**ABSTRACT**: In this paper, we give theoretical results for fourth atom-bond connectivity index $ABC_4(G)$ and fifth geometric-arithmetic connectivity index $GA_5(G)$, by considering $G$ as line graph of subdivision of convex polytopes $T_n$ and $S_n$.

**Key Words**: topological indices, line graph, subdivision, convex polytopes

1. **Introduction and Preliminaries**

Euler (1701-1783) is considered to be the father of graph theory with his work towards the solution of the problems (see [6, 8, 24]). Graph theory has provided the chemist with a variety of very useful tools. Usually, graphs are categorized into structural graphs, molecular graphs and reaction graphs etc. Some of the graph theory concepts corresponds to the terms in chemistry e.g. point as an atom, line as a covalent bond, degree as atom valency and path as chemical substructure etc. Topological indices are real numbers that are presented as
graph parameters (e.g. the degree of vertices, distances, etc.) introduced during studies conducted on the molecular graphs in chemistry and can describe some physical and chemical properties of molecules. Topological representation of an object tells us about the number of elements composing it and their connectivity (see [3]). Topological indices are invariant under graph isomorphisms. They have significant role in the quantitative structure-property relationship (QSPR) and quantitative structure-activity relationship (QSAR) investigations (see [4, 7, 17]).

Let $G$ be a connected graph with vertex set $V(G)$ and edge set $E(G) \subseteq V(G) \times V(G)$. Let $p = |V(G)|$, the order of $G$ and $q = |E(G)|$, the size of $G$. The degree $d_v$ of any vertex $v$ is defined as the number of vertices joining to that vertex $v$ and the degree $d_e$ of an edge $e \in E(G)$ is defined as the number of its adjacent vertices in $V(L(G))$, where $L(G)$ is the line graph whose vertices are the edges of $G$ and they are adjacent if and only if they have a common end point in $G$. In structural chemistry, line graph of a graph $G$ is very useful. The first topological index on the basis of line graph was introduced by Bertz in 1981 (see [5]). For more details on line graph see the articles [14, 16, 18, 19, 20, 23]. The subdivision $S(G)$ of a graph $G$ can be obtained by inserting an additional vertex between each pair of vertices of $G$. For more details on the topological indices of $L(S(G))$ we