

WEAKLY μ -NORMAL GENERALIZED TOPOLOGIES

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Abstract: In this paper, a new kind of normal spaces called $w\mu$ -normal spaces are introduced and studied in generalized topological spaces. Some of their properties of $w\mu$ -normal spaces are proved, and their characterization with the help of a suitable form of Urysohn's Lemma is discussed.

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1. Introduction

In 1970, the concept of generalized closed set in a topological space was introduced by N. Levine [8] in order to extend many of the important properties of closed sets to a larger family. After that, the concept of generalized closed set has been investigated by many mathematicians because the notion of generalized closed set is a natural generalization of closed set (see, [8], [10], [12] for details). It is also well known that separation axioms are one of the basic subjects of study in general topology and in several branches of mathematics. In

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literature separation axioms have been studied by different mathematicians. In 1973, Singal et al. introduced the concept of almost regular [13], almost normal [14] and mildly normal [15] spaces. The aim of this paper is to unify such types of existing spaces by using the concept of generalized topology introduced by A. Császár.

According to [1], a generalized topology (briefly GT) on a set X is a subset μ of the power set $\exp X$ such that $\emptyset \in \mu$ and the union of the elements of an arbitrary subset of μ belongs to μ . The elements of μ are said to be μ -open, their complements μ -closed. Clearly each topology is a GT.

As a generalization of the concept of normal topology, it is natural to say that a GT μ is normal iff, whenever V and V' are μ -closed sets such that $V \cap V' = \emptyset$, there exists μ -open sets U and U' satisfying $V \subset U$, $V' \subset U'$ and $U \cap U' = \emptyset$. In the literature, there are many papers discussing properties of normal GT's for some particular GT μ (cf. e.g. [3], [4], [6], [7], [9], [11])

In the present paper, our purpose is to indicate some general properties of a $w\mu$ -normal GT and in particular, to give a characterization of them with the help of a form of the well-known Urysohn's Lemma. (see e.g. [5])

2. Preliminaries

For $A \subset X$, we denote by $c_\mu(A)$ the intersection of all μ -closed sets containing A , i.e., the smallest μ -closed set containing A ; and by $i_\mu(A)$ the union of all μ -open sets contained in A , i.e., the largest μ -open set contained in A . A set $A \subset X$ is said to be μ -semi open [2] if $A \subset c_\mu(i_\mu(A))$. We denote by $\sigma(\mu)$ the class of all μ -semi open sets.

Definition 1. [16] Let μ be a GT on a topological space X . Then $A \subset X$ is called weakly μ -closed set or simply an $w\mu$ -closed set if $c_\mu(A) \subset U$ whenever $A \subset U$ and $U \in \sigma(\mu)$. The complement of an $w\mu$ -closed set is called an $w\mu$ -open set. The family of all $w\mu$ -open sets in a space X is denoted by $w\mu(X)$.

3. Fundamental Properties of $w\mu$ -Normal GT's

Definition 2. Let μ be a GT on X . Then (X, μ) is said to be weakly μ -normal (for short, $w\mu$ -normal) if for each $w\mu$ -closed sets U and V of X with $U \cap V = \emptyset$, there exists two disjoint $w\mu$ -open sets U' and V' such that $U \subset U'$ and $V \subset V'$.

A first obvious remark rises from the fact that it is possible for a GT μ that X is not $w\mu$ -open.

Proposition 3. If μ is a GT such that $X \notin w\mu$ then μ is $w\mu$ -normal.

Proof. There are now no disjoint $w\mu$ -closed sets, since $X - V, X - V' \in w\mu$, $V \cap V' = \emptyset$ would imply $X = (X - V) \cup (X - V') \in w\mu$. $\square$

An almost obvious equivalent characterization is:

Proposition 4. A GT μ on X is $w\mu$ -normal iff $V \subset U, X - V, U \in w\mu$ imply the existence of $U', X - V' \in w\mu$ such that $V \subset U' \subset V' \subset U$.

Proof. If μ is $w\mu$ -normal then, for the disjoint $w\mu$ -closed sets V and $X - U$, we choose the disjoint $w\mu$ -open sets $U' \supset V$ and $X - V' \supset X - U$.

