FAINTLY SEMICONTINUOUS MULTIFUNCTIONS VIA IDEALS

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Abstract: The aim of this paper is to introduce and study upper and lower faintly semi-\(I\)-continuous multifunctions as a generalization of upper and lower semi-\(I\)-continuous multifunctions, respectively.

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1. Introduction

It is well known that various types of functions play a significant role in the theory of classical point set topology. A great number of papers dealing with such functions have appeared, and a good number of them have been extended to the setting of multifunctions [1, 9, 10, 11, 12]. This implies that both,
functions and multifunctions are important tools for studying other properties of spaces and for constructing new spaces from previously existing ones. The concept of ideals in topological spaces has been introduced and studied by Kuratowski [7] and Vaidyanathaswamy [15]. An ideal $\mathcal{I}$ on a topological space $(X, \tau)$ is a nonempty collection of subsets of $X$ which satisfies (i) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. Given a topological space $(X, \tau)$ with an ideal $\mathcal{I}$ on $X$ and if $\mathcal{P}(X)$ is the set of all subsets of $X$, a set operator $(\cdot)^* : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$, called the local function [15] of $A$ with respect to $\tau$ and $\mathcal{I}$, is defined as follows: for $A \subset X$, $A^*(\tau, \mathcal{I}) = \{x \in X | U \cap A \notin \mathcal{I} \text{ for every } U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau | x \in U\}$. A Kuratowski closure operator $\text{Cl}^*(\cdot)$ for a topology $\tau^*(\tau, \mathcal{I})$ called the $*$-topology, finer than $\tau$ is defined by $\text{Cl}^*(A) = A \cup A^*(\tau, \mathcal{I})$ when there is no chance of confusion, $A^*(\mathcal{I})$ is denoted by $A^*$. If $\mathcal{I}$ is an ideal on $X$, then $(X, \tau, \mathcal{I})$ is called an ideal topological space. Recently, Akdag and Canan [1] introduced and studied the concept of semi-$\mathcal{I}$-continuous multifunctions in topological spaces. In this paper, we introduce and study upper and lower faintly semi-$\mathcal{I}$-continuous multifunctions on ideal topological space.

2. Preliminaries

For a subset $A$ of a topological space $(X, \tau)$, $\text{Cl}(A)$ and $\text{Int}(A)$ denote the closure of $A$ with respect to $\tau$ and the interior of $A$ with respect to $\tau$, respectively. A subset $S$ of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be semi-$\mathcal{I}$-open [4] if $S \subset \text{Cl}^*(\text{Int}(S))$. The complement of a semi-$\mathcal{I}$-closed set is said to be a semi-$\mathcal{I}$-open set. The semi-$\mathcal{I}$-closure and the semi-$\mathcal{I}$-interior, that can be defined in the same way as $\text{Cl}(A)$ and $\text{Int}(A)$, respectively, will be denoted by $s\mathcal{I}\text{Cl}(A)$ and $s\mathcal{I}\text{Int}(A)$, respectively. The family of all semi-$\mathcal{I}$-open (resp. semi-$\mathcal{I}$-closed) sets of $(X, \tau, \mathcal{I})$ is denoted by $\text{STO}(X)$ (resp. $\text{STC}(X)$). The family of all semi-$\mathcal{I}$-open (resp. semi-$\mathcal{I}$-closed) sets of $(X, \tau, \mathcal{I})$ containing a point $x \in X$ is denoted by $\text{STO}(X, x)$ (resp. $\text{STC}(X, x)$). A subset $N$ of a topological space $(X, \tau)$ is said to be semi-$\mathcal{I}$-neighborhood of a point $x \in X$, if there exists a semi-$\mathcal{I}$-open set $V$ such that $x \in V \subset N$. A point $x \in X$ is called a $\theta$-cluster point of $A$ [14] if $\text{Cl}(V) \cap A \neq \emptyset$ for every open set $V$ of $X$ containing $x$. The set of all $\theta$-cluster points of $A$ is called the $\theta$-closure of $A$ and is denoted by $\text{Cl}_\theta(A)$. If $A = \text{Cl}_\theta(A)$, then $A$ is said to be $\theta$-closed [14]. The complement of $\theta$-closed set is said to be $\theta$-open [14]. The union of all $\theta$-open sets contained in a subset $A$ is called the $\theta$-interior of $A$ and is denoted by $\text{Int}_\theta(A)$. It follows from [14] that the collection of $\theta$-open sets
in a topological space \((X, \tau)\) forms a topology \(\tau_\theta\) on \(X\). By a multifunction \(F: (X, \tau, \mathcal{I}) \to (Y, \sigma)\), we mean a point-to-set correspondence from \(X\) into \(Y\), also we always assume that \(F(x) \neq \emptyset\) for all \(x \in X\). For a multifunction \(F: (X, \tau) \to (Y, \sigma)\), the upper and lower inverse of any subset \(A\) of \(Y\) by \(F^+(A)\) and \(F^-(A)\), respectively, that is \(F^+(A) = \{x \in X : F(x) \subseteq A\}\) and \(F^-(A) = \{x \in X : F(x) \cap A \neq \emptyset\}\). In particular, \(F^-(y) = \{x \in X : y \in F(x)\}\) for each point \(y \in Y\). A multifunction \(F: (X, \tau, \mathcal{I}) \to (Y, \sigma)\) is said to be lower semi-\(\mathcal{I}\)-continuous [1] (resp. upper semi-\(\mathcal{I}\)-continuous) multifunction if \(F^-(V) \in S\mathcal{I}O(X, \tau)\) (resp. \(F^+(V) \in S\mathcal{I}O(X, \tau)\)) for every \(V \in \sigma\).

