SYMMETRIES OF THE GENERALIZED FISHER EQUATION WITH $t$–DEPENDENT COEFFICIENT

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Abstract: There are many tools for analyzing PDEs. In the equivalence theory, the symmetry methods like the Lie symmetry and Fushchych methods are tools for solving PDEs. Indeed, these methods can determine classical and non-classical invariants and then by reformulating the equations according to these invariants, they can reduce the order of PDEs and convert them to ODEs.

In this paper, we consider the generalized version of FKPP equation (GFKPP) with $t$-dependent coefficient

$$f(t)u_{tt}(x,t) + u_t(x,t) = u_{xx}(x,t) + u(x,t) - u^2(x,t),$$

Where $f(t)$ is a smooth function of $t$. In this study, the Lie symmetry and Fushchych methods applied on the GFKPP and obtained the symmetry groups, differential invariants & invariant solutions. After that, the Fourier transform method (FTM) (as the harmonic analysis method) applied on the GFKPP, and calculated solutions. Finally, the results of the equivalence methods (Lie and Fushchych methods) compared with FTM.

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1. Introduction

Sophus Lie developed the theory of Lie symmetry groups of differential equations at the 19th century [3]. The Lie symmetry groups are invertible point transformations of the dependent and independent variables of PDEs. The symmetry group methods offer an ultimate tools for analyzing the differential equations. The most important of applications of Lie groups in the theory of differential equations such as: the reduction of order of ordinary differential equations, construction of invariant solutions, mapping solutions to other solutions and the detection of linearizing transformations, were discussed in the literature (for many other applications of Lie symmetries, we refer the reader to [13], [14]).

The Fisher’s equation (also known as the Fisher Kolmogorov equation and the Fisher KPP equation, named after R. A. Fisher and A. N. Kolmogorov), is the PDE: $u_t = u_{xx} + u - u^2$ that describes the spatial spread of an advantageous allele and explored its travelling wave solutions and nowadays, proposed as a model of diffusion in biomathematics [8].

The experimental observation of an initially flat liquid-film interface that evolves with time to a propagating diffusion front with a constant front velocity are specific characteristics of the Fisher Kolmogorov Petrovskii Piskounov (FKPP) equation obeying a traveling wave solution.

These equations occur, for example, in ecology, physiology, combustion, crystallization, plasma physics, and in general phase transition problems, this equation is a well known and widely applied nonlinear reaction-diffusion equations [1], [11], and is traditionally applied to model the spread of genes in population genetics [9].

The proof of existence of travelling wave solutions and analysis of their properties is often done by the phase space method. Some important PDEs such as Balitsky-Kovchegov equation, transformed to FKPP equation in certain conditions [7].

The generalized version of FKPP equation with $t$-dependent coefficient as follows

$$f(t)u_{tt} + u_t = u_{xx} + u - u^2,$$

So far, this version of the equation was solved with numerical methods and any explicit solution was not found. By the Lie symmetry method, the generalized FKPP equation will be converted to ODEs and ODEs will be solved. All symmetries and generalized vector fileds will be determined.
The remainder of this paper is organized as follows. In Section 2, we recall some results needed to construct the Lie point, partial, conditional symmetries of a given system of differential equations and FTM. In Section 3, we classify the FKPP equation to 3 cases. In Section 4, we solve the FKPP equation in every one of 3 cases with the Lie symmetry method and obtain vector fields, Lie symmetry groups, invariant solutions and reduced ODEs. In Sections 5-7, we apply the Fushchych method on the FKPP equation and obtain standard CSs, partial symmetries and weak CSs. In Section 8, we apply FTM on the GFKPP and obtain solutions. Finally, some results and conclusions are presented and future works are expressed.

