X, \tau, I) \rightarrow (Y, \sigma, \{\emptyset\})$  is upper nearly  $(I, \{\emptyset\})$ -continuous, then  $f$  is upper nearly  $I$ -continuous.

**Example 3.3.**  $f : (X, \tau, I) \rightarrow (Y, \sigma)$  upper nearly  $I$ -continuous but  $f : (X, \tau, I) \rightarrow (Y, \sigma, \{\emptyset\})$  is not upper nearly  $(I, \{\emptyset\})$ -continuous.

Now consider  $(X, \tau, I)$  and  $(Y, \sigma, J)$  two ideals topological spaces, If  $J \neq \{\emptyset\}$ , then the concepts of upper nearly  $(I, J)$ -continuous and upper nearly  $I$ -continuous are independent, as we can see in the following examples.

**Example 3.4.** In the Example 3.1, the multifunction  $f$  is upper nearly  $(I, J)$ -continuous on  $X$  but is not upper nearly  $I$ -continuous on  $X$ .

**Example 3.5.** In the Example 3.2, the multifunction  $f$  is upper nearly  $I$ -continuous on  $X$  but is not upper nearly  $(I, J)$ -continuous on  $X$ .

**Example 3.6.** Let  $\mathbb{R}$  be the set of real numbers with the discrete topology  $\tau_d$  and  $I = \{\emptyset\} = J$ . Consider the multifunction  $F : (\mathbb{R}, \tau_d, I) \rightarrow (\mathbb{R}, \tau_d, J)$  defined as follows:  $F(x) = \{x\}$  for all  $x \in \mathbb{R}$ . It is easy to see that:  $F$  is upper (resp. lower) nearly  $(I, J)$ -continuous on  $X$ .

**Remark 3.1.** It is easy to see that if  $F : (X, \tau, I) \rightarrow (Y, \sigma, J)$  is a multifunction and  $JO(Y) \subset \sigma$ . If  $F$  is upper (lower) nearly  $I$ -continuous, then  $F$  is upper (lower) nearly  $(I, J)$ -continuous. Even more, if  $F : (X, \tau, I) \rightarrow (Y, \sigma, J)$  is a multifunction and  $JO(Y) \subset \sigma$ , we can find upper (resp. lower) nearly  $(I, J)$ -continuous on  $X$  that are not upper (lower) nearly  $I$ -continuous.

The following theorem characterize the upper nearly  $(I, J)$  continuous multifunctions.

**Theorem 9.** For a multifunction  $F : (X, \tau, I) \rightarrow (Y, \sigma, J)$ , the following statements are equivalent:

1.  $F$  is upper nearly  $(I, J)$ -continuous.
2.  $F^+(V)$  is  $I$ -open for each  $J$ -open set  $V$  of  $Y$  having  $N$ -closed complement.
3.  $F^-(K)$  is  $I$ -closed for every  $N$ -closed and  $J$ -closed subset  $K$  of  $Y$ .
4.  $I Cl(F^-(B)) \subset F^-(J Cl(B))$  for every subset  $B$  of  $Y$  having  $N$ -closed  $J$ -closure.
5.  $F^+(Jint(B)) \subset Iint(F^+(B))$  for every subset  $B$  of  $Y$  such that  $Y \setminus Jint(B)$  is  $N$ -closed.

*Proof.* (1) $\Rightarrow$ (2): Let  $x \in F^+(V)$  and  $V$  be any  $J$ -open set of  $Y$  having  $N$ -closed complement. From (1), there exists an  $I$ -open set  $U_x$  containing  $x$  such that  $U_x \subset F^+(V)$ . It follows that  $F^+(V) = \bigcup_{x \in F^+(V)} U_x$ . Since any union

of  $I$ -open sets is  $I$ -open,  $F^+(V)$  is  $I$ -open in  $(X, \tau)$ .

