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PROXIMATE AND APPROXIMATE SEQUENCES

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Abstract: In this paper the author defines and discuses the concept of approximate sequences. First, in a separate section, he discusses proximate and approximate sequences.

After that he presents some properties of approximate sequences that are anoloqous to similar results for fundamental sequences, established in previous paper [1] titled equivalence of intrinsic shape, based on V-continuous functions and shape (N. Shekutkovski, Z. Misajleski, G. Markoski, M. Shoptrajanov, Bulletin mathematique, 2013, No. 1, 39-48). The author gives an optional definition of the function $r_{\mathcal{V}}$, with the help of intersections, which in [1] is defined using notion of depth. Also he shows that in a compact metric space there exists a cofinal sequence of finite regular coverings. In addition he shows that it is possible to choose the images of functions of approximate as subsets of the union of elements of such a sequence of coverings. Furthermore, analogue theorems of [1], which refers to approximate instead for fundamental sequences, he present and prove. Finally, the author shows that shape category constructed with the classes of approximate sequences, is equivalent with the intrinsic shape category constructed with the classes of proximate sequences.

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1. Introduction

"A covering" refers to a covering consisting of open sets. Let X and Y be compact metric spaces. We repeat the intrinsic approaches to shape.

The following definitions and statements were given in [2]:

Definition 1. Let $\mathcal{V}$ be a finite covering of Y. A function $f : X \to Y$ is $\mathcal{V}$ $-$ **continuous** if for every point $x \in X$, there exists a neighborhood U_x of x, and $V \in \mathcal{V}$, such that $f(U_x) \subseteq V$.

If $f : X \to Y$ is $\mathcal{W}$ $-$continuous, then $f : X \to Y$ is $\mathcal{V}$ $-$ continuous if for any $\mathcal{V}$, such that $\mathcal{W} \prec \mathcal{V}$. If $\mathcal{V}$ is a finite covering of Y, and $V \in \mathcal{V}$, then star of V is then open set $st(\mathcal{V}) = \{W | W \in \mathcal{V}, W \cap V \neq \emptyset\}$. We form a new covering of $st(\mathcal{V}) = \{st(V) | V \in \mathcal{V}\}$.

Definition 2. The functions $f, g : X \to Y$ are $\mathcal{V}$ $-$ **homotopic**, if there exists a function $F : X \times I \to Y$ such that: 1) F is $st(\mathcal{V})$ $-$ continuous, 2) $F|_{\partial_I \times X}$ is $\mathcal{V}$ $-$ continuous and 3) $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ (F is connecting f and g).

The relation of $\mathcal{V}$ $-$ homotopy is an equivalence relation. Usually, the condition 2) of the previous condition is formulated as: $2'$) there exists an neighborhood N of $\partial I = \{0, 1\}$ such that $F|_{X \times N}$ is $\mathcal{V}$ $-$continuous.

Definition 3. A cofinal sequence of finite coverings $\mathcal{V}_1 \succ \mathcal{V}_2 \succ ... \succ \mathcal{V}_n \succ ...$ is a sequence of finite covering of spaces, such that for any covering $\mathcal{V}$, there exists n, such that $\mathcal{V}_n \prec \mathcal{V}$.

In a compact metric space there exists such a sequence. This fact allows working with proximate sequences instead with proximate nets.

Definition 4. The sequence (f_n) of functions $f_n : X \to Y$ is a **proximate sequence from** X **to** Y, if there exists a cofinal sequence of finite coverings of $Y, \mathcal{V}_1 \succ \mathcal{V}_2 \succ ... \succ \mathcal{V}_n \succ ...$, and for all indexes $m \geq n$, f_n and f_m are $\mathcal{V}_n$ $-$ homotopic.

In this case we say that (f_n) is a proximate sequence over $(\mathcal{V}_n)$. If (f_n) and (f'_n) are proximate sequences from X to Y , than there exists a cofinal sequence of finite coverings $\mathcal{V}_1 \succ \mathcal{V}_2 \succ ... \succ \mathcal{V}_n \succ ...$ such that (f_n) and (f'_n) are proximate sequences over $(\mathcal{V}_n)$.

Definition 5. Two proximate sequences (f_n) and (f'_n) are **homotopic** if there exists a cofinal sequence of finite coverings $\mathcal{V}_1 \succ \mathcal{V}_2 \succ ... \succ \mathcal{V}_n \succ ...$ of Y, such that (f_n) and (f'_n) are $\mathcal{V}_n$ $-$ homotopic for all integers n.

We say that (f_n) and (f'_n) are homotopic over $(\mathcal{V}_n)$.