2. Preliminaries

The Lie symmetry method: In this section, we remember the normal procedure for determining symmetries for any system of PDEs (see [3], [13] and [12]). First, let us consider the general case of a nonlinear system of partial differential equations of order \(n\)th in \(p\) independent and \(q\) dependent variables, that is given as a system of equations:

\[
\Delta_\nu(x, u^{(n)}) = 0, \quad \nu = 1, \ldots, l, \tag{1}
\]

that involving \(x = (x^1, \ldots, x^p)\), \(u = (u^1, \ldots, u^q)\) and the derivatives of \(u\) with respect to \(x\) up to \(n\), where \(u^{(n)}\) represents all the derivatives of \(u\) of any orders from 0 to \(n\). We consider a one-parameter Lie group of infinitesimal transformations that acts on the independent and dependent variables of the system (1):

\[
(\tilde{x}^i, \tilde{u}^j) = (x^i, u^j) + s(\xi^i, \eta^j) + O(s^2), \quad i = 1 \cdots p, \ j = 1 \cdots q,
\]

Where \(s\) is the parameter of the transformation and \(\xi^i, \eta^j\) are the infinitesimals of the transformations for the independent and dependent variables (respectively). The infinitesimal generator \(v\) associated with the above group of transformations can be written as \(v = \sum_{i=1}^{p} \xi^i \partial_{x^i} + \sum_{j=1}^{q} \eta^j \partial_{u^j}\). A symmetry of a differential equation is a transformation which maps solutions of the equation to another solutions. The invariance of the system (1) under the infinitesimal transformations leads to the invariance conditions (Theorem 2.36 of [13]):

\[
\text{Pr}^{(n)}_\nu[\Delta_\nu(x, u^{(n)})] = 0, \quad \Delta_\nu(x, u^{(n)}) = 0, \quad \nu = 1, \ldots, l,
\]

Where \(\text{Pr}^{(n)}_\nu\) is called the \(n^{th}\) order prolongation of the infinitesimal generator given by \(\text{Pr}^{(n)}_\nu v = v + \sum_{a=1}^{q} \sum_{j} \phi_{a}^{(n)}(x, u^{(n)}) \partial_{u^j_a}\), where \(J = (j_1, \ldots, j_k), 1 \leq \].
\[ j_k \leq p, \ 1 \leq k \leq n \text{ and the sum is over all } J\text{'s of order } 0 < \#J \leq n. \text{ If } \#J = k, \text{ the coefficient } \phi^J_j \text{ of } \partial u^J_j \text{ will only depend on } k\text{-th and lower order derivatives of } u, \text{ and } \phi^J_j(x, u^{(n)}) = D_J(\alpha - \sum_{i=1}^p \xi_i u^\alpha_i) + \sum_{i=1}^p \xi_i u^\alpha_{j,i}, \text{ where } u^\alpha_i := \partial u^\alpha / \partial x^i \text{ and } u^\alpha_{j,i} := \partial u^\alpha_J / \partial x^i. \]

One of the most important and useful properties of these infinitesimal symmetries is that, they form a Lie algebra under the usual Lie bracket. The first advantage of symmetry group methods is to construct new solutions from older solutions. The second is when a nonlinear system of differential equations admits infinite symmetries, so it is possible to transform it to a linear system. Neither the first advantage nor the second will be investigated here, but symmetry group method will be applied to the PDE to be connected directly to some order differential equations. To do this, a particular linear combinations of infinitesimals are considered and their corresponding invariants are determined.

The PDE is exposed in the coordinates \((x, t, u)\), so to reduce this equation is to search for its form in specific coordinates. These coordinates will be constructed by searching for independent invariants \((y, v)\) corresponding to the infinitesimal generator. So using the chain rule, the expression of equation in the new coordinate allows us to reduce the order of this equation.

**The Fushchych method:** Let us now consider the Fushchych method. A fundamental and comprehensive method has been introduced by Fushchych [10]: let us say that \(X\) is a conditional symmetry of the equation \(\Delta \nu(x, u^{(n)}) = 0\) in the sense of Fushchych, if there is a supplementary equation \(E = 0\) such that \(X\) is an exact symmetry of the system \(\Delta \nu_i(x, u^{(n)}) = 0, \ E = 0\). The easiest and more general case is obtained by choosing as supplementary equation the side condition or invariant surface condition as follows

\[ X_Q u = \xi_i \frac{\partial u}{\partial x_i} - \phi = 0, \]

Where \(X_Q\) is the evolutionary form of symmetry [13].