Conversely, if $V \subset U, X - V, U \in w\mu$ implies the existence of $U', X - V' \in w\mu$ satisfying $V \subset U' \subset V' \subset U$, then, given disjoint $w\mu$ -closed sets W, W' we can choose $U', X - V' \in w\mu$ such that $W \subset U' \subset V' \subset X - W'$ and then $U', X - V'$ are disjoint $w\mu$ -open sets satisfying $W \subset U', W' \subset X - V'$ so that μ is $w\mu$ -normal. $\square$

Consider now a GT μ on X and a map $g : X_0 \rightarrow X$. Similarly to the particular case of topologies, we say that the sets $g^{-1}(M) : M \in w\mu$ constitute the inverse image $w\mu_0 = g^{-1}(w\mu)$ of μ .

Proposition 5. If μ is a GT on X then $g^{-1}(\mu)$ is a GT on X_0

Proof. $\bigcup_{i \in I} g^{-1}(M_i) = g^{-1}(U_{i \in I} M_i)$. $\square$

Of course, the $w\mu_0$ -closed sets are those of the form $g^{-1}(V)$ where V is $w\mu$ -closed.

In the particular case when $X_0 \subset X$ and $g(x) = x$ for $x \in X_0$ we say that $w\mu_0$ is the restriction of $w\mu$ to X_0 , we write now $w\mu_0 = w\mu|_{X_0}$. In this case $g^{-1}(A) = A \cap X_0$ for $A \subset X$.

Proposition 6. If μ is a $w\mu$ -normal GT on X and $g : X_0 \rightarrow X$ is surjective then $g^{-1}(\mu)$ is $w\mu$ -normal.

Proof. Suppose V_0, V'_0 are disjoint $w\mu_0$ -closed sets. Then $V_0 = g^{-1}(V)$, $V'_0 = g^{-1}(V')$, and $x \in V \cap V'$ would imply $x = g(y)$ for some $y \in X_0$ so that $y \in V_0 \cap V'_0$ would hold. Therefore V and V' are disjoint, there are disjoint $w\mu$ -open sets U, U' such that $V \subset U, V' \subset U'$, and then $U_0 = g^{-1}(U)$, $U'_0 = g^{-1}(U')$ are disjoint $w\mu_0$ -open sets such that $V_0 \subset U_0, V'_0 \subset U'_0$. $\square$

For the case of a restriction, the above statement cannot be used because then g is not surjective in general. However, we have:

Proposition 7. If μ is a $w\mu$ -normal GT on X and $X_0 \subset X$ is $w\mu$ -closed then then the restriction $w\mu_0 = w\mu|_{X_0}$ is $w\mu$ -normal.

Proof. If V_0, V_0^1 are disjoint $w\mu_0$ -closed sets then $V_0 = V \cap X_0$ for some $w\mu$ -closed set V , but so V_0 is $w\mu$ -closed itself. Similarly V_0^1 is $w\mu$ -closed. Therefore there exists disjoint $w\mu$ -open sets U, U' satisfying $V_0 \subset U, V_0^1 \subset U'$ and then $U \cap X_0, U' \cap X_0$ are disjoint, $w\mu_0$ -open and satisfy $V_0 \subset U \cap X_0, V_0^1 \subset U' \cap X_0$. $\square$

4. Urysohn's Lemma for $w\mu$ -Normal GT's

Definition 8. If μ is $w\mu$ -normal GT on X , ν is a GT on Y , then $f : X \rightarrow Y$ is said to be (μ, ν) -weakly continuous iff $f^{-1}(N) \in w\mu(X)$ for each $N \in \nu$.

Let $\beta \subset \exp Y$ be arbitrary. Then:

Lemma 9. The family $\nu \subset \exp Y$ composed of $\emptyset$ and all sets $N \subset Y$ of the form $N = \bigcup_{i \in I} B_i$, where $B_i \in \beta$ and $I \neq \emptyset$ is arbitrary, is a GT on Y .

We say that the base β generates the GT ν .

