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GENERATING SETS OF SEMIGROUPS OF PARTIAL TRANSFORMATIONS PRESERVING A ZIG-ZAG ORDER ON $\mathbb{N}$

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Abstract: This paper deals with the monoid $PF_{\mathbb{N}}$ of all partial transformations on $\mathbb{N}$ preserving a zig-zag order on $\mathbb{N}$. We determine the relative rank of $PF_{\mathbb{N}}$ modulo a set containing all idempotents and all surjections in $PF_{\mathbb{N}}$. Moreover, we show that all transformations in $PF_{\mathbb{N}}$ with finite rank can be generated by the idempotents with finite rank and the full transformation γ_0 with infinite rank, where γ_0 maps each natural number n to $n+2$.

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1. Introduction

An order-preserving (or isotone) mapping (or transformation) is the most natural morphism in the class of all partially ordered sets. It plays a role of a fun-

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damental notion in some theories (e.g., the theory of fixed points). Hence, the interest it causes seems to be not very strange. The set of all order-preserving full (partial, respectively) transformations forms a monoid under the usual composition of mappings. One of the very natural questions arising in relation to order-preserving transformations is: What are the minimal generating sets and what is the rank (i.e. what is the minimal size of a generating set)? The rank of the monoid of all order-preserving full transformations on an n-element chain is n and the rank of the monoid of all order-preserving partial transformations on an n-element chain is $2n-1$ [4].

In natural way, the interest focuses on sets which are "next" to chains with respect to a simplicity of their structure. Such posets are fences. Fences were first studied by J. D. Currie, T. I. Visentin, and A. Rutkowski. The exact numbers for such order-preserving full transformations on an n-element fence, where n is even, have been calculated (with the help of generating functions) in [1] and in [9], the author presented the exact formulas for even as well as for odd n. A minimal generating set as well as the rank of the monoid of all order-preserving transformations on an n-element fence was given in [3]. Moreover, Dimitrova and Koppitz have investigated the monoid of all order-preserving partial injections on an n-element fence. They determined the rank, whenever n is even [2]. Note that the monoid of all order-preserving (full) transformations on a fence is not regular. In [10], regular elements of this monoid are determined, whenever the fence is finite.

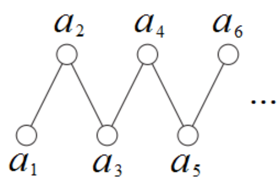
Fences on the countable infinite set $\mathbb{N}$ of the natural numbers were considered in [7]. The authors studied regular subsemigroups of the monoid of all order-preserving partial transformations on a fence on the set $\mathbb{N}$. In particular, they presented uncountable many maximal regular subsemigroups.

In [5], the authors studied rank properties of order-preserving transformations on infinite partially ordered sets. It is clear that the rank of a semigroup S is not of interest in the case where S is infinite. For any semigroup S with an order $k > \aleph_0$, all one can say is that the rank of S is k. Even the order of the monoid $PF_{\mathbb{N}}$ of all order-preserving partial transformations on the fence on $\mathbb{N}$ is $2^{\aleph_0}$. So its rank is also $2^{\aleph_0}$. There is, however, another notation, that yields some interesting results. For a semigroup S and a subset A of S, the relative rank of S modulo A is the minimal size of a subset B of S such that $A \cup B$ generates S. The notion of relative rank was introduced by Ruškuc [8]. In [6], Howie, Ruškuc and Higgins considered the relative ranks of the monoid of full transformations on an infinite set modulo some distinguished subsets.

In the present paper, we determine the relative rank of $PF_{\mathbb{N}}$ modulo a set containing all idempotents, all surjective transformations, all transformations with defect $\{1\}$, and a particular set of transformations with both infinite rank and infinite defect.

2. Preliminaries

Signs $<$ and $\leq$ denote the natural orders in the set of integers. For a finite set $A \subseteq \mathbb{N}$ by $\max A$ (by $\min A$, respectively), we mean the maximal (the minimal, respectively) element in A with respect to $<$. We will write $A < B$ (for subsets A and B of $\mathbb{N}$), whenever $a < b$ for all $a \in A$ and all $b \in B$. Signs $\prec$ and $\preceq$ correspond to the order in a (countable infinite) fence, i.e. a partially ordered set with a Hasse-diagram isomorphic to that presented in the figure below.



Elements of fences are always labeled as in that figure (i.e. "from left to right"). The definition of the partial order $\preceq$ is self-explanatory. As the matter of fact, if no misunderstanding occurs, we consider the set $\mathbb{N}$ of all natural numbers as underlying set of the countable infinite fence. Under these conditions, we observe that any natural number $n > 1$ is comparable with exactly two elements in $\mathbb{N}$, namely with $n-1$ and $n+1$, that means either $n-1 \prec n \succ n+1$ or $n-1 \succ n \prec n+1$. But 1 is only comparable with 2. We will write $A \parallel B$ (for subsets A and B of $\mathbb{N}$) if no element in A is comparable with any element in B.