### 3. Faintly Semi-\(\mathcal{I}\)-Continuous Multifunctions

**Definition 3.1.** A multifunction \(F: (X, \tau, \mathcal{I}) \to (Y, \sigma)\) is said to be:

1. upper faintly semi-\(\mathcal{I}\)-continuous at \(x \in X\) if for each \(\theta\)-open set \(V\) of \(Y\) containing \(F(x)\), there exists \(U \in S\mathcal{I}O(X)\) containing \(x\) such that \(F(U) \subseteq V\);

2. lower faintly semi-\(\mathcal{I}\)-continuous at \(x \in X\) if for each \(\theta\)-open set \(V\) of \(Y\) such that \(F(x) \cap V \neq \emptyset\), there exists \(U \in S\mathcal{I}O(X)\) containing \(x\) such that \(F(u) \cap V \neq \emptyset\) for every \(u \in U\);

3. upper (lower) faintly semi-\(\mathcal{I}\)-continuous if it has this property at each point of \(X\).

**Remark 3.2.** It is clear that every upper semi-\(\mathcal{I}\)-continuous multifunction is upper faintly semi-\(\mathcal{I}\)-continuous. But the converse is not true in general, as the following example shows.

**Example 3.3.** Let \(X = \{a, b, c\}\), \(\tau = \{\emptyset, \{b\}, X\}\), \(\sigma = \{\emptyset, \{a\}, X\}\) and \(\mathcal{I} = \{\emptyset, \{a\}\}\). The multifunction \(F: (X, \tau, \mathcal{I}) \to (X, \sigma)\) defined by \(F(x) = \{x\}\) for all \(x \in X\) is upper faintly semi-\(\mathcal{I}\)-continuous but is not upper semi-\(\mathcal{I}\)-continuous.

**Definition 3.4.** A sequence \((x_\alpha)\) is said to semi-\(\mathcal{I}\)-converge to a point \(x\) if for every semi-\(\mathcal{I}\)-open set \(V\) containing \(x\), there exists an index \(\alpha_0\) such that for \(\alpha \geq \alpha_0\), \(x_\alpha \in V\). This is denoted by \(x_\alpha \overset{\mathcal{I}}{\to} x\).