This corresponds to the usual (properly called) conditional symmetry (CS) (also called Q-conditional symmetry). With the above condition, Indeed we are looking precisely for solutions which are invariant under \(X\). To avoid unessential complications with notations, we will consider from now on only the case of a single PDE: \(\Delta \nu = 0\) for a single unknown function \(u(x)\). The generalization to more general cases is completely straightforward.

**Definition 1.** Consider a PDE \(\Delta \nu = 0\). A projectable vector field \(X\) is a **standard** conditional symmetry for the equation, if it is a symmetry for the
system $\Delta_\nu = 0$, $X_Q(u) = 0$. In other hands, a reduced equation in $(p - 1)$ independent variables is existed, which gives $X$-invariant solutions of $\Delta_\nu = 0$.

**Definition 2.** A vector field $X$ that is a symmetry of the system

$$\Delta_\nu = 0, \quad \Pr^{(1)} X[\Delta_\nu(x, u^{(n)})] = 0, \cdots, \Pr^{(\sigma-1)} X[\Delta_\nu(x, u^{(1)})] = 0,$$

is called a *partial* symmetry (of order $\sigma$).

**Definition 3.** A vector field $X$ that is a symmetry of the system

$$\Delta_\nu = 0, \quad X_Q(u) = 0,$$

$$\Pr^{(1)} X[\Delta_\nu(x, u^{(n)})] = 0, \cdots, \Pr^{(\sigma-1)} X[\Delta_\nu(x, u^{(1)})] = 0,$$

is called a *weak* CS (of order $\sigma$). This corresponds to the existence of a system of $\sigma$ reduced equations, which gives $X$-invariant solutions of $\Delta_\nu = 0$.

**Theorem 4.** Any vector field $X$ is either an exact, or a standard CS, or a weak CS. Similarly, any $X$ is either an exact or a partial symmetry. For more details and examples, we refer the reader to [6].

**The Fourier transform method:** In the Fourier analysis, the Fourier transform method (FTM) is impressive and beneficial tools for analyzing ODEs & PDEs. In this method (like the Laplace transform method), we take FT from every side of ODE, and then by solving resulting algebraic equations, obtain FT. After that we apply the inverse FT formula for obtaining solutions. In the case 2 variables and in PDEs (for example, $(x, t)$), we assume $x$ is variable and $t$ is constant, then based on $x$ we take FT from both side of equation. In the following some formula will be presented (for the proofs and more details refer to [4]).

**Definition 5.** The Fourier transform of $u(x, t)$ (assuming that $x$ is variable and $t$ is constant) is defined by

$$F(u(x, t))(\omega) := \hat{u}(\omega, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, t) \exp(-i\omega x) dx$$

Where parameter $\omega$ is corresponding to $x$.

**Theorem 6.** Assume that $\Delta_\nu(x, t, u^{(2)}) = 0$ is a second order PDE with independent variables $(x, t)$ so that $t$ is constant and $x$ is variable, then

1) $F(\partial_t u(x, t))(\omega) = \frac{d}{dt} \hat{u}(\omega, t)$. 
2) $F(\partial_t^n u(x,t))(\omega) = \frac{d^n}{dt^n} \hat{u}(\omega,t)$.

3) $F(\partial_x u(x,t))(\omega) = (i\omega) \hat{u}(\omega,t)$.

4) $F(\partial_x^n u)(x,t) = (i\omega)^n \hat{u}(\omega,t)$.

The inverse formula for FT is defined as follows

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(\omega,t) \exp(i\omega x) d\omega.$$  

3. The Classification

Now, we consider the generalized FKPP equation with $t$-dependent coefficient as follows

$$f(t) u_{tt} + u_t = u_{xx} + u - u^2,$$

where $f(t)$ is a function of independent variable $t$. Based on the form of $f(t)$, we may be faced with the following three states

<table>
<thead>
<tr>
<th>$f$</th>
<th>The correspondent equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$u_t = u_{xx} + u - u^2$</td>
</tr>
<tr>
<td>$c$</td>
<td>$cu_{tt} + u_t = u_{xx} + u - u^2$</td>
</tr>
<tr>
<td>$f(t)$</td>
<td>$f(t)u_{tt} + u_t = u_{xx} + u - u^2$</td>
</tr>
</tbody>
</table>