Let us remember that a (partial) transformation $\alpha : \mathbb{N} \to \mathbb{N}$ is order-preserving if it satisfies $x \prec y \Rightarrow x\alpha \preceq y\alpha$ for each x and y in the domain of α. As already mentioned, the set $PF_{\mathbb{N}}$ of all these order-preserving partial transformations on $\mathbb{N}$ forms a monoid with the identical mapping $id_{\mathbb{N}}$ on $\mathbb{N}$ as identity. We denote by $\operatorname{dom} \alpha$, $\operatorname{im} \alpha$, and $d(\alpha) := \mathbb{N} \setminus \operatorname{im} \alpha$ the domain of α, the image (or range) of α, and the defect of α, respectively. The kernel of α is the equivalence relation $\ker \alpha := \{(x,y) : x, y \in \operatorname{dom} \alpha, x\alpha = y\alpha\}$. For any set A, we denote by $|A|$ the size (or cardinality) of A. In particular, we put $\operatorname{rank} \alpha := |\operatorname{im} \alpha|$ and $D(\alpha) := |d(\alpha)|$, where $\operatorname{rank} \alpha$ is said to be the rank of α. For a set $A \subseteq PF_{\mathbb{N}}$, we mean by $\langle A \rangle$ the subsemigroup of $PF_{\mathbb{N}}$ generated by A. Using these notions,

$$\operatorname{rank}\,(PF_{\mathbb{N}} : A) := \min\{|B| : \langle A \cup B \rangle = PF_{\mathbb{N}}\}$$

is the relative rank of $PF_{\mathbb{N}}$ modulo A.

A transformation $\alpha \in PF_{\mathbb{N}}$ is called idempotent if $\alpha\alpha = \alpha$ and let us denote by E the family of all idempotents in $PF_{\mathbb{N}}$. It is well known that any transformation is idempotent if and only if its restriction to its image is the identity (on its image). By $Sur^*(PF_{\mathbb{N}})$, we denote the set

$$Sur^*(PF_{\mathbb{N}}) := \{\alpha \in PF_{\mathbb{N}} : \mathbb{N} \setminus \{1\} \subseteq \text{im } \alpha\}.$$

Thus $Sur^*(PF_{\mathbb{N}})$ contains all surjective partial transformations in $PF_{\mathbb{N}}$ as well as all partial transformations in $PF_{\mathbb{N}}$ with defect $\{1\}$.

Recall, a subset Σ of $\mathbb{N}$ is said to be convex if $a, b \in \Sigma$ and $a < c < b$ implie $c \in \Sigma$. For a set $\Delta \subseteq \mathbb{N}$, let $f^*(\Delta)$ be the set of all convex subsets of $\mathbb{N} \setminus \Delta$ and let $f(\Delta)$ be the set of all maximal elements in $f^*(\Delta)$ with respect to the set inclusion. Then we put:

Definition 1. Let

1. $f_3(\Delta) := \{\Sigma \in f(\Delta) : |\Sigma| \geq 3\}$;
2. $F := \{\alpha \in PF_{\mathbb{N}} : D(\alpha) = \text{rank } \alpha = |\{|\Sigma| : \Sigma \in f(\text{im } \alpha)\}| = \aleph_0\}$;
3. $F_< := \{\alpha \in F : |f_3(\text{dom } \alpha)| < \aleph_0 \text{ and } \{1, 2\} \cap \text{im } \alpha \neq \emptyset\}$.

The set $F_<$ is a particular set of transformations with infinite rank as well as with infinite defect. From now on, whenever the capital letter A is used, it denotes the set

$$A := E \cup Sur^*(PF_{\mathbb{N}}) \cup F_<.$$

It is easy to verify that

$$\gamma_0 := \begin{pmatrix} 1 & 2 & 3 & \cdots \\ 3 & 4 & 5 & \cdots \end{pmatrix}$$

(i.e. $x\gamma_0 := x + 2$ for all $x \in \mathbb{N}$) belongs to $PF_{\mathbb{N}}$.

We will determine the relative rank of $PF_{\mathbb{N}}$ modulo A, in the next section.

3. Relative Rank of $PF_{\mathbb{N}}$

First, we will show that all transformations in $PF_{\mathbb{N}}$ with finite rank can be generated by the idempotents with finite rank and one full transformation with infinite rank, namely γ_0.

We observe that γ_0 is injective and that both γ_0 and γ_0^{-1} belong to $PF_{\mathbb{N}}$. This provides $(\gamma_0^{-1})^k \in PF_{\mathbb{N}}$ (for any $k \in \mathbb{N}$), where $(\gamma_0^{-1})^k = (\gamma_0^k)^{-1}$. Let

$$E_f := \{\alpha \in E : \text{rank } \alpha \in \mathbb{N}\}$$

be the set of all idempotents in $PF_{\mathbb{N}}$ with finite rank.

Lemma 2. $\langle E_f, \gamma_0 \rangle = \{\alpha \in PF_{\mathbb{N}} : \text{rank } \alpha \in \mathbb{N}\} \cup \{\gamma_0^k : k \in \mathbb{N}\}$.