Let $(f_n) : X \to Y$ be a proximate sequence over $(\mathcal{V}_n)$ and $(g_k) : Y \to Z$ be a proximate sequence over $(\mathcal{W}_n)$. For a covering $\mathcal{W}_k$ of Z , there exists a covering $\mathcal{V}_{n_k}$ of Y such that $g(\mathcal{V}_{n_k}) \prec \mathcal{W}_k$. Then, the composition is the proximate sequence $(h_k) = (g_k f_{n_k}) : X \to Z$. In [2] is proven that compact metric spaces and homotopy classes of proximate sequences $[(f_n)]$ form the shape category i.e. isomorphic spaces in this category has the same shape.

Let X, Y and Z, are compact metric space, that lie in the Hilbert space $Q \in AR$.

The following definitions and statements were used in [3]:

Definition 6. **Approximate sequence from** X **to** Y is a sequence of maps $\bar{f}_n : X \to Q$, $n \in N$, such that for every neighborhood V of Y in Q , there exist a index n_0, such that for every $n \geq n_0$, $\bar{f}_n$ is homotopic to $\bar{f}_{n+1}$ in V.

We denote approximate sequence with $(\bar{f}_n)_Q$ from X to Y or $(\bar{f}_n)$ from X to Y. Original notion in [3] is $(\bar{f}_n : X \to Y)_Q$ or $(\bar{f}_n : X \to Y)$.

Definition 7. Approximate sequences $(\bar{f}_n)$ and $(\bar{f}'_n)$ from X to Y are **homotopic**, if for every neighborhood V of Y, there exist $n_0 \in N$, such that $\bar{f}_n$ is homotopic to $\bar{f}'_n$ in V for all $n \geq n_0$, $n \in N$.

The relation of homotopy of approximate sequences is an equivalence relation.

Definition 8. **Fundamental sequence from** X **to** Y is a sequence of maps $\bar{f}_n : Q \to Q$, $n = 1, 2, 3...$, where for every neighborhood V of Y , there exist neighborhood U of X and there exist $n_0 \in N$, such that $\bar{f}_n|_U$ is homotopic to $\bar{f}_{n+1}|_U$ in V for all $n \geq n_0$.

Fundamental sequence is denoted with $(\bar{f}_n)_{Q,Q}$ from X to Y.

Definition 9. Fundamental sequences $(\bar{f}_n)_{Q,Q}$ and $(\bar{f}'_n)_{Q,Q}$ from X to Y are **homotopic**, if for every neighborhood V of Y in Q, there exist neighborhood U of X in Q and $n_0 \in N$, such that $\bar{f}_n|_U$ is homotopic to $\bar{f}'_n|_U$ in V, for all $n \geq n_0$, $n \in N$.

The relation of homotopy of fundamental sequences is an equivalence relation.

For every approximate sequence $(\bar{f}_n)$ from X to Y, because Q is absolute retract, there exist fundamental sequence $(\hat{f}_n)_{Q,Q}$ from X to Y, such that $\hat{f}_n|_X = \bar{f}_n$.

For every approximate sequence $(\bar{f_n})$ from X to Y and for every fundamental sequence $(\bar{g_n})_{Q,Q}$ from Y to Z, maps $\bar{g_n}\bar{f_n} : X \to Q$, define approximate sequence $(\bar{g_n}\bar{f_n})$ from X to Z, which is called composition of the approximate sequences $(\bar{g_n})$ and $(\bar{f_n})$. If approximate sequences $(\bar{f_n})$ and $(\bar{f'_n})$ from X to Y are homotopic, and fundamental sequences $(\bar{g_n})_{Q,Q}$ and $(\bar{g'_n})_{Q,Q}$ from Y to Z are homotopic, then the compositions $(\bar{g_n}\bar{f_n})$ and $(\bar{g'}_n\bar{f'}_n)$ from X to Z are homotopic. Following the last claims, it can be concluded that for every approximate sequence class $[(\bar{f_n})] : X \to Y$ and every fundamental class $\lfloor(\bar{g_n})_{Q,Q}\rfloor : Y \to Z$ every compositions $(\hat{g_n}\bar{f_n})$ belong to same class, which is denoted with $\lfloor(\bar{g_n})_{Q,Q}\rfloor[(\bar{f_n})]$. In this way, if we take the compact metric spaces to be objects, and the classes of approximate sequences to be morphisms, then we get a new category, which is called **shape** Sh.

2. Proximate and Approximate Sequences

All spaces in this section are compact metric spaces.