**Theorem 3.5.** For a multifunction \(F: (X, \tau, \mathcal{I}) \to (Y, \sigma)\), the following statements are equivalent:

1. \(F\) is upper faintly semi-\(\mathcal{I}\)-continuous;
2. For each \( x \in X \) and for each \( \theta \)-open set \( V \) such that \( x \in F^+(V) \), there exists a semi-\( I \)-open set \( U \) containing \( x \) such that \( U \subset F^+(V) \);

3. For each \( x \in X \) and for each \( \theta \)-closed set \( V \) such that \( x \in F^+(Y \setminus V) \), there exists a semi-\( I \)-closed set \( H \) such that \( x \in X \setminus H \) and \( F^-(V) \subset H \);

4. \( F^+(V) \) is a semi-\( I \)-open set for any \( \theta \)-open set \( V \) of \( Y \);

5. \( F^-(V) \) is a semi-\( I \)-closed set for any \( \theta \)-closed set \( V \) of \( Y \);

6. \( F^-(Y \setminus V) \) is a semi-\( I \)-closed set for any \( \theta \)-open set \( V \) of \( Y \);

7. \( F^+(Y \setminus V) \) is a semi-\( I \)-open set for any \( \theta \)-closed set \( V \) of \( Y \);

8. For each \( x \in X \) and for each net \( (x_\alpha) \) which \( I \)-converges to \( x \in X \) and for each \( \theta \)-open set \( V \) of \( Y \) such that \( x \in F^+(V) \), the net \( (x_\alpha) \) is eventually in \( F^+(V) \).

Proof. (1)\( \Leftrightarrow \) (2): Clear. (2)\( \Leftrightarrow \) (3): Let \( x \in X \) and \( V \) be a \( \theta \)-open set of \( Y \) such that \( x \in F^+(Y \setminus V) \). By (2), there exists a semi-\( I \)-open set \( U \) containing \( x \) such that \( U \subset F^+(Y \setminus V) \). Then \( F^-(V) \subset X \setminus U \). Take \( H = X \setminus U \). We have \( x \in X \setminus H \) and \( H \) is semi-\( I \)-open. The converse is similar. (1)\( \Leftrightarrow \) (4): Let \( x \in F^+(V) \) and \( V \) be a \( \theta \)-open set of \( Y \). By (1), there exists a semi-\( I \)-open set \( U_x \) containing \( x \) such that \( U_x \subset F^+(V) \). It follows that \( F^+(V) = \bigcup_{x \in F^+(V)} U_x \).

Since any union of semi-\( I \)-open sets is semi-\( I \)-open, \( F^+(V) \) is semi-\( I \)-open. The converse can be shown similarly. (4)\( \Leftrightarrow \) (5)\( \Leftrightarrow \) (6)\( \Leftrightarrow \) (7)\( \Leftrightarrow \) (8): Clear. (1)\( \Rightarrow \) (8): Let \( (x_\alpha) \) be a net which semi-\( I \)-converges to \( x \) in \( X \) and let \( V \) be any \( \theta \)-open set of \( Y \) such that \( x \in F^+(V) \). Since \( F \) is an upper faintly semi-\( I \)-continuous multifunction, it follows that there exists a semi-\( I \)-open set \( U \) of \( X \) containing \( x \) such that \( U \subset F^+(V) \). Since \( (x_\alpha) \) semi-\( I \)-converges to \( x \), it follows that there exists an index \( \alpha_0 \in J \) such that \( x_\alpha \in U \) for all \( \alpha \geq \alpha_0 \). From here, we obtain that \( x_\alpha \in U \subset F^+(V) \) for all \( \alpha \geq \alpha_0 \). Thus, the net \( (x_\alpha) \) is eventually in \( F^+(V) \). (8)\( \Rightarrow \) (1): Suppose that (1) is not true. There exists a point \( x \) and a \( \theta \)-open set \( V \) with \( x \in F^+(V) \) such that \( U \not\subset F^+(V) \) for each semi-\( I \)-open set \( U \) of \( X \) containing \( x \). Let \( x_U \in U \) and \( x_U \not\in F^+(V) \) for each semi-\( I \)-open set \( U \) of \( X \) containing \( x \). Then for each semi-\( I \)-neighborhood net \( (x_U) \), \( x_U \not\leftrightarrow x \), but \( (x_U) \) is not eventually in \( F^+(V) \). This is a contradiction. Thus, \( F \) is an upper faintly semi-\( I \)-continuous multifunction. \( \square \)

**Theorem 3.6.** For a multifunction \( F : (X, \tau, I) \to (Y, \sigma) \), the following statements are equivalent:
1. $F$ is lower faintly semi-$\mathcal{I}$-continuous;