Proof. Let $\alpha \in PF_{\mathbb{N}}$ with rank $\alpha = l \in \mathbb{N}$, say im $\alpha = \{a_1 < a_2 < \cdots < a_l\} \subseteq \mathbb{N}$. Let $m := a_l$. We define a partial transformation γ by

$$x\gamma := \begin{cases} x(\gamma_0^m)^{-1}\alpha & \text{if} \quad x \in (\text{dom } \alpha)\gamma_0^m \\ x & \text{if} \quad x \in \text{im } \alpha. \end{cases}$$

Since im $\alpha <$ (dom $\alpha)\gamma_0^m$, the mapping γ is well defined. We show that $\gamma \in PF_{\mathbb{N}}$. For this let $x, y \in \text{dom } \gamma = \text{im } \alpha \cup (\text{dom } \alpha)\gamma_0^m$ with $x \prec y$. Since $\min[(\text{dom } \alpha)\gamma_0^m] \geq 2m + 1 = 2a_l + 1 = 2(\max \text{im } \alpha) + 1$, we obtain that either $x, y \in \text{im } \alpha$ or $x, y \in (\text{dom } \alpha)\gamma_0^m$. If $x, y \in \text{im } \alpha$, then $x\gamma = x \prec y = y\gamma$. If $x, y \in (\text{dom } \alpha)\gamma_0^m$, then $x(\gamma_0^m)^{-1}\alpha \prec y(\gamma_0^m)^{-1}\alpha$ since $\alpha, (\gamma_0^m)^{-1} \in PF_{\mathbb{N}}$, i.e. $x\gamma \prec y\gamma$. Altogether, we have shown that $\gamma \in PF_{\mathbb{N}}$. Clearly, γ is idempotent and has finite rank. Therefore, $\gamma \in E_f$. We show now that $\alpha = \gamma_0^m\gamma$. For this let $x \in \text{dom } \alpha$. Then $x\gamma_0^m\gamma = x\gamma_0^m(\gamma_0^m)^{-1}\alpha = x\alpha$. Since dom $\gamma = \text{im } \alpha \cup (\text{dom } \alpha)\gamma_0^m$, we have dom $\gamma_0^m\gamma = (\text{dom } \gamma)(\gamma_0^m)^{-1} = (\text{im } \alpha \cup (\text{dom } \alpha)\gamma_0^m)(\gamma_0^m)^{-1} = (\text{dom } \alpha)\gamma_0^m(\gamma_0^m)^{-1} = \text{dom } \alpha$. This shows that $\alpha = \gamma_0^m\gamma \in \langle E_f, \gamma_0 \rangle$. Clearly, $\{\gamma_0^k : k \in \mathbb{N}\} = \langle \gamma_0 \rangle \subseteq \langle E_f, \gamma_0 \rangle$. This shows that $\{\alpha \in PF_{\mathbb{N}} : \text{rank } \alpha \in \mathbb{N}\} \cup \{\gamma_0^k : k \in \mathbb{N}\} \subseteq \langle E_f, \gamma_0 \rangle$. The converse inclusion follows form the fact that the product of two order-preserving transformations is an order-preserving transformation with finite rank, whenever one of the both transformations has a finite rank. $\square$

Corollary 3. *The relative rank of the semigroup* $\{\alpha \in PF_{\mathbb{N}} : \text{rank } \alpha \in \mathbb{N}\} \cup \{\gamma_0^k : k \in \mathbb{N}\}$ *modulo* E_f *is* 1.

The only surjective transformation generated by γ_0 and elements of E is the identity $id_{\mathbb{N}}$. In fact, take a surjective $\alpha \in PF_{\mathbb{N}} \setminus \{id_{\mathbb{N}}\}$. Then there are $\alpha_1, \alpha_2, \ldots, \alpha_n \in \{\gamma_0\} \cup (E \setminus \{id_{\mathbb{N}}\})$ such that $\alpha = \alpha_1\alpha_2 \cdots \alpha_n$. This implies im $\alpha_n = \mathbb{N}$, i.e. $\alpha_n \in E$. But the only idempotent surjective transformation is $id_{\mathbb{N}}$, a contradiction.

Now, we will show that all transformations in $PF_{\mathbb{N}}$ with finite defect as well as all $\alpha \in F$ with $|f_3(\text{dom } \alpha)| = \aleph_0$ can be generated by the full transformation γ_0, by elements of the set E of all idempotents, and by elements of the set

$$Sur(PF_{\mathbb{N}}) := \{\alpha \in PF_{\mathbb{N}} : \text{im } \alpha = \mathbb{N}\}.$$

For this let

$$D_f := \{\alpha \in PF_{\mathbb{N}} : D(f) \text{ is finite}\}$$

and

$$F_{\aleph_0} := \{\alpha \in F : |f_3(\text{dom } \alpha)| = \aleph_0\}.$$

Lemma 4. $F_{\aleph_0} \cup D_f \subseteq \langle Sur(PF_{\mathbb{N}}), E, \gamma_0 \rangle$.