(2) $\Rightarrow$ (3): Let  $K$  be any  $N$ -closed and  $J$ -closed set of  $Y$ . Then by (2),  $F^+(Y \setminus K) = X \setminus F^-(K)$  is an  $I$ -open set. Then it is obtained that  $F^-(K)$  is an  $I$ -closed set.

(3) $\Rightarrow$ (4): Let  $B$  be any subset of  $Y$  having  $N$ -closed  $J$ -closure. By (3), we have

$$F^-(B) \subset F^-(J Cl(B)) = I Cl(F^-(J Cl(B))).$$

Hence  $I Cl(F^-(B)) \subset I Cl(F^-(J Cl(B))) = F^-(J Cl(B))$ .

(4) $\Rightarrow$ (5): Let  $B$  be a subset of  $Y$  such that  $Y \setminus J\text{int}(B)$  is  $N$ -closed. Then by (4), we have

$$\begin{aligned} X \setminus \text{Int}(F^+(B)) &= I \text{Cl}(X \setminus F^+(B)) = I \text{Cl}(F^-(Y \setminus B)) \subset F^-(J \text{Cl}(Y \setminus B)) \\ &= F^-(Y \setminus J\text{int}(B)) = X \setminus F^+(J\text{int}(B)). \end{aligned}$$

Therefore, we obtain  $F^+(J\text{int}(B)) \subset \text{Int}(F^+(B))$ .

(5) $\Rightarrow$ (1): Let  $x \in X$  and  $V$  be any  $J$ -open set of  $Y$  containing  $F(x)$  and having  $N$ -closed complement. Then by (5),  $x \in F^+(V) = F^+(J\text{int}(V)) \subset \text{Int}(F^+(V))$ . There exists an  $I$ -open set  $U$  containing  $x$  such that  $U \subset F^+(V)$ ; hence  $F(U) \subset V$ . This shows that  $F$  is upper nearly  $I$ -continuous.  $\square$

**Theorem 10.** For a multifunction  $F : (X, \tau, I) \rightarrow (Y, \sigma, J)$ , the following statements are equivalent:

1.  $F$  is lower nearly  $(I, J)$ -continuous.
2.  $F^-(V)$  is  $I$ -open for each  $J$ -open set  $V$  of  $Y$  having  $N$ -closed complement.
3.  $F^+(K)$  is  $I$ -closed for every  $N$ -closed and  $J$ -closed set  $K$  of  $Y$ .
4.  $I \text{Cl}(F^+(B)) \subset F^+(J\text{cl}(B))$  for every subset  $B$  of  $Y$  having  $N$ -closed closure.
5.  $F^-(J\text{int}(B)) \subset \text{Int}(F^-(B))$  for every subset  $B$  of  $Y$  such that  $Y \setminus J\text{int}(B)$  is  $N$ -closed.

*Proof.* The proof is similar to that of Theorem 9.  $\square$

**Corollary 11.** A multifunction  $F : (X, \tau, I) \rightarrow (Y, \sigma, J)$  is upper nearly  $(I, J)$ -continuous (resp. lower nearly  $(I, J)$ -continuous) if  $F^-(K)$  is  $I$ -closed (resp.  $F^+(K)$  is  $I$ -closed) for every  $N$ -closed set  $K$  of  $Y$ .

*Proof.* Let  $G$  be any  $J$ -open set of  $Y$  having  $N$ -closed complement. Then  $Y \setminus G$  is  $N$ -closed. By the hypothesis,

$$X \setminus F^+(G) = F^-(Y \setminus G) = \text{Int}(F^-(Y \setminus G)) = I \text{Cl}(X \setminus F^+(G)) = X \setminus \text{Int}(F^+(G))$$

and hence,  $F^+(G) = \text{Int}(F^+(G))$ . It follows from Theorem 9, that  $F$  is upper nearly  $(I, J)$ -continuous. The proof of lower nearly  $(I, J)$ -continuous is entirely similar.  $\square$