Definition 10. Covering $\mathcal{V}$ of Y is **regular** if it satisfies the following conditions:

1) If $V \in \mathcal{V}$ then $V \cap Y \neq \emptyset$ and

2) If U, $V \in \mathcal{V}$ and $U \cap V \neq \emptyset$, then $U \cap V \in \mathcal{V}$.

Let $\mathcal{V} = \{V_i | i = 1, 2, ..., n\}$ be a regular covering of Y and $\bar{f} : X \to \bigcup\limits_{V \in \mathcal{V}} V$ is continuous function i.e. map. A function: $r_{\mathcal{V}} : \bigcup\limits_{V \in \mathcal{V}} V \to Y$ will be defined such that the set $\bigcup\limits_{V \in \mathcal{V}} V$ in which $\mathcal{V}$ has n elements, we will divide into $2^n - 1$ disjoint nonempty subsets the most, and in each one one element from Y, will be chosen. Disjoint subsets are of the type $U_{i_1} \cap U_{i_2} \cap ... \cap U_{i_k} \setminus \bigcup\limits_{j_1, j_2, ..., j_{k+1} \in \{1,2,...n\}} U_{j_1} \cap U_{j_2} \cap ... \cap U_{j_{k+1}}$ for some combination $\{i_1, i_2, ...i_k\}$ of n elements, class k and all combinations $\{j_1, j_2, ...j_{k+1}\}$ of n elements, class $k + 1$. Linking will be defined inductively as follows:

n) If $y \in U_1 \cap U_2 \cap ... \cap U_n$ we put

$$r_{\mathcal{V}}(y) = [U_1 \cap U_2 \cap ... \cap U_n]$$

for some element $[U_1 \cap U_2 \cap ... \cap U_n]$ from $Y \cap U_1 \cap U_2 \cap ... \cap U_n$.

$n-1$) If $y \in U_{i_1} \cap U_{i_2} \cap ... \cap U_{i_{n-1}} \setminus (U_1 \cap U_2 \cap ... \cap U_n), i_1, i_2, ..., i_{n-1} \in \{1, 2, ..., n\}$ then we put

$$r_{\mathcal{V}}(y) = [U_{i_1} \cap U_{i_2} \cap ... \cap U_{i_{n-1}}]$$

for some element $[U_1 \cap U_2 \cap ... \cap U_n]$ from $Y \cap U_1 \cap U_2 \cap ... \cap U_{n-1}$

$0 < k < n$) If

$y \in U_{i_1} \cap U_{i_2} \cap ... \cap U_{i_k} \setminus \bigcup_{j_i, j_2, ..., j_{k+1} \in \{1,2,...n\}} U_{j_1} \cap U_{j_2} \cap ... \cap U_{j_{k+1}}$

we put

$$r_{\mathcal{V}}(y) = [U_1 \cap U_2 \cap ... \cap U_n]$$

for some element $[U_{i_1} \cap U_{i_2} \cap ... \cap U_{i_k}]$ from $Y \cap U_{i_1} \cap U_{i_2} \cap ... \cap U_{i_k}$. Because all sets of the type

$$U_{i_1} \cap U_{i_2} \cap ... \cap U_{i_k} \setminus \bigcup_{j_i, j_2, ..., j_{k+1} \in \{1,2,...n\}} U_{j_1} \cap U_{j_2} \cap ... \cap U_{j_{k+1}},$$

are pairwise disjoint, it follows that $r_{\mathcal{V}}$ is well defined. $r_{\mathcal{V}}$ is also $\mathcal{V}-$ continuous. We can also define $r_{\mathcal{V}}$ in the following way: $r_{\mathcal{V}}(y) = [V]$ if V is the smallest set in $\mathcal{V}$ which contains y.

If $\mathcal{V}$ is a regular covering of Y and $\bar{f} : X \to \bigcup_{V \in \mathcal{V}} V$ is a map, we can **define function** $f : X \to Y$ with

$$f(x) = r_{\mathcal{V}} \bar{f}(x) \qquad \text{for all} \quad x \in X.$$

The function f is well defined. Since $\bar{f}$ is continuous, the function f is $\mathcal{V}$ $-$continuous. We will say that the function f is **obtained** from a continuous function $\bar{f}$ and covering $\mathcal{V}$.