Proof. First, let $\alpha \in D_f$. Then there is $n \in \mathbb{N}$ with $\max(d(\alpha)) = n$. Let

$$\begin{aligned} B \ &: \ = \{x \in \text{im } \alpha : x < n\} \text{ and} \\ \overline{B} \ &: \ = \{x \in \mathbb{N} : x > n\}. \end{aligned}$$

It is easy to see that im $\alpha = B \cup \overline{B}$. We define a partial transformation γ by

$$x\gamma := \begin{cases} x(\gamma_0^n)^{-1}\alpha & \text{if } \ x \in B\alpha^{-1}\gamma_0^n \\ x & \text{if } \ x \in B \cup \overline{B}\alpha^{-1}\gamma_0^n. \end{cases}$$

Clearly, dom $\gamma = B \cup (\text{dom } \alpha)\gamma_0^n$. Since $B < B\alpha^{-1}\gamma_0^n < \overline{B}\alpha^{-1}\gamma_0^n$, the partial transformation γ is well defined. Let $x \in B\alpha^{-1}\gamma_0^n$. Then $x(\gamma_0^n)^{-1} \in B\alpha^{-1}\gamma_0^n(\gamma_0^n)^{-1} = B\alpha^{-1}$ and $x(\gamma_0^n)^{-1}\alpha \in B\alpha^{-1}\alpha = B$. Hence, $x\gamma = x(\gamma_0^n)^{-1}\alpha \in B$. This shows that im $\alpha = B \cup \overline{B}\alpha^{-1}\gamma_0^n$, i.e. α is idempotent.
Now we show that $\gamma \in PF_{\mathbb{N}}$. For this let $x, y \in \text{dom } \gamma = B \cup (\text{dom } \alpha)\gamma_0^n$ with $x \prec y$. Since $1\gamma_0^n = 2n+1$, we calculate that either $x, y \in B$ or $x, y \in (\text{dom } \alpha)\gamma_0^n = B\alpha^{-1}\gamma_0^n \cup \overline{B}\alpha^{-1}\gamma_0^n$. Since $B < \{n\} < \overline{B}$, we have $B\alpha^{-1} \parallel \overline{B}\alpha^{-1}$ and thus $B\alpha^{-1}\gamma_0^n \parallel \overline{B}\alpha^{-1}\gamma_0^n$ since $(\gamma_0^n)^{-1}, \alpha \in PF_{\mathbb{N}}$. This provides $x, y \in B$ or $x, y \in B\alpha^{-1}\gamma_0^n$ or $x, y \in \overline{B}\alpha^{-1}\gamma_0^n$. If $x, y \in B$ or $x, y \in \overline{B}\alpha^{-1}\gamma_0^n$ then $x\gamma = x \prec y = y\gamma$. Suppose now that $x, y \in B\alpha^{-1}\gamma_0^n$. Note, $x \prec y$ and $\alpha, (\gamma_0^n)^{-1} \in PF_{\mathbb{N}}$ implies $x(\gamma_0^n)^{-1}\alpha \prec y(\gamma_0^n)^{-1}\alpha$, i.e. $x\gamma \prec y\gamma$. Altogether, we have shown that $\gamma \in E$.
Now we define a partial transformation δ by

$$x\delta := \begin{cases} x & \text{if } \ x \in \{1, \ldots, n\} \\ x(\gamma_0^n)^{-1}\alpha & \text{if } \ x \in \overline{B}\alpha^{-1}\gamma_0^n. \end{cases}$$

Since $\{1, \ldots, n\} < (\text{dom } \alpha)\gamma_0^n$, the partial transformation δ is well defined. We show that $\delta \in PF_{\mathbb{N}}$. For this let $x, y \in \text{dom } \delta = \{1, \ldots, n\} \cup \overline{B}\alpha^{-1}\gamma_0^n$ with $x \prec y$. We observe that $n+1 \notin \text{dom } \delta$. Thus, either $x, y \in \{1, \ldots, n\}$ or $x, y \in \overline{B}\alpha^{-1}\gamma_0^n$. If $x, y \in \{1, \ldots, n\}$ then $x\delta = x \prec y = y\delta$. If $x, y \in \overline{B}\alpha^{-1}\gamma_0^n$ then $x\delta = x(\gamma_0^n)^{-1}\alpha \prec y(\gamma_0^n)^{-1}\alpha = y\delta$ since $\alpha, (\gamma_0^n)^{-1} \in PF_{\mathbb{N}}$.
Now we show that δ is surjective. We have

$$\begin{aligned}\text{im }\delta = \{1,\ldots,n\}\delta \cup (\overline{B}\alpha^{-1}\gamma_0^n)\delta &= \{1,\ldots,n\}\delta \ \cup\ \overline{B}\alpha^{-1}\gamma_0^n(\gamma_0^n)^{-1}\alpha \\ &= \{1,\ldots,n\}\ \cup\ \overline{B}id_{\text{im }\alpha} = \{1,\ldots,n\} \cup \overline{B} = \mathbb{N},\end{aligned}$$