**Definition 12.** A multifunction  $F : (X, \tau, I) \rightarrow (Y, \sigma, J)$  is said to be:

1. upper  $(I, J)$ -continuous at a point  $x \in X$  if for each  $J$ -open set  $V$  containing  $F(x)$ , there exists an  $I$ -open set  $U$  containing  $x$  such that  $F(U) \subset V$ .
2. lower  $(I, J)$ -continuous at a point  $x \in X$  if for each  $J$ -open set  $V$  of  $Y$  meeting  $F(x)$ , there exists an  $I$ -open set  $U$  of  $X$  containing  $x$  such that  $F(u) \cap V \neq \emptyset$  for each  $u \in U$ .
3. upper (resp. lower)  $(I, J)$ -continuous on  $X$  if it has this property at every point of  $X$ .

**Example 3.7.** *The Multifunction defined in Example 3.6 is upper nearly  $(I, J)$ -continuous on  $X$  but is not upper  $(I, J)$ -continuous on  $X$ .*

**Remark 3.2.** *Every upper (resp. lower)  $(I, J)$ -continuous multifunction on  $X$  is upper (resp. lower) nearly  $(I, J)$ -continuous multifunction on  $X$ , but the converse is not necessarily true, as we can see in the following example.*

**Example 3.8.** *The Multifunction defined in Example 3.1 is upper nearly  $(I, J)$ -continuous on  $X$  but is not upper  $(I, J)$ -continuous.*

**Theorem 13.** *For a multifunction  $F : (X, \tau, I) \rightarrow (Y, \sigma, J)$ , the following statements are equivalent:*

1.  $F$  is upper  $(I, J)$ -continuous.
2.  $F^+(V)$  is  $I$ -open for each  $J$ -open set  $V$  of  $Y$ .
3.  $F^-(K)$  is  $I$ -closed for every  $J$ -closed subset  $K$  of  $Y$ .
4.  $I \text{ Cl}(F^-(B)) \subset F^-(J \text{ Cl}(B))$  for every subset  $B$  of  $Y$ .
5. For each point  $x \in X$  and each  $J$ -open set  $V$  containing  $F(x)$ ,  $F^+(V)$  is an  $I$ -open containing  $x$ .
6. For each point  $x \in X$  and each  $J$ -open set containing  $F(x)$ , there exist an  $I$ -open set  $U$  containing  $x$  such that  $F(U) \subseteq V$ .

*Proof.* The proof is similar to that of Theorem 9. □

**Theorem 14.** *Let  $F : (X, \tau, I) \rightarrow (Y, \sigma, J)$  and  $G : (Y, \sigma, J) \rightarrow (Z, \beta, K)$  be multifunctions. If  $F$  is upper nearly  $(I, J)$ -continuous (upper  $(I, J)$ -continuous) and  $G$  upper  $(I, J)$ -continuous (upper nearly  $(I, J)$ -continuous), then  $F \circ G : (X, \tau, I) \rightarrow (Z, \beta, K)$  is upper nearly  $(I, J)$ -continuous.*

**Definition 15.** An ideal topological space  $(X, \tau, I)$  is said to be  $I$ -compact[3] if every cover of  $X$  by  $I$ -open sets have a finite subcover.



**Definition 16.** A multifunction  $F : (X, \tau, I) \rightarrow (Y, \sigma, J)$  is said to be:

1. upper  $(I, J)$ -irresolute at a point  $x \in X$  if for each  $I$ -open set  $U$  containing  $x$ , there exists an  $I$ -open set  $V$  containing  $F(x)$  such that  $F(U) \subset V$ .
2. lower  $(I, J)$ -irresolute at a point  $x \in X$  if for each  $J$ -open set  $V$  of  $Y$  meeting  $F(x)$ , there exists an  $I$ -open set  $U$  containing  $x$  such that  $U \subseteq F^{-}(V)$ .
3. upper (resp. lower)  $(I, J)$ -irresolute on  $X$  if it has this property at every point of  $X$ .