E.g Consider $Y = R$ and $\beta = \{(-\infty, t) : t \in R\} \cup \{(t, +\infty) : t \in R\}$. Then the GT on R generated by the base β will be denoted by ν .

Lemma 10. Let μ be a GT on X and the GT ν on Y be generated by the base β . Then a map $f : X \rightarrow Y$ is (μ, ν) -weakly continuous iff $f^{-1}(B) \in w\mu(X)$ for each $B \in \beta$.

Now we are able to prove the following variant of Urysohn's Lemma:

Theorem 11. Let μ be a $w\mu$ -normal GT on X and V, V' be disjoint $w\mu$ -closed sets. Then there exists a (μ, ν) -weakly continuous function $f : X \rightarrow R$ such that $f(x) = 0$ for $x \in V$ and $f(x) = 1$ for $x \in V'$.

Proof. Let D denote the collection of all real numbers of the form $m/2^n$, $n = 0, 1, \dots, m \in Z$. We first define $w\mu$ -open sets $U(r)$ and $w\mu$ -closed sets $V(r)$ for $r \in D$ satisfying

$$U(r) \subset V(r) \quad \text{for } r \in D \quad (1)$$

and

$$V(r) \subset U(s) \quad \text{for } r, s \in D \quad (r < s). \quad (2)$$

First put for $r \in D$

$$U(r) = \emptyset \quad (r \leq 0), \quad V(r) = \emptyset \quad (r < 0), \quad (3)$$

$$U(r) = X \quad (r > 1), \quad V(r) = X \quad (r \geq 1), \quad (4)$$

$$V(0) = V \quad U(1) = X - V'. \quad (5)$$

So the sets $U(r)$ and $V(r)$ are defined for $r \in D$, $r \leq 0$ and $r \geq 1$ in a way that (1) and (2) are valid.

We have to define $U(r)$ and $V(r)$ for $r \in D$, $0 < r < 1$. Let D_n denote the set composed of $m/2^n$ for $n = 0, 1, 2, \dots$, and $m = 0, 1, \dots, 2^n$. We shall define $U(r)$ and $V(r)$ for $r \in D_n$, $n = 0, 1, 2, \dots$. The set $D_0 = \{0, 1\}$ is settled by (3) and (5). This will be the starting point of a recursion.

Suppose that $U(r)$ and $V(r)$ are defined for $r \in D_k$, $k = 0, 1, \dots, n$ in such a way that (1) and (2) are fulfilled. Then, in particular, $V(m/2^n) \subset U((m+1)/2^n)$. Define the $w\mu$ -open set $U((2m+1)/2^{n+1})$ and the $w\mu$ -closed set $V((2m+1)/2^{n+1})$ using 4 so that

$$V(m/2^n) \subset U((2m+1)/2^{n+1}) \subset V((2m+1)/2^{n+1}) \subset U((m+1)/2^n).$$

This being done for $m = 0, 1, \dots, 2^n - 1$, the set D_{n+1} is settled. At the end of this recursion, $U(r)$ and $V(r)$ is known for $r \in D$ and (1), (2) are valid.

Define now

$$f(x) = \inf \{r \in D : x \in V(r)\}$$

By (3) and (4) $0 \leq f(x) \leq 1$ and $f(x) = 0$ for $x \in V(0) = V$ by (5), $f(x) = 1$ for $x \in V'$ since $r \in D$, $r < 1$ implies $V(r) \subset U(1) = X - V'$, thus $x \notin V(r)$, by (5) again.

We have to show that f is (μ, ν) -weakly continuous. According to (2) it suffices to examine whether $f^{-1}((-\infty, t))$ and $f^{-1}((t, +\infty))$ belongs to μ .

If x belongs to the first set, i.e. if $f(x) < t$ then there is $r \in D$ such that $r < t$ and $x \in V(r)$, and then $x \in U(s)$ for $s \in D$, $r < s < t$, according to (2). Now $y \in U(s)$ implies $y \in V(s)$ by (1), therefore $f(y) \leq s < t$, so that $U(s)$ is a $w\mu$ -open set satisfying $x \in U(s) \subset f^{-1}((-\infty, t))$: the latter set is the union of $w\mu$ -open sets and so $w\mu$ -open itself.