since $\overline{B} \subseteq \text{im }\alpha$. Thus, $\delta \in Sur(PF_{\mathbb{N}})$.
It remains to show that $\alpha = \gamma_0^n\gamma\delta$. Let $x \in \text{dom }\alpha = B\alpha^{-1} \cup \overline{B}\alpha^{-1}$. If $x \in B\alpha^{-1}$ then $x\gamma_0^n\gamma\delta = x\gamma_0^n(\gamma_0^n)^{-1}\alpha\delta = x\alpha\delta = x\alpha$ since $x\alpha \in B\alpha^{-1}\alpha = B \subseteq \{1,\ldots,n\}$. If $x \in \overline{B}\alpha^{-1}$ then $x\gamma_0^n\gamma\delta = x\gamma_0^n\delta = x\gamma_0^n(\gamma_0^n)^{-1}\alpha = x\alpha$. From $\text{dom }\gamma = (\text{dom }\alpha)\gamma_0^n \cup B$, we obtain $\text{dom }\gamma_0^n\gamma\delta \subseteq \text{dom }\gamma_0^n\gamma = (\text{dom }\gamma)(\gamma_0^n)^{-1} = ((\text{dom }\alpha)\gamma_0^n \cup B)(\gamma_0^n)^{-1} = \text{dom }\alpha$.
Altogether, we have shown that $\alpha = \gamma_0^n\gamma\delta \in \langle Sur(PF_{\mathbb{N}}), E, \gamma_0\rangle$.

Let now $\alpha \in F_{\aleph_0}$. Then $D(\alpha) = |f_3(\text{dom }\alpha)| = \aleph_0$ and there is a bijection

$$g : d(\alpha) \to f_3(\text{dom }\alpha).$$

Since $|\Sigma| \geq 3$ for all $\Sigma \in f_3(\text{dom }\alpha)$, there is an injection

$$h : f_3(\text{dom }\alpha) \to \mathbb{N} \setminus \text{dom }\alpha$$

with $h(\Sigma)+1, h(\Sigma), h(\Sigma)-1 \in \Sigma$ for all $\Sigma \in f_3(\text{dom }\alpha)$. We define a partial transformation β by

$$x\beta := \begin{cases} x\alpha & \text{if } \ x \in \text{dom }\alpha \\ g^{-1}(h^{-1}(x)) & \text{if } \ x \in h(g(d(\alpha))). \end{cases}$$

Clearly, β is well defined. We show that $\beta \in PF_{\mathbb{N}}$. For this let $x, y \in \text{dom }\beta = \text{dom }\alpha \cup h(g(d(\alpha)))$ with $x \prec y$. We observe that $z+1, z-1 \notin \text{dom }\alpha \cup h(g(d(\alpha)))$ for any $z \in h(g(d(\alpha)))$. Hence, $x, y \in \text{dom }\alpha$ and we have $x\beta = x\alpha \prec y\alpha = y\beta$ since $\alpha \in PF_{\mathbb{N}}$. Moreover, we have $\text{im }\beta = (\text{dom }\alpha \cup h(g(d(\alpha))))\beta = (\text{dom }\alpha)\beta \cup (h(g(d(\alpha))))\beta = (\text{dom }\alpha)\alpha \cup g^{-1}(h^{-1}(h(g(d(\alpha))))) = \text{im }\alpha \cup d(\alpha) = \mathbb{N}$. This shows that $\beta \in Sur(PF_{\mathbb{N}})$.
Finally, we note that the identity $id_{\text{im }\alpha}$ on $\text{im }\alpha$ is idempotent and order-preserving. It is easy to verify
$\alpha = \beta id_{\text{im }\alpha} \in \langle Sur(PF_{\mathbb{N}}), E\rangle \subseteq \langle Sur(PF_{\mathbb{N}}), E, \gamma_0\rangle$. □

We have to note that $F_< \cap \langle Sur^*(PF_{\mathbb{N}}), E, \gamma_0\rangle \neq \emptyset$, but not all elements in $F_<$ can be generated by elements of the set $Sur^*(PF_{\mathbb{N}}) \cup E \cup \{\gamma_0\}$. For example, let us take the partial transformation η with $\text{dom }\eta = 2\mathbb{N}$ and $(2i)\eta := i^2$ for $i \in \mathbb{N}$. In order to check that η can not be generated by elements of $Sur^*(PF_{\mathbb{N}}) \cup E \cup \{\gamma_0\}$, we note that if $\{a_i : i \in \mathbb{N}\}$ is a subset of $\mathbb{N}$ such that $\{|a_i - a_{i+1}| : i \in \mathbb{N}\}$ is finite then $\{|a_i\alpha - a_{i+1}\alpha| : i \in \mathbb{N}\}$ is finite, whenever

$\alpha \in Sur^*(PF_{\mathbb{N}}) \cup E \cup \{\gamma_0\}$. On the other hand, we have $\{|2i - 2(i+1)| : i \in \mathbb{N}\}$ is a singleton set but $\{|(2i)\eta - (2(i+1))\eta| : i \in \mathbb{N}\}$ is infinite. We will show that the remaining elements in $PF_{\mathbb{N}}$ can be generated by elements in $F_< \cup Sur^*(PF_{\mathbb{N}}) \cup E \cup \{\gamma_0\} = A \cup \{\gamma_0\}$.