**Theorem 17.** Let  $F : (X, \tau, I) \rightarrow (Y, \sigma, J)$  be a surjective  $(I, J)$ -irresolute multifunction such that  $F(x)$  is  $J$ -compact for each  $x \in X$ . If  $(X, \tau, I)$  is  $I$ -compact, then  $(Y, \sigma, J)$  is  $J$ -compact.

*Proof.* Let  $\{V_i : i \in \Delta\}$  be a  $J$ -open cover of  $Y$ . For each  $x \in X$ , there exists a finite subset  $\Delta(x)$  of  $\Delta$  such that  $F(x) \subseteq \bigcup\{V_i : i \in \Delta(x)\}$ . Consider  $V(x) = \bigcup\{V_i : i \in \Delta(x)\}$ . Then  $F(x) \subseteq V(x) \in JO(Y)$ . using the fact that  $F$  is  $(I, J)$ -irresolute, then there exist an  $U(x) \in IO(X)$  such that  $F(U(x)) \subset V(x)$ . Now using the that  $F$  is surjective, then the collection  $\{U(x) : x \in X\}$  is an  $I$ -open cover of  $X$ . In consequence, there exists a finite number of points of  $X$ , say,  $x_1, x_2, \dots, x_n$  such that  $X = \bigcup_{i=1}^n U(x_i)$ . It follows that  $F(X) = F(\bigcup_{i=1}^n U(x_i)) \subseteq \bigcup_{i=1}^n F(U(x_i)) \subseteq \bigcup_{i=1}^n V(x_i) \subseteq \bigcup_{i=1}^n \bigcup_{i \in \Delta(x_i)} U(x_i)$ . It follows that  $Y$  is  $J$ -compact.  $\square$

**Definition 18.** A multifunction  $f : (X, \tau, I) \rightarrow (Y, \sigma, J)$  is said to be:

1. upper almost nearly  $(I, J)$ -continuous at a point  $x \in X$  if for each  $J$ -open set  $V$  containing  $F(x)$  and having  $N$ -closed complement, there exists an  $I$ -open set  $U$  containing  $x$  such that  $F(U) \subset \text{int}(J Cl(V))$ .
2. lower almost nearly  $(I, J)$ -continuous at a point  $x \in X$  if for each  $J$ -open set  $V$  of  $Y$  meeting  $F(x)$  and having  $N$ -closed complement, there exists an  $I$ -open set  $U$  of  $X$  containing  $x$  such that  $F(u) \cap \text{int}(J Cl(V)) \neq \emptyset$  for each  $u \in U$ .
3. upper (resp. lower) almost nearly  $(I, J)$ -continuous on  $X$  if it has this property at every point of  $X$ .

**Example 3.9.** Let  $X = \mathbb{R}$  the set of real numbers with the topology  $\tau = \{\emptyset, \mathbb{R}, \mathbb{R} \setminus \mathbb{Q}\}$ ,  $Y = \mathbb{R}$  with the topology  $\sigma = \{\emptyset, \mathbb{R}, \mathbb{Q}\}$  and  $I = \{\emptyset\} = J$ . Define  $F : (X, \tau, I) \rightarrow (Y, \sigma, J)$  as follows:  $F(x) = \mathbb{Q}$  if  $x \in \mathbb{Q}$  and  $F(x) = \mathbb{R} \setminus \mathbb{Q}$

if  $x \in \mathbb{R} \setminus \mathbb{Q}$ . It is easy to see that  $F$  is upper (resp. lower) almost nearly  $(I, J)$ -continuous on  $X$ .

It is clear that every upper (resp. lower)  $(I, J)$ -continuous multifunction is upper (resp. lower) nearly  $(I, J)$ -continuous multifunction and every upper (resp. lower) nearly  $(I, J)$ -continuous multifunction is upper (resp. lower) almost nearly  $(I, J)$ -continuous multifunction but the converse in both cases is not true in general as shown in the following examples.