It is worth pointing out that the states 1,2 were investigated in [15] and so in this paper, we consider only case 3. In fact, the Lie, partial and conditional symmetries and generator vector fields will be determined with the Lie symmetry and Fushchych methods. Then the differential classical and non-classical invariants of this case will be calculated and by reformulating PDEs according to the differential invariants, we have order reduction and the PDE will be converted to ODE. Indeed, in some cases explicit solution will be obtained.
4. The Lie Symmetries

The generated vector field of symmetry group is \( \mathbf{v} := \tau D_t + \xi D_x + \phi D_u \), based on the Lie symmetry method, the second order prolongation of \( \mathbf{v} \) be calculated and then

\[
\Pr^{(2)}\mathbf{v}[\Delta(x, u^{(2)})] = 0, \quad \Delta(x, u^{(2)}) = 0,
\]

After calculation, the determining equations will be appeared.

The second order prolongation of \( \mathbf{v} \) is

\[
\Pr^{(2)}\mathbf{v} = v + \Phi^x \partial_{u_x} + \Phi^t \partial_{u_t} + \Phi^{xx} \partial_{u_{xx}} + \Phi^{xt} \partial_{u_{xt}} + \Phi^{tt} \partial_{u_{tt}}.
\]

Then \( \Pr^{(2)}\mathbf{v}[\Delta] = 0 \) leads to \( \Phi^t - \Phi^{xx} - \Phi + 2u\Phi = 0 \). Therefore, we have

\[
\begin{align*}
\tau f'(t)u_{tt} + \Phi_t + \Phi_u u_t - u_x \xi_t - u_x u_t \xi_u - u_t \tau_t - u_t^2 \tau_u \\
- \Phi_{xx} - 2\Phi_{xu} u_x + u_x \xi_{xx} + 2u_x^2 \xi_{xu} + u_t \tau_{xx} + 2u_t u_x \tau_{xu} \\
-u_x^2 \Phi_{uu} + u_x^3 \xi_{uu} + u_t u_x^2 \tau_{uu} - u_{xx} \Phi_u + 3u_{xx} u_x \xi_u + 2u_{xx} \xi_x \\
+ u_{xx} u_t \tau_u + 2u_x \tau_x + 2u_{xt} u_x \tau_u + f(t)(\Phi_{tt} + 2u_t \Phi_{tu} - u_x \xi_{tt}) \\
- 2u_t u_x \xi_{tu} - u_t \tau_{tt} - 2u_t^2 \tau_{tu} + u_t^2 \Phi_{uu} - u_x u_t^2 \xi_{uu} - u_t^3 \tau_{uu} \\
- 2u_x \xi_t - 2u_{xt} \xi_u + u_{tt} \Phi_u - u_x u_{tt} \xi_u - 3u_t u_{tt} \tau_u - 2u_{xx} \tau_t)
\end{align*}
\]

Now \( f(t)u_{tt} + u_t = u_{xx} + u - u^2 \). After calculation, we have the determining equations as follows

\[
\begin{align*}
\tau_u = 0, \quad \tau_{uu} = 0, \\
\tau_{xx} + 2f(t)\Phi_{tu} + \Phi_u = \tau_t + f(t)\tau_{tt}, \\
\xi_u = 0, \quad \xi_{uu} = 0, \quad 2\xi_x = \Phi_u, \\
\Phi_{uu} = 0, \quad f'(t)\tau + f(t)\Phi_u = 2f(t)\tau_t, \\
\Phi_t + f(t)\Phi_{tt} = \Phi_{xx}, \quad \xi_t + f(t)\xi_{tt} + 2\Phi_{xx} = \xi_{xx}, \\
2\tau_x = 2f(t)\xi_t, \quad \Phi_u + 2f(t)\Phi_{tu} - \tau_{xx} = f(t)\tau_{tt} + \tau_t.
\end{align*}
\]