2. For each $x \in X$ and for each $\theta$-open set $V$ such that $x \in F^-(V)$, there exists a semi-$\mathcal{I}$-open set $U$ containing $x$ such that $U \subset F^-(V)$;

3. For each $x \in X$ and for each $\theta$-closed set $V$ such that $x \in F^-(Y \setminus V)$, there exists a semi-$\mathcal{I}$-closed set $H$ such that $x \in X \setminus H$ and $F^+(V) \subset H$;

4. $F^-(V)$ is a semi-$\mathcal{I}$-open set for any $\theta$-open set $V$ of $Y$;

5. $F^+(V)$ is a semi-$\mathcal{I}$-closed set for any $\theta$-closed set $V$ of $Y$;

6. $F^+(Y \setminus V)$ is a semi-$\mathcal{I}$-closed set for any $\theta$-open set $V$ of $Y$;

7. $F^-(Y \setminus V)$ is a semi-$\mathcal{I}$-open set for any $\theta$-closed set $V$ of $Y$;

8. For each $x \in X$ and for each net $(x_\alpha)$ which semi-$\mathcal{I}$-converges to $x \in X$ and for each $\theta$-open set $V$ of $Y$ such that $x \in F^-(V)$ the net $(x_\alpha)$ is eventually in $F^-(V)$.

Proof. The proof is similar to that of Theorem 3.5. \hfill \square

**Lemma 3.7.** [4] Let $A$ and $B$ be subsets of an ideal topological space $(X, \tau, \mathcal{I})$. If $A \in S\text{TO}(X)$ and $B \in \tau$, then $A \cap B \in S\text{TO}(B)$.

**Theorem 3.8.** Let $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a multifunction and $U \in \tau$. If $F$ is a lower (upper) faintly semi-$\mathcal{I}$-continuous multifunction, then multifunction $F|_U : U \to Y$ is a lower (upper) faintly semi-$\mathcal{I}$-continuous multifunction.

Proof. Let $V$ be any $\theta$-open set of $Y$, $x \in U$ and $x \in F|_U^-(V)$. Since $F$ is a lower faintly semi-$\mathcal{I}$-continuous multifunction, there exists a semi-$\mathcal{I}$-open set $G$ containing $x$ such that $G \subset F^-(V)$. From here by Lemma 3.7, we obtain that $x \in G \cap U \in S\text{TO}(A)$ and $G \cap U \subset F|_U^-(V)$. This shows that the restriction multifunction $F|_U$ is a lower faintly semi-$\mathcal{I}$-continuous. The proof of the upper faintly semi-$\mathcal{I}$-continuity of $F|_U$ can be done by the similar manner. \hfill \square

**Theorem 3.9.** A multifunction $F : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is lower faintly semi-$\mathcal{I}$-continuous if $G_F : (X, \tau, \mathcal{I}) \to (X \times Y, \tau \times \sigma)$ is lower faintly semi-$\mathcal{I}$-continuous.

Proof. Suppose that $G_F$ is lower faintly semi-$\mathcal{I}$-continuous. Let $x \in X$ and $V$ be any $\theta$-open set of $Y$ such that $x \in F^-(V)$. Then $X \times V$ is $\theta$-open in $X \times Y$ and $G_F(x) \cap (X \times V) = (\{x\} \times F(x)) \cap (X \times V) = \{x\} \times (F(x) \cap V) \neq \emptyset$. Since $G_F$ is lower faintly semi-$\mathcal{I}$-continuous, there exists a semi-$\mathcal{I}$-open $U$
containing \( x \) such that \( U \subset G_{\alpha}^+(X \times V) \); hence \( U \subset F^-(V) \). This shows that \( F \) is lower faintly semi-\( I \)-continuous. \( \square \)

**Theorem 3.10.** Suppose that \((X, \tau, \mathcal{I})\) is an ideal topological space and \((X_\alpha, \tau_\alpha)\) are topological spaces where \( \alpha \in J \). Let \( F : (X, \tau, \mathcal{I}) \to \prod_{\alpha \in J} X_\alpha \) be a multifunction from \((X, \tau, \mathcal{I})\) to the product space \( \prod_{\alpha \in J} X_\alpha \) and let \( P_\alpha : \prod_{\alpha \in J} X_\alpha \to X_\alpha \) be the projection multifunction for each \( \alpha \in J \) which is defined by \( P_\alpha((x_\alpha)) = \{x_\alpha\} \). If \( F \) is an upper (lower) faintly semi-\( I \)-continuous multifunction, then \( P_\alpha \circ F \) is an upper (lower) faintly semi-\( I \)-continuous multifunction for each \( \alpha \in J \).