Example 1. Let X be a compact metric space, $Y = I \times I \subseteq R^2$, $\bar{f}_1 : X \to W$ and $\bar{f}_2 : X \to \bigcup_{i=1}^{3} V_i$, are maps, and $\bar{f}_{1,2} : X \times I \to R^2$ is homotopy connecting $\bar{f}_1$ and $\bar{f}_2$, where $W = (-1, 2) \times (-1, 2)$, $\mathcal{V}_1 = \{W\}$, $V_1 = (-\frac{1}{2}, \frac{3}{2}) \times (\frac{1}{3}, \frac{2}{3})$, $V_2 = (-1, 2) \times (\frac{1}{3}, 2)$, $V_3 = (\frac{-1}{2}, \frac{3}{2}) \times (-1, \frac{2}{3})$ $\mathcal{V}_2 = \{V_1, V_2, V_3\}$, $1 \equiv (\frac{1}{2}, \frac{1}{2})$, $2 \equiv (\frac{3}{4}, \frac{1}{2})$ and $3 \equiv (\frac{1}{4}, \frac{1}{2})$. Then $f_1(x) = 1$ because $\bar{f}_1 \in W$,

$$f_2(x) = \begin{cases} 1 & \bar{f}_2(x) \in V_1 \\ 2 & \bar{f}_2(x) \in V_2 \setminus V_1 \\ 3 & \bar{f}_2(x) \in V_3 \setminus V_1 \end{cases} \quad \text{for } x \in X \text{ and}$$

$$f_{1,2}(x,t) = \begin{cases} 1 & \bar{f}_{1,2}(x,t) \in V_1, t \in [0,1) \quad \text{and} \quad \bar{f}_{1,2}(x,1) \in V_1 \\ 2 & \bar{f}_{1,2}(x,t) \in V_2 \setminus V_1 \\ 3 & \bar{f}_{1,2}(x,t) \in V_3 \setminus V_1 \end{cases} \quad \text{for } x \in X.$$

So, $f_{1,2}(x,0) = f_1(x)$ and $f_{1,2}(x,1) = f_2(x)$ i.e. $f_{1,2}$ is connecting f_1 and f_2. Function f_2 is $\mathcal{V}_2$-continuous. Namely, if $\bar{f}_2(x) \in V_1$, ($\bar{f}_2$ is a map), there exists a neighborhood U_1 in X, such that $x \in U_1$ and $\bar{f}_2(U_1) \subseteq V_1$. It follows $f_2(U_1) = \{1\} \subseteq V_1$. If $\bar{f}_2(x) \in V_2 \backslash V_1$, there exists a neighborhood U_2 in X, such that $x \in U_2$ and $\bar{f}_2(U_2) \subseteq V_2$. It follows $f_2(U_2) = \{1,2\} \subseteq V_2$. If $\bar{f}_2(x) \in V_3 \backslash V_1$, there exists U_3 in X, such that $x \in U_3$ and $\bar{f}_2(U_3) \subseteq V_3$. It follows $f_2(U_3) = \{1,3\} \subseteq V_3$. Clearly the functions f_1 and $f_{1,2}$ are $\mathcal{V}_1$-continuous. Function $f_{1,2}$ will be homotopy as well as when $f_{1,2}(x,t) = 1$, for every $(x,t) \in X \times I$.

Definition 11. Cofinal sequence of finite **regular** coverings is cofinal sequence of finite coverings whose coverings are regular.

Theorem 12. *In a compact metric space, there exists a cofinal sequence of finite regular coverings of the space.*

Proof. In a compact metric space Y embedded in a Hilbert cube, there exists a cofinal sequence of finite coverings $\mathcal{V}_1 \succ \mathcal{V}_2 \succ ... \succ \mathcal{V}_n \succ ....$ On each of the coverings $\mathcal{V}_n$, $n \in N$ we add intersections of its elements. Then from every covering $\mathcal{V}_n$, $n \in N$, we reject the elements whose intersection with Y is an empty set. The cofinality is not violated because if one element has empty intersection with Y, then all sets V_m from $\mathcal{V}_m$, $m > n$ which are contained in some sets in $\mathcal{V}_n$, also have an empty intersection with Y. In that way we get a cofinal sequence of finite regular coverings of Y. □

Theorem 13. *For the approximate sequence $(\bar{f}_n)$ from X to Y, there exists a cofinal sequence of finite regular coverings $\mathcal{V}_1 \succ \mathcal{V}_2 \succ ... \succ \mathcal{V}_n \succ ...$ of Y in Q such that $\mathrm{Im}\bar{f}_n, \mathrm{Im}\bar{f}_{n+1} \subseteq \bigcup_{V \in \mathcal{V}_n} V$.*

Proof. From Theorem 12 it follows that there exists a cofinal sequence of finite regular coverings of $\mathcal{V}'_1 \succ \mathcal{V}'_2 \succ ... \succ \mathcal{V}'_n \succ ...$ of Y in Q. Because Y is metric space, we can choose a sequence of neighborhoods V_n of Y in Q such that $d(Y, Q \backslash V_n) < \frac{1}{n}$.