By calculating determining equations, the component of \( \mathbf{v} \) are \( \xi = C_1 \), \( \tau = 0 \), and \( \Phi = C_3(t x + x^3/6) + C_5(t + x^2/2) + C_2 x + C_3 \). Therefore

**Theorem 7.** Every infinitesimal generator of one parameter Lie group of symmetries of FKPP has the form \( \mathbf{v} = \sum_{i=1}^5 a_i \mathbf{v}_i \), where \( a_i \)s are arbitrary constants and

\[
\begin{align*}
\mathbf{v}_1 &= \partial_x, \quad \mathbf{v}_2 = x \partial_u, \quad \mathbf{v}_3 = \partial_u,
\end{align*}
\]
\[ \mathbf{v}_4 = (tx + \frac{x^3}{6})\partial_u, \quad \mathbf{v}_5 = (t + \frac{x^2}{2})\partial_u. \]

These vector fields span a solvable 5-dimensional Lie algebra \( \mathfrak{g} \) with the following commutator table

\[
\begin{array}{c|ccccc}
\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 \\
\mathbf{v}_1 & 0 & -\mathbf{v}_3 & 0 & \mathbf{v}_5 & \mathbf{v}_2 \\
\mathbf{v}_2 & -\mathbf{v}_3 & 0 & 0 & 0 & 0 \\
\mathbf{v}_3 & 0 & 0 & 0 & 0 & 0 \\
\mathbf{v}_4 & -\mathbf{v}_5 & 0 & 0 & 0 & 0 \\
\mathbf{v}_5 & -\mathbf{v}_2 & 0 & 0 & 0 & 0 \\
\end{array}
\]

For \( \mathbf{v}_1 = \partial_x \), associated characteristics system is \( dx/1 = dt/0 = du/0 \). Hence, two functionally independent invariants are \( y = t, v = u \). Based on the differential invariants \((y,v)\), the FKPP equation is reduced to ODE \( f(t)u''(t) + u'(t) = u(t) - u(t)^2 \). If \( f(t) = 0 \) \((f(t) = c)\), then the results are exactly similar to the case 1 (case 2). If \( f(t) = t \) then the Lie algebra \( \mathfrak{g} \) is a solvable 2-dimensional with the following commutator table

\[
\begin{array}{c|cc}
\mathbf{v}_1 & \mathbf{v}_2 \\
\mathbf{v}_1 & 0 & -\mathbf{v}_2 \\
\mathbf{v}_2 & -\mathbf{v}_2 & 0 \\
\end{array}
\]

Which

\[ \mathbf{v}_1 = x\partial_x + 2t\partial_t + 2u\partial_u, \quad \mathbf{v}_2 = \partial_x. \]

5. The Non-Classical Symmetries: SCSs

For solving FKPP equation by the Fushchych method, first let’s recall the generalized vector field

\[ \mathbf{v}_Q := \sum Q_\alpha [u] \frac{\partial}{\partial u_\alpha}, \]

This vector field also called evolutionary vector field and \( Q \) is called characteristics [13].

By calculating, the evolutionary vector fields are

\[ X_{Q_1} = u_x D_u, \quad X_{Q_2} = (u - u^2 + u_{xx})D_u, \]
Where \( f(t) \) is an arbitrary function of \( t \). Thus for calculating the standard CSs, we should solve the following equations

\[
\begin{align*}
X Q_1[u] &= 0, \\
X Q_2[u] &= 0,
\end{align*}
\Rightarrow
\begin{align*}
u_x &= 0 \\
u - u^2 + u_{xx} &= 0.
\end{align*}
\]

After solving the equations \( u_x = 0 \) and \( f(t)u_{tt} + u_t = 0 \), We have \( u = \int F_1(x) \exp(-\int \frac{dt}{f(t)})dt + F_2(x) \). Where \( F_1(x), F_2(x) \) are obtained by replacing \( u(x,t) \) in the PDE and solving the resulting equation.

6. The Non-Classical Symmetries: PSs

Similar to the standard Lie symmetry method, the generated vector field of partial symmetries is assumed to be of the form \( v := \tau D_t + \xi D_x + \phi D_u \).