Lemma 5. $F \subseteq \langle A, \gamma_0 \rangle$.

Proof. Let $\alpha \in F$. If $|f_3(\text{dom } \alpha)| = \aleph_0$ then $\alpha \in F_{\aleph_0}$, i.e.

$$\alpha \in \langle Sur(PF_{\mathbb{N}}), E, \gamma_0 \rangle \subseteq \langle A, \gamma_0 \rangle$$

by Lemma 4. Suppose that $|f_3(\text{dom } \alpha)| < \aleph_0$. If $\{1,2\} \cap \text{im } \alpha \neq \emptyset$ then $\alpha \in F_< \subseteq \langle A, \gamma_0 \rangle$. Admit now that $\{1,2\} \cap \text{im } \alpha = \emptyset$. Then $n := \min(\text{im } \alpha) > 2$ and there is $k \in \mathbb{N}$ with $2k \in \{n-1, n-2\}$. We consider the partial transformation

$$\gamma := \alpha(\gamma_0^k)^{-1}.$$

We have dom γ = dom α since dom $(\gamma_0^k)^{-1} = \{x \in \mathbb{N} : x \geq 2k+1\} \supseteq \{x \in \mathbb{N} : x \geq n\} \supseteq \text{im } \alpha$. Further, we have $x\gamma = x\alpha - 2k$ for all $x \in \text{im } \alpha$. Using this fact, we can argue that $D(\alpha) = \text{rank } \alpha = |\{|\Sigma| : \Sigma \in f(\text{im } \alpha)\}| = \chi_0$ provides $D(\gamma) = \text{rank } \gamma = |\{|\Sigma| : \Sigma \in f(\text{im } \gamma)\}| = \chi_0$. Hence, $\gamma \in F$ and it is easy to verify that $|f_3(\text{dom } \gamma)| < \aleph_0$. Let $y \in n\alpha^{-1}$. Then $y\gamma = y\alpha - 2k = n - 2k \in \{n-(n-1), n-(n-2)\} = \{1,2\}$. Therefore, $\gamma \in F_<$ and we can conclude $\alpha = \alpha(\gamma_0^k)^{-1}\gamma_0^k = \gamma\gamma_0^k \in \langle F_<, \gamma_0 \rangle \subseteq \langle A, \gamma_0 \rangle$. □

Finally, we will show that A together with γ_0 generate the set

$$Inf(PF_{\mathbb{N}}) := \{\alpha \in PF_{\mathbb{N}} : \text{rank } \alpha = D(\alpha) = \aleph_0\}.$$

Lemma 6. $Inf(PF_{\mathbb{N}}) \subseteq \langle A, \gamma_0 \rangle$.

Proof. Let $\alpha \in Inf(PF_{\mathbb{N}})$. If $\alpha \in F$, then $\alpha \in \langle A, \gamma_0 \rangle$ by Lemma 5. Admit now that $\alpha \notin F$. Then $\{|\Sigma| : \Sigma \in f(\text{im } \alpha)\}$ is finite and let $m := \max\{|\Sigma| : \Sigma \in f(\text{im } \alpha)\}$. Since rank $\alpha = D(\alpha) = \aleph_0$, there is a partition

$$\{A_i : i \in \mathbb{N}\}$$

of im α such that A_i is a finite convex set with $A_i < A_{i+1}$ and $\max A_i + 1, \min A_i - 1 \notin \text{im } \alpha$ for all $i \in \mathbb{N}$. We put

$$a_i := \max A_i \text{ and } b_i := \min A_i$$

for all $i \in \mathbb{N}$. Let β be the partial transformation defined recursively by

$$x\beta := x\alpha \quad \text{for } x \in A_1\alpha^{-1}$$

and

$$x\beta := 4m\max(A_{i-1}\alpha^{-1}\beta) + x\alpha \text{ for } x \in A_i\alpha^{-1}, \ \ 2 \leq i \in \mathbb{N}.$$

We put

$$c_0 := 0 \text{ and } c_i := \max(A_i\alpha^{-1}\beta)$$

for all $i \in \mathbb{N}$. We observe that $0 < c_i < c_{i+1}$ for $i \in \mathbb{N}$.
First, we check that $\beta \in PF_{\mathbb{N}}$. For this let $x, y \in \text{dom } \beta = \text{dom } \alpha$ with $x \prec y$. Then there is $i \in \mathbb{N}$ such that $x, y \in A_i\alpha^{-1}$ since $A_k \parallel A_l$ and thus $A_k\alpha^{-1} \parallel A_l\alpha^{-1}$, whenever $k < l \in \mathbb{N}$. From $x \prec y$, $\alpha \in PF_{\mathbb{N}}$, and $4mc_{i-1} \in 2\mathbb{N} \cup \{0\}$, it follows that $x\alpha \prec y\alpha$ and $4mc_{i-1} + x\alpha \prec 4mc_{i-1} + y\alpha$, i.e. $x\beta \prec y\beta$.
Now we show that $\beta \in F$. It is easy to verify that rank β = rank $\alpha = \aleph_0$. We define sets

$$\begin{aligned} D_0 &:= \{x \in \mathbb{N} : x < b_1\} \text{ and} \\ D_i &:= \{x \in \mathbb{N} : c_i < x < 4mc_i + b_{i+1}\} \text{ for } i \in \mathbb{N}. \end{aligned}$$