If $x \in f^{-1}((t, +\infty))$ then $t < f(x)$ so that $x \notin V(r)$ whenever $r \in D$ such that $r < f(x)$. Choose r such that $t < r < f(x)$. For the $w\mu$ -open set

$X - V(r)$, necessarily $x \in X - V(r) \subset f^{-1}((t, +\infty))$: in fact $y \in X - V(r)$ implies $y \in V(s)$ $s \in D$, $s < r$ is impossible since then $y \in V(s) \subset U(r) \subset V(r)$ would hold by (1) and (2). Therefore $y \notin V(s)$ for these s , $f(y) \geq r > t$. Again, $f^{-1}((t, +\infty))$ is the union of $w\mu$ -open sets and so $w\mu$ -open itself. $\square$

The statement in 11 is sufficient for the normality of $w\mu$:

Theorem 12. If μ is a GT on X with the property that, if V, V' are disjoint $w\mu$ -closed sets, there exists a (μ, ν) -weakly continuous function $f : X \rightarrow R$ satisfying $f(x) = 0$ for $x \in V$ and $f(x) = 1$ for $x \in V'$, then μ is $w\mu$ -normal.

Proof. The disjoint sets $f^{-1}((-\infty, 1/2))$ and $f^{-1}((1/2, +\infty))$ are $w\mu$ -open and contain V and V' respectively. $\square$

References

- [1] A. Császár, Generalized topology, generalized continuity, *Acta Math. Hungar*, **96** (2002), 351-357.
- [2] A. Császár, Generalized open sets in generalized topologies, *Acta Math. Hungar*, **106** (2005), 53-66.
- [3] J. Dontchev and T. Noiri, Quasi normal spaces and πg -closed sets, *Acta Math. Hungar*, **89** (2000), 211-219.
- [4] E. Ekici, On γ -normal spaces, *Bull. Math. Soc. Sci. Math. Roumanie*, **50** (98) (2007), 259-272.
- [5] R. Engelking, *General topology*, Heldermann Verlag (Berlin, 1989).
- [6] M. Ganster, S. Jafari and G. B. Navalagi, On semi- g -regular and semi- g -normal spaces, *Demonstratio Math*, **35** (2002), 415-421.
- [7] J. K. Kohli and A. K. Das, New normality axioms and decompositions of normality, *Glasnik Math*, **37** (2002), 163-173.
- [8] N. Levine, Generalized closed sets in topology, *Rend. Circ. Mat. Palermo*, **19** (2) (1970), 89-96.
- [9] S. N. Maheshwari and R. Prasad, On s -normal spaces, *Bull. Math. Soc. Sci. Math. R. S. Roumanie (N. S.)*, **22** (68) (1978), 27-29.
- [10] H. Maki, J. Umehara and T. Noiri, Every topological space is pre- $t_{1/2}$, *Mem. Fac. Sci. Kochi Univ. Ser. A(Math.)*, **17** (1996), 33-42.
- [11] T. Noiri, Semi-normal spaces and some functions, *Acta Math. Hungar*, **65** (1994), 305-311.
- [12] N. Palaniappan and K. Chandrasekhara Rao, Regular generalized closed sets, *Kyungpook Math. J.*, **33** (2) (1993), 211-219.
- [13] M. K. Singal and S. P. Arya, On almost-regular spaces, *Glasnik Mat.*, **4** (24) (1969), 89-99.

- [14] M. K. Singal and S. P. Arya, Almost normal, almost completely regular spaces, *Glasnik Mat.*, **5** (25) (1970), 141-152.
- [15] M. K. Singal and A. R. Singal, Mildly normal spaces, *Kyungpook Math. J.*, **13** (1973), 27-31.
- [16] A. Vadivel, Regular weakly μ -closed sets, almost weakly μ -regular, almost weakly μ -normal and mildly weakly μ -normal in Topological Spaces, *Journal of Advanced Studies in Topology*, **1** (2) (2010), 37-44.