For neighborhood V_1 of Y there exists number $n_1 \in N$ such that for all $n \geq n_1$ there exists a homotopy $\bar{f}_{n,n+1} : X \times I \to V_1$ which is connecting $\bar{f}_n$ and $\bar{f}_{n+1}$. We are choosing $\mathcal{V}_1 = \mathcal{V}_2 = ... = \mathcal{V}_{n_1-1} = \{Q\}$.

For neighborhood V_2 of Y there exists a number $n_2 > n_1$, $n_2 \in N$ such that for all $n \geq n_2$ there exists a homotopy $\bar{f}_{n,n+1} : X \times I \to V_2$ which is connecting

$\bar{f}_n$ and $\bar{f}_{n+1}$. We are choosing $\mathcal{V}_{n_1} = \mathcal{V}_{n_1+1} = ... = \mathcal{V}_{n_2-1} = \{V_1 \cap \mathcal{V}'_{n_1}\}$, where $V_1 \cap \mathcal{V}'_{n_1} = \{V_1 \cap V'_{n_1} \mid V'_{n_1} \in \mathcal{V}'_{n_1}\}$.

Continuing inductively we get that for neighborhood V_k of Y there exists a number $n_k > n_{k-1}$, $n_k \in N$ such that for all $n \geq n_k$ there exists a homotopy $\bar{f}_{n,n+1} : X \times I \to V_k$ which is connecting $\bar{f}_n$ and $\bar{f}_{n+1}$. We are choosing $\mathcal{V}_{n_{k-1}} = \mathcal{V}_{n_{k-1}+1} = ... = \mathcal{V}_{n_k-1} = \{V_{k-1} \cap \mathcal{V}'_{n_{k-1}}$. In that way we get a cofinal sequence of finite regular coverings $\mathcal{V}_1 \succ \mathcal{V}_2 \succ .. \succ {}_n \succ ...$ of Y with property images of maps $\bar{f}_{n,n+1} : X \to Q$ from approximate sequence $(\bar{f}_n)$ to be in $\bigcup\limits_{V \in \mathcal{V}_n} V$ i.e. $\text{Im}\bar{f}_{n,n+1} \subseteq \bigcup\limits_{V \in \mathcal{V}_n} V$ for every $n \in N$. Specially $\text{Im}\bar{f}_n \subseteq \bigcup\limits_{V \in \mathcal{V}_n} V$, for every $n \in N$. □

Similarly, for two approximate sequences $(\bar{f}_n)$ and $(\bar{f}'_n)$ from X to Y, there exists a cofinal equence of finite regular coverings $\mathcal{V}_1 \succ \mathcal{V}_2 \succ ... \succ \mathcal{V}_n \succ ...$ of Y in Q such that images of functions $\bar{f}_n$ and $\bar{f}'_n$ are subsets of $\bigcup\limits_{V \in \mathcal{V}_n} V$. Next theorem is proven in [1].

Theorem 14. *If Y is compact metric space embedded in Hilbert cube Q, $\mathcal{V}$ and $\mathcal{W}$ are regular coverings of Y in Q such that $\mathcal{W} \prec \mathcal{V}$ and $Z = \bigcup\limits_{W \in \mathcal{W}} W$. Then $r_{\mathcal{V}}|_Z$ (the restriction of $r_{\mathcal{V}}$ to Z) and $r_{\mathcal{W}}$ are $\mathcal{V}$-homotopic.*

Proof. Let us consider the function $R : Z \times I \to Y$ defined by

$$R(x,t) = \left\{ \begin{array}{ll} r_{\mathcal{V}}(x) & (x,t) \in Z \times [0,1) \\ r_{\mathcal{W}}(x) & (x,1) \in Z \times \{1\}. \end{array} \right.$$

If $(x,t) \in Z \times [0,1)$ then $R(x,t) = r_{\mathcal{V}}(x)$ and R is $\mathcal{V}$-continuous in (x,t).