Clearly, $f(\text{im } \beta) = \{D_0\} \cup \{D_i : i \in \mathbb{N}\}$ and we calculate

$$|D_i| = |4mc_i + b_{i+1} - c_i - 1| = |(4m-1)c_i + b_{i+1} - 1| <$$

$$< |(4m-1)c_{i+1} + b_{i+2} - 1| = |4mc_{i+1} + b_{i+2} - c_{i+1} - 1| = |D_{i+1}|$$

since $c_i < c_{i+1}$ and $b_{i+1} < b_{i+2}$, for all $i \in \mathbb{N}$. This shows that $\{|D_i| : i \in \mathbb{N}\}$ is infinite and $D(\beta) = \aleph_0$. Together with rank $\beta = \aleph_0$, we obtain $\beta \in F$.
Let $k \in \mathbb{N}$ such that $u := 2(k-1) \in \{b_1 - 1, b_1 - 2\}$. Note that $f(\text{im } \alpha)$ is countable infinite and $D_0 \in f(\text{im } \alpha)$, say $f(\text{im } \alpha) = \{D_0\} \cup \{B_i : i \in \mathbb{N}\}$, whenever $D_0 \neq \emptyset$ (and $f(\text{im } \alpha) = \{B_i : i \in \mathbb{N}\}$ otherwise). For $i \in \mathbb{N}$, let

$$C_i := \{c_i + 2p : 1 \leq p \leq |B_i|\}.$$

Since $c_i + 2|B_i| \leq c_i + 2m < 2mc_i + 2mc_i + b_{i+1} = 4mc_i + b_{i+1}$, we can conclude that $C_i \subseteq D_i$ and $c_i + 2|B_i| + 1 \notin \text{im } \beta$. For $x \in B_i$, it holds $u < b_1 \leq x$, i.e. $x - u \in \mathbb{N}$. We put

$$B_i - u := \{x - u : x \in B_i\}.$$

Since $|C_i| = |B_i| = |B_i - u|$, there is a bijection

$$f_i : C_i \to B_i - u.$$

We define now a partial transformation γ by

$$\begin{aligned} x\beta\gamma &:= x\alpha - u \quad \text{for} \quad x \in \text{dom } \alpha = \text{dom } \beta; \\ x\gamma &:= f_i(x) \quad \text{for} \quad x \in C_i,\ i \in \mathbb{N}. \end{aligned}$$

Since im $\beta \cap C_i = \emptyset$ for all $i \in \mathbb{N}$, γ is well defined. We verify that $\gamma \in PF_{\mathbb{N}}$. For this let $x, y \in \text{dom } \gamma = \text{im } \beta \cup \bigcup\{C_i : i \in \mathbb{N}\}$ with $x \prec y$. Since $z-1, z+1 \notin \text{im } \beta \cup \bigcup\{C_i : i \in \mathbb{N}\}$ for any $z \in \bigcup\{C_i : i \in \mathbb{N}\}$, we obtain that $x, y \in \text{im } \beta$. Clearly, there is $i \in \mathbb{N}$ with $x, y \in A_i\alpha^{-1}\beta$. Thus, there are $x_0, y_0 \in A_i\alpha^{-1}$ with $x_0\beta = x$ and $y_0\beta = y$. From $x \prec y$, it follows $4mc_i + x_0\alpha \prec 4mc_i + y_0\alpha$, i.e. $x_0\alpha \prec y_0\alpha$ and $x_0\alpha - u \prec y_0\alpha - u$ since both $4mc_i$ and u are even. Therefore, $x\gamma = x_0\beta\gamma = x_0\alpha - u \prec y_0\alpha - u = y_0\beta\gamma = y\gamma$. This shows that $\gamma \in PF_{\mathbb{N}}$.
Now we show that $\gamma \in Sur^*(PF_{\mathbb{N}})$. For this let $x \in \mathbb{N}$ with $x \geq 2$. If $x+u \in \text{im } \alpha$, then there is $z \in \text{dom } \alpha = \text{dom } \beta$ with $z\alpha = x+u$ and $z\beta\gamma = z\alpha - u = (x+u)-u = x$. If $x+u \notin \text{im } \alpha$, then $x+u \geq 2+u \geq b_1$, i.e. $x+u \notin D_0$. Hence, there is $i \in \mathbb{N}$ with $x+u \in B_i$, i.e. $x = x+u-u \in B_i - u$ and $f_i^{-1}(x)\gamma = f_i(f_i^{-1}(x)) = x$. Altogether, we have $\mathbb{N} \setminus \{1\} \subseteq \text{im } \gamma$.
Finally, we show that $\alpha = \beta\gamma$ if $k=1$ and $\alpha = \beta\gamma\gamma_0^{k-1}$ if $k \geq 2$. Let $x \in \text{dom } \alpha = \text{dom } \beta$. If $k=1$ then $u=0$, i.e. $x\beta\gamma = x\alpha$. If $k \geq 2$ then $x\beta\gamma\gamma_0^{k-1} = (x\alpha - u)\gamma_0^{k-1} = (x\alpha - 2(k-1))\gamma_0^{k-1} = x\alpha - 2(k-1) + 2(k-1) = x\alpha$. Since dom $\beta\gamma$ (dom $\beta\gamma\gamma_0^{k-1}$, respectively) is contained in dom β = dom α, we have shown that $\alpha = \beta\gamma$ and $\alpha = \beta\gamma\gamma_0^{k-1}$, respectively. □