6.1. The First Order Partial Symmetries

The first order prolongation of \( v \) is \( \text{Pr}^{(1)}v = v + \Phi^x \partial_x + \Phi^t \partial_t \), then \( \text{Pr}^{(1)}v[\Delta] = 0 \) which leads to

\[
\xi f'(t)u_{tt} + \phi_t + u_t \phi_u - u_x \xi_t - u_x u_t \xi_u - u_t \tau_t - u_t^2 \tau_u = 0.
\]

By replacing \( f(t)u_{tt} + u_t = u_{xx} + u - u^2 \) after calculation, the determining equations are appeared as follows

\[
\xi_t = \xi f'(t) = \tau_u = \xi_u = \phi_u - \tau_t = \phi_t = 0.
\]

If \( f(t) \) is an arbitrary function of \( t \) then

\[
\xi = F_1(x), \quad \tau = 0, \quad \Phi = F_2(x).
\]

Thus

**Theorem 8.** Every infinitesimal generator of one parameter Lie group of symmetries of FKPP has the form \( v = \sum_{i=1}^{2} a_i v_i \), where \( a_1, a_2 \) are arbitrary constants and

\[
v_1 = F_1(x)\partial_x, \quad v_2 = F_2(x)\partial_u.
\]

These vector fields span a solvable Lie algebra \( g \) which is infinite dimensional.
6.2. The Second Order Partial Symmetries

Second order prolongation of \( v \) is

\[
\text{Pr}^{(2)} v = v + \Phi^x \partial_{ux} + \Phi^t \partial_{ut} + \Phi^{xx} \partial_{uxx} + \Phi^{xt} \partial_{uxt} + \Phi^{tt} \partial_{utt},
\]

If \( f(t) \) be an arbitrary function of \( t \) then \( \xi = C_1, \tau = 0 \) and \( \Phi = C_4 (tx + x^3/6) + C_5 (t + x^2/2) + C_2 x + C_3. \) Thus

**Theorem 9.** Every infinitesimal generator of one parameter Lie group of symmetries of FKPP has the form \( v = \sum_{i=1}^{5} a_i v_i, \) where \( a_i \)s are arbitrary constants and \( v_1 = \partial_x, v_2 = x \partial_u, v_3 = \partial_u, v_4 = (tx + x^3/6) \partial_u, \) and \( v_5 = (t + x^2/2) \partial_u. \) These vector fields span a trivial solvable 5-dimensional Lie algebra \( g. \)

7. The Non-Classical Symmetries: WCSs

By definition of weak CS for finding \( n-\)th order weak CS, we should intersect first to \( n-\)th order partial symmetries with standard CS (that is, the WCS from \( n-\)th order is \( n-\)th order partial symmetry that, satisfies the SCS conditions or it is SCS that satisfies \( n-\)th order PS conditions).

7.1. The First Order Weak Conditional Symmetries

We should solve the following equations

\[
\begin{align*}
u_{xx} + u - u^2 &= 0, \\
f(t)u_{tt} + u_t &= 0, \\
\xi u_x + \tau u_t &= \Phi.
\end{align*}
\]

Thus

\[
u = \int F_1 \exp(-\int \frac{t}{f(t)} dt) dt + F_2,\]

\[(0) u_x + \tau (F_1 \exp(-\int \frac{t}{f(t)} dt)) = \Phi.\]

Now intersect with the first order partial symmetries and we have

\[
\begin{align*}
\xi &= F_1(x), \\
\tau &= 0, \\
\Phi &= 0.
\end{align*}
\]

Therefore
Theorem 10. Every infinitesimal generator of one parameter Lie group of symmetries of FKPP has the form \( \mathbf{v} = \alpha \mathbf{v}_1 \), where \( \alpha \) are arbitrary constants and
\[
\mathbf{v}_1 = F_1(x) \partial_x,
\]
This vector field spans a solvable Lie algebra \( \mathfrak{g} \) which is infinite dimensional.