If $(x,1) \in Z \times \{1\}$, then $R(x,1) = r_{\mathcal{W}}(x) = [W]$, where W is the smallest set in $\mathcal{W}$ that contains x. From $\mathcal{W} \prec \mathcal{V}$, it follows that $W \subseteq V \in \mathcal{V}$ and we can choose V to be the smallest set in $\mathcal{V}$, with the property $W \subseteq V$. Then $r_{\mathcal{V}}(V) \in V \cap Y$ and

$$R(W \times 1) = r_{\mathcal{W}}(W) \in W \cap Y \subset V \cap Y. \tag{1}$$

We consider the neighborhood $W \times [0,1]$ of $(x,1)$ and $(w,t) \in W \times [0,1)$. There is a smallest set $V_{\mathcal{W}}$ in $\mathcal{V}$ such that $w \in V_{\mathcal{W}}$. So it follows that: $R(w,t) = r_{\mathcal{V}}(w) = [V_{\mathcal{W}}] \in V_{\mathcal{W}} \cap Y$, for all $t \in [0,1)$. Also, $V_{\mathcal{W}} \subseteq V$ for all $x \in W$. Then, it follows

$$R(W \times [0,1)) \subseteq V \cap Y. \tag{2}$$

Finally from (1) and (2),

$$R(W \times [0,1]) = R(W \times [0,1)) \cup R(W \times 1) \subseteq V \cap Y$$

it follows that R is $\mathcal{V}$-continuous at $(x,1) \in Z \times \{1\}$ and $R(x,0) = r_{\mathcal{V}}(x)$, $R(x,1) = r_{\mathcal{W}}(x)$. ☐

Theorem 15. *Let $(\bar{f}_n)$ from X to Y be an approximate sequence over cofinal sequence of finite regular covering $\mathcal{V}_1 \succ \mathcal{V}_2 \succ ... \succ \mathcal{V}_n \succ ...$, let $f_n : X \to \bigcup\limits_{V \in \mathcal{V}_n} V$, $n \in N$ be functions defined by $f_n = r_{\mathcal{V}_n} \bar{f}_n$ and let $f_{n,n+1} : X \times I \to \bigcup\limits_{V \in \mathcal{V}_n} V$, $n \in N$ be $f_{n,n+1} = r_{\mathcal{V}_n} \bar{f}_{n,n+1}$. Then the sequence (f_n) is a proximate sequence from X to Y.*

Proof. The existence of cofinal sequence of finite regular covering $\mathcal{V}_1 \succ \mathcal{V}_2 \succ ... \succ \mathcal{V}_n \succ ...$, such that $\mathrm{Im}\bar{f}_n, \mathrm{Im}\bar{f}_{n,n+1} \subseteq \bigcup\limits_{V \in \mathcal{V}_n} V$ is provided by Theorem 12 and Theorem 13. The functions f_n and $f_{n,n+1}$ are $\mathcal{V}_n$-continuous by construction and

$$f_{n,n+1}(x,0) = r_{\mathcal{V}_n} \bar{f}_{n,n+1}(x,0) = r_{\mathcal{V}_n} \bar{f}_n(x) = f_n(x),$$

$$f_{n,n+1}(x,1) = r_{\mathcal{V}_n} \bar{f}_{n,n+1}(x,1) = r_{\mathcal{V}_n} \bar{f}_{n+1}(x). \tag{3}$$

Taking into consideretion the fact that $\mathcal{V}_{n+1} \prec \mathcal{V}_n$ from the theorem 14 it follows that $r_{\mathcal{V}_n}|_{\bigcup\limits_{V \in \mathcal{V}_{n+1}} V}$ (the restriction of $r_{\mathcal{V}_n}$ to $\bigcup\limits_{V \in \mathcal{V}_n}$) and $r_{\mathcal{V}_{n+1}}$ are $\mathcal{V}_n$-homotopic, by a homotopy $R : \bigcup\limits_{V \in \mathcal{V}_{n+1}} V \times I \to Y$ i.e.

$$R(x,0) = r_{\mathcal{V}_n}(x), \quad R(x,1) = r_{\mathcal{V}_{n+1}}(x), \qquad \text{for all} \quad x \in \bigcup_{V \in \mathcal{V}_{n+1}} V.$$

So, the $\mathcal{V}_n$-homotopy $R\bar{f}_{n+1} : X \times I \to Y$ satisfies

$$R\bar{f}_{n+1}(x,0) = r_{\mathcal{V}_n} \bar{f}_{n+1}(x)$$

$$R\bar{f}_{n+1}(x,1) = r_{\mathcal{V}_{n+1}} \bar{f}_{n+1}(x) = f_{n+1}(x). \tag{4}$$

Since $\mathcal{V}_n$-homotopy is an equivalence relation by (3) and (4) it follows that $f_n(x)$ and $f_{n+1}(x)$ are $\mathcal{V}_n$-homotopic.