**Proof.** Take any \( \alpha_0 \in J \). Let \( V_{\alpha_0} \) be a \( \theta \)-open set in \((X_{\alpha_0}, \tau_{\alpha_0})\). Then 
\[
(P_{\alpha_0} \circ F)^+(V_{\alpha_0}) = F^+(P_{\alpha_0}^+(V_{\alpha_0})) = F^+(V_{\alpha_0} \times \prod_{\alpha \neq \alpha_0} X_\alpha) \quad \text{(resp.} \quad (P_{\alpha_0} \circ F)^-(V_{\alpha_0}) = F^-(P_{\alpha_0}^-(V_{\alpha_0})) = F^-(V_{\alpha_0} \times \prod_{\alpha \neq \alpha_0} X_\alpha)).
\]
Since \( F \) is an upper (lower) faintly semi-\( I \)-continuous multifunction and since \( V_{\alpha_0} \times \prod_{\alpha \neq \alpha_0} X_\alpha \) is a \( \theta \)-open set, it follows that \( F^+(V_{\alpha_0} \times \prod_{\alpha \neq 0} X_\alpha) \) (resp. \( F^-(V_{\alpha_0} \times \prod_{\alpha \neq 0} X_\alpha) \)) is a semi-\( I \)-open set in \((X, \tau)\). This shows that \( P_\alpha \circ F \) is an upper (lower) faintly semi-\( I \)-continuous multifunction. Hence, we obtain that \( P_{\alpha_0} \circ F \) is an upper (lower) faintly semi-\( I \)-continuous multifunction for each \( \alpha \in J \). \( \square \)

**Theorem 3.11.** Suppose that for each \( \alpha \in J \), \((X, \tau, \mathcal{I})\) is an ideal topological space and \((Y_\alpha, \sigma_\alpha)\) is a topological space. Let \( F_\alpha : X_\alpha \to Y_\alpha \) be a multifunction for each \( \alpha \in J \) and let \( F : \prod_{\alpha \in J} X_\alpha \to \prod_{\alpha \in J} Y_\alpha \) be defined by 
\[
F((x_\alpha)) = \prod_{\alpha \in J} F_\alpha(x_\alpha)
\]
from the product space \( \prod_{\alpha \in J} X_\alpha \) to the product space \( \prod_{\alpha \in J} Y_\alpha \). If \( F \) is an upper (lower) faintly semi-\( I \)-continuous multifunction, then each \( F_\alpha \) is an upper (lower) faintly semi-\( I \)-continuous multifunction for each \( \alpha \in J \).

**Proof.** Let \( V_\alpha \) be a \( \theta \)-open set of \( Y_\alpha \). Then \( V_\alpha \times \prod_{\alpha \neq \beta} Y_\beta \) is a \( \theta \)-open set. Since \( F \) is an upper (lower) faintly semi-\( I \)-continuous multifunction, it follows that 
\[
F^+(V_\alpha \times \prod_{\alpha \neq \beta} Y_\beta) = F_\alpha^+(V_\alpha) \times \prod_{\alpha \neq \beta} X_\beta \quad \text{(resp.} \quad F^-(V_\alpha \times \prod_{\alpha \neq \beta} Y_\beta) = F_\alpha^-(V_\alpha) \times \prod_{\alpha \neq \beta} X_\beta \))
\]
is a semi-\( I \)-open set. Consequently, we obtain that \( F_\alpha^+(V_\alpha) \) (resp. \( F_\alpha^-(V_\alpha) \)) is an semi-\( I \)-open set. Thus, we show that \( F_\alpha \) is an upper (lower) faintly semi-\( I \)-continuous multifunction. \( \square \)
References