7.2. The Second Order Weak Conditional Symmetries

We should solve the following equations
\[
u_{xx} + u - u^2 = 0, \quad f(t)u_{tt} + u_t = 0, \quad \xi u_x + \tau u_t = \Phi.
\]
Thus
\[
u = \int \{F_1 \exp\left(- \int \frac{t}{f(t)} dt\right)\} dt + F_2,
\]
\[
C_1 u_x + (0)\left(\frac{tF_1}{f(t)} \exp\left(- \int \frac{t}{f(t)} dt\right)\right) = \Phi.
\]
and now by intersecting with the second order partial symmetries, we get
\[
\xi = C_1, \quad \tau = 0, \quad \Phi = 0.
\]
Therefore

Theorem 11. Every infinitesimal generator of one parameter Lie group of symmetries of FKPP has the form \( \mathbf{v} = \alpha \mathbf{v}_1 \), where \( \alpha \) is arbitrary constants and
\[
\mathbf{v}_1 = \partial_x,
\]
This vector field spans a solvable 1-dimensional Lie algebra \( \mathfrak{g} \).

8. The Fourier Transform Method

We consider the GFKPP with \( f(t) \) as follows
\[
f(t)u_{tt} + u_t = u_{xx} + F(u)
\]
By applying FTM we have
\[
f(t)\frac{d^2}{dt^2}\hat{u}(\omega, t) + \frac{d}{dt}\hat{u}(\omega, t) = (i\omega)^2\hat{u}(\omega, t) + \hat{F}(u(\omega, t))
\]
Thus

\[ f(t) \frac{d^2}{dt^2} \hat{u} + \frac{d}{dt} \hat{u} + \omega^2 \hat{u} - \hat{F}(u(\omega, t)) = 0 \]

This equation is not solvable with analytic methods. In fact for an arbitrary function \( f(t) \), this method is not effective.

9. Conclusions and Future Works

In this paper, we applied the Lie symmetry and Fushchych methods on the generalized FKPP equation with \( t \)-dependent coefficient and calculated the vector fields, symmetry groups, exact and non-exact solutions. For demonstrating the performance of equivalence methods, we applied a famous harmonic analysis method (the Fourier transform method) on the FKPP and obtained the solutions. Now, we compared its results with the results of Lie symmetry and Fushchych methods. The results of applying the Lie symmetry method, Fushchych method and FTM on the GFKPP with \( f(t) \) are presented in the tables 1-3:

<table>
<thead>
<tr>
<th>VF</th>
<th>dim(( g ))</th>
<th>Invariants</th>
<th>( u(x,t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t )</td>
<td>( x \partial_x + 2t \partial_t )</td>
<td>2</td>
<td>( (\frac{x^2}{t}, \frac{u}{t}) ), ( (t, u) )</td>
</tr>
<tr>
<td>( f(t) )</td>
<td>( \partial_x, x \partial_u, \partial_t, tx + \frac{x^3}{6} \partial_u, t + \frac{x^2}{2} \partial_u )</td>
<td>5</td>
<td>( (t, u) ), ( f(t)u''(t) + u'(t) = u(t) - u(t)^2 )</td>
</tr>
</tbody>
</table>

Table 1. The Lie symmetry method results.

<table>
<thead>
<tr>
<th>( \text{SCS}(u(x,t)) )</th>
<th>( \dim(\text{g}) ) of PS, WCS(1)&amp;WCS(2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(t) )</td>
<td>( u = \int { F(x) \exp(-\int \frac{dt}{f(t)}) } dt + c )</td>
</tr>
</tbody>
</table>

Table 2. The Fushchych method results.

<table>
<thead>
<tr>
<th>( f(t) )</th>
<th>Reduced equation for ( \hat{u}(x,t) )</th>
<th>( u_h(x,t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(t) )</td>
<td>( f(t) \frac{d^2}{dt^2} \hat{u} + \frac{d}{dt} \hat{u} + \omega^2 \hat{u} = F^3 )</td>
<td>Algorithm restricted</td>
</tr>
</tbody>
</table>

Table 3. The results of FTM.
Thus, the Lie and Fushchych methods are more advantageous than other methods like FTM. In the future works, we want to combine the geometric and harmonic analysis methods for solving ODEs & PDEs. Indeed, we will propose the novel methods based on the Lie and Fushchych data with applying the harmonic and Fourier analysis approaches.

References