Therefore (f_n) is a proximate sequence over $\mathcal{V}_1 \succ \mathcal{V}_2 \succ ... \succ \mathcal{V}_n \succ ...$. ☐

Theorem 16. *If two approximate sequences* $(\bar{f}_n)$ *and* $(\bar{f'}_n)$ *from* X *to* Y *are homotopic, then the obtained from the proximate sequences* (f_n) *and* (f'_n) *from* X *to* Y *are also homotopic.*

Proof. Let the approximate sequences $(\bar{f}_n)$ and $(\bar{f'}_n)$ be homotopic with homotopy $\bar{F}_n : X \times I \to \bigcup_{V \in \mathcal{V}_n} V$ i.e. $\bar{F}_n(x, 0) = \bar{f}_n(x)$ and $\bar{F}_n(x, 1) = \bar{f'}_n(x)$. We define $F_n : X \times I \to Y$ by

$$F_n(x, t) = r_{\mathcal{V}_n} \bar{F}_n(x, t).$$

Then $F - n$ is $\mathcal{V}_n$-continuous and $st(\mathcal{V}_n)$-continuous for all points of $X \times \partial I$ and $F_n(x, 0) = r_{\mathcal{V}_n} \bar{f}_n(x) = f_n(x), \quad F_n(x, 1) = r_{\mathcal{V}_n} \bar{f'}_n(x) = f'_n(x).$ □

Following the previous theorem we can define the function $\Phi_{X,Y}$ from the classes of continuous functions from X to Y to the classes of $\mathcal{V}$-continuous functions from X to Y, i.e $\Phi_{X,Y}[(\bar{f}_n)] = [(f_n)]$ for every class of approximate sequences $[(\bar{f})]$ from X to Y.

We will describe a functor $\Phi : SH \to InSh$. The functor is consists of:

1) Function $\Phi : CPt(Q) \to Cpt(Q)$ defined with $\Phi(X) = X$, for every compact metric space X.

2) Functions $\Phi : Mor_{Sh}(X, Y) \to Mor_{InSh}(X, Y)$ for every compact metric spaces X and Y defined with $\Phi([(\bar{f}_n)]) = \Phi_{XY}[(\bar{f}_n)]$ for every class of approximate sequences $[(\bar{f}_n)]$ from X to Y.

In order to prove that Φ is a functor, we have to prove the following theorem.

Theorem 17. *If* $(\bar{f}_n)$ *from* X *to* Y *and* $(\bar{g}_n)$ *from* Y *to* Z *are two approximate sequences, then*

1) $\Phi([(\bar{g}_n)(\bar{f})]) = \Phi([(\bar{g}_n)])\Phi([(\bar{f}_n)])$;

2) $\Phi([(\bar{1}_X)]) = 1_{\Phi(X)}$.

Proof. 1) Let a proximate sequence (g_n) from Y to Z be obtained from approximate sequence $(\bar{g}_n)$, with the cofinal sequence of finite regular coverings $(\mathcal{W}_k)$ of Z in Q.
Suppose (f_{n_k}) from X to Y is a proximate subsequence of the proximate sequence (f_n) obtained from approximate sequence $(\bar{f}_n)$. The subsequence is

chosen such that $f_{n_k}(\mathcal{V}_{n_k}) \prec \mathcal{W}_k$.

The approximate sequences $(\bar{f}_n)$ and $(\bar{f}_{n_k})$ belong to the same class. If we take into consideration a theorem from [2],then (f_{n_k}) and (f_n) belong to the same class and if $\bar{g}_k\bar{f}_{n_k} = \bar{h_k}$ we have to prove

$$\lfloor r_{\mathcal{W}_k}\bar{h}_k\rfloor = \lfloor(r_{\mathcal{W}_k}\bar{g}_k)(r_{\mathcal{V}_{n_k}}\bar{f}_{n_k}\rfloor$$

i.e. $r_{\mathcal{W}_k}\bar{h}_k$ and $(r_{\mathcal{W}_k}\bar{g}_k)(r_{\mathcal{V}_{n_k}}\bar{f}_{n_k})$ are homotopic.