**Example 3.10.** Let  $\mathbb{R}$  with the finite complement topology  $\tau_c$  and with the discrete topology, take  $I = \{\emptyset\} = J$ . Consider the multifunction  $F : (\mathbb{R}, \tau_c, I) \rightarrow (\mathbb{R}, \tau_d, J)$  defined as follows:  $F(x) = \{x\}$  for all  $x \in \mathbb{R}$ . It is easy to see that:  $F$  is upper (resp. lower) nearly  $(I, J)$ -continuous on  $X$  but is not upper (resp. lower)  $(I, J)$ -continuous on  $X$ .

**Example 3.11.** The multifunction  $F$  defined in Example 3.9 is upper (resp. lower) nearly almost  $(I, J)$ -continuous on  $X$  but is not upper (resp. lower) nearly  $(I, J)$ -continuous on  $X$ .

At this point, there is a question. Given a multifunction  $F : (X, \tau, I) \rightarrow (Y, \sigma)$ . It is possible to write a characterization for upper (resp. lower) nearly almost  $(I, J)$ -continuous on  $X$ .

## References

- [1] M.E. Abd El-Monsef, E.F. Lashien, A.A. Nasef, On  $I$ -open sets and  $I$ -continuous functions *Kyungpook Math. J.*, **32**, No. 1 (1992), 21-30.
- [2] M. Akdag, On upper and lower  $I$ -continuous multifunctions, *Far East J. Math. Sci.*, **25**, No. 1 (2007), 49-57.
- [3] A. Al-Omari and M. S. M. Noorani, Contra- $I$ -continuous and almost  $I$ -continuous functions, *Int. J. Math. Math. Sci.*, bf 9 (2007), 169-179, doi: 10.1155/2007/40469.
- [4] D. Andrijevic, Semi-preopen sets, *Mat. Vesnik*, **38** (1986), 24-32.
- [5] C. Arivazhagi and N. Rajesh, Nearly  $J$ -continuous multifunctions, *Bol. Soc. Paran. Mat.*, **37**, No. 1 (2019), 33-38.
- [6] D. Carnahan, Locally nearly compact spaces, *Boll. Un. Mat. Ital.*, **4**, No. 6 (1972), 143-153.
- [7] C. Carpintero, E. Rosas, J. Moreno, More on upper and lower almost nearly  $I$ -continuous multifunctions, *International Journal of Pure and Applied Mathematics*, To appear.
- [8] E. Ekici, Nearly continuous multifunctions, *Acta Math. Univ. Comenianae*, **72** (2003), 229-235.
- [9] E. Ekici, Almost nearly continuous multifunctions, *Acta Math. Univ. Comenianae*, **73** (2004), 175-186.

- [10] D.S. Jankovic, T.R. Hamlett, New topologies from old via ideals, *Amer. Math. Monthly*, **97**, No. 4 (1990), 295-310, **doi:** 10.2307/2324512.
- [11] N. Levine, Semi open sets and semi-continuity in topological spaces, *Amer. Math. Monthly*, **70** (1963), 36-41.
- [12] K. Kuratowski, *Topology*, Academic Press, New York, 1966.
- [13] A.S. Mashhour, M.E. Abd El-Monsef, S, N. El-Deep, on precontinuous and weak precontinuous mappings, *Proced. Phys. Soc. Egypt*, **53** (1982), 47-53.
- [14] R.E. Simithson, Almost and weak continuity for multifunctions, *Bull. Calcutta Math. Soc.*, bf **70** (1978), 383-390.
- [15] M. Stone, Applications of the theory of boolean rings to general topology, *Trans. Amer. Math. Soc.*, **41** (1937), 374-381.
- [16] R. Vaidyanathaswamy, The localisation theory in set topology, *Proc. Indian Acad. Sci.*, **20** (1945), 51-61.
- [17] I. Zorlutuna, *I*-continuous multifunctions, *Filomat*, **27**, No. 1 (2013), 155-162, **doi:** 10.2298/FIL1301165Z.