Now, we can prove the main result of the paper.

Theorem 7. $PF_{\mathbb{N}} = \langle A, \gamma_0 \rangle$.

Proof. Clearly, $\langle A, \gamma_0 \rangle \subseteq PF_{\mathbb{N}}$. Conversely, let $\alpha \in PF_{\mathbb{N}}$. If rank α is finite then $\alpha \in \langle E_f, \gamma_0 \rangle \subseteq \langle A, \gamma_0 \rangle$ by Lemma 2. If rank $\alpha = \aleph_0$ and $D(\alpha) < \aleph_0$ then $\alpha \in D_f \subseteq \langle A, \gamma_0 \rangle$ by Lemma 4. If rank $\alpha = \aleph_0$ and $D(\alpha) = \aleph_0$ then $\alpha \in Inf(PF_{\mathbb{N}}) \subseteq \langle A, \gamma_0 \rangle$ by Lemma 6. □

In fact, $\gamma_0 \notin \langle A \rangle$. Otherwise, there are $\alpha_1, \alpha_2, \ldots, \alpha_n \in A \setminus \{id_{\mathbb{N}}\}$ (for some $n \in \mathbb{N}$) such that $\gamma_0 = \alpha_1\alpha_2 \cdots \alpha_n$. Clearly, the kernel of α_1 is the diagonal, i.e. $x\alpha_1 \neq y\alpha_1$, whenever $x \neq y$ and $x, y \in \text{dom } \alpha_1 = \mathbb{N}$. Thus, $\alpha_1 \notin F_<$, i.e. $\alpha_1 \in Sur^*(PF_{\mathbb{N}}) \cup E$. But an idempotent with the diagonal on $\mathbb{N}$ as kernel is the identity. Hence, $\alpha_1 \in Sur^*(PF_{\mathbb{N}})$, i.e. im $\alpha_1 = \mathbb{N}$ or im $\alpha_1 = \mathbb{N} \setminus \{1\}$. But this dives $\alpha_1 = id_{\mathbb{N}}$ and $1\alpha_1 = 2\alpha_1$, respectively, a contradiction. Because of $\gamma_0 \notin \langle A \rangle$, Theorem 7 provides immediately:

Corollary 8. rank $(PF_{\mathbb{N}} : A) = 1$.

References

[1] J. D. Currie, T. I. Visentin, The number of order-preserving maps of fences and crowns, *Order* **8(2)** (1991), 133-142.

[2] I. Dimitrova, J. Koppitz, On the semigroup of all partial fence-preserving injections on a finite set, *Journal of algebra and its applications*, **16(11)** (2017), 1750223.

[3] V. H. Fernades, J. Koppitz, T. Musunthia, On the semigroup of all order-preserving transformations on a finite set, (2017), preprint.

[4] G. M. S Gomes, J. M. Howie, On the ranks of certain semigroups of order-preserving transformations, *Semigroup Forum*, **45** (1992), 272-282.

[5] P. M. Higgens, J. D. Mitchell, N. Ruškuc, Rank properties of endomorphisms of infinite partially ordered sets, *Bulletin of the London Mathematical Society*, **38** (2006), 177-191.

[6] J. M. Howie, N. Ruškuc and P. M. Higgins, On relative ranks of full transformation semigroups, *Comm. Algebra*, **26** (1998), 733–748.

[7] J. Koppitz, L. Lohapan, Regular semigroups of partial transformations preserving a fence $\mathbb{N}$, *Novi Sad Journal of Mathematics*, (2016), submitted.

[8] N. Ruškuc, On the rank of completely 0-simple semigroups, *Math. Proc. Cambridge Philos. Soc.*, **116** (1994), 325–338.

[9] A. Rutkowski, The formula for the number of order-preserving self-mappings of a fence, *Order* **9(2)** (1992), 127-137.

[10] R. Tanyawong, R. Srithus, R. Chinram, Regular subsemigroups of the semigroups of transformations preserving a fence, *Asian-European Journal of Mathematics*, **9(1)** (2016), 1650003.