Let $x \in X$. By definition

$$r_{\mathcal{W}_k}\bar{h}_k(x) = [W], \tag{5}$$

where W is the smallest set W in $\mathcal{W}$ such that $\bar{g}_k\bar{f}_{n_k}(x) = \bar{h}_k(x) \in W$. Since $(\bar{g}_k\mathcal{V}_{n_k}) \prec \mathcal{W}_k$ there exists V' in $\mathcal{V}$ such that $\bar{g}_k(V') \subseteq W$. On the other hand, by definition

$$r_{\mathcal{V}_{n_k}}\bar{f}_{n_k}(x) = [V],$$

where V is the smallest set containing x, then $V \subseteq V'$ and it follows $\bar{g}_k(V) \subseteq W$. Then

$$(r_{\mathcal{W}_k}\bar{g}_k)(r_{\mathcal{V}_{n_k}}\bar{f}_{n_k})(x) = r_{\mathcal{W}_k}(\bar{g}_k[V]) \in W. \tag{6}$$

Then, from (5) and (6) $r_{\mathcal{W}_k}\bar{h}_k = h_k$ and $(r_{\mathcal{W}_k}\bar{g}_k)(r_{\mathcal{V}_{n_k}}\bar{f}_{n_k}) = g_kf_{n_k}$ are $\mathcal{W}_k$-near and since h_k is $\mathcal{W}_k$-continuous, by Lemma 1.1 from [4], we get that h_k and $g_kf_{n_k}$ are $\mathcal{W}_k$-homotopic.

2)One representive of the identical morphism in Sh is the class of approximate sequences, $(\bar{1}_n)$ from X to X, where $\bar{1}_n : X \to \bigcup_{V\in\mathcal{V}_n} V$, $n \in N$ defined with $\bar{1}_n(x) = x$, $x \in X$ are identical maps in X. Then $\Phi([(\bar{1}_n)]) = [(1_n)]$, where $1_n : X \to X$, $n \in N$ are copies of the identical map. $[(1_n)]$ is also the identical morphism in $InSh$, since for proximate sequences (f_n) from X to Y and (g_n) from Y to X it is true that $(f_n)(1_n) = (f_n)$ and $(1_n)(g_n) = (g_n)$. It follows $\Phi([(1_X)]) = 1_{\Phi(X)}$. □

The next theorem is also valid.

Theorem 18. *The functor Φ is a bijection.*

Proof. 1) Suppose (g_n) is a proximate sequence. We will reformulate of theorem 1 in [7]: For every proximate sequence (g_n) from X to Y there exist an approximate sequence $(\bar{f}_n)$ from X to Y and a cofinal sequence of coverings $\mathcal{V}_1 \succ \mathcal{V}_2 \succ ... \succ \mathcal{V}_n \succ ...$ such that $\bar{f}_n$ and g_n are $\mathcal{V}_n$-close for all integers. All approximate sequences obtained from (f_n) are also homotopic.

One proximate sequence (f_n) obtained from approximate sequence $(\bar{f}_n)$ consists of $\mathcal{V}_n$-close functions f_n and $\bar{f}_n$. Therefore (f_n) and (g_n) are homotopic and it follows that $\Phi[(\bar{f}_n)] = [(g_n)]$, i.e. the functor is surjective.

2) Suppose the proximate sequences (f_n) and (f'_n) from X to Y are obtained from approximate sequences $(\bar{f}_n)$ and $(\bar{f'}_n)$ from X to Y, respectively. Suppose (f_n) and (f'_n) are homotopic, i.e. f_n and f'_n are connected by homotopy F_n connecting $\bar{f}_n$ and $\bar{f'}_n$ for all natural numbers n. If we take into consideration a theorem 1 from [4], it follows that there exists a continuous homotopy $(\bar{F}_n)$ connecting $(\bar{f}_n)$ and $(\bar{f'}_n)$ for all natural numbers n,i.e. the functor is injective. $\square$

According to the previous theorems it follows that shape categories constructed with the classes of approximate and proximate sequences are equivalent.

References

[1] N. Shekutkovski, Z. Misajleski, G. Markoski, M. Shoptrajanov, Equvalence of intristic shape, based on V-continuous functions and shape, *Bulletin mathematique*, **1**, (2013), 39–48.

[2] N. Shekutkovski, Intristic definition of strong shape for compact metric spaces, *Topology Proceedings*, **39**, (2012), 27–39.

[3] K. Borsuk, *Teorija shejpov*, Izdateljstvo Mir, Moskva, (1976).

[4] J. M. R. Sanjurjo, A non continuous description of the shape, *Quart. J. Math. Oxford Ser.*, **40**, (1989), 351–359.

[5] N. Shekutkovski, Z. Misajleski, Intristic shape based on ε-continuity and on continuiuty up to a covering are equivalent, *Proceedings of FMNS 2011*, **1**, (2011), 77–82.

[6] N. Shekutkovski, Z. Misajleski, Intristic shape based on ε-continuity and on contiuity up to a covering are equivalent (II), *Proceedings of FMNS 2013*, **1**, (2013), 87–94.

[7] N. Shekutkovski, Z. Misajleski, Equivalence of intristic shape and shape, *God. Zb. Inst. Mat.*, **42**, (2013), 69–80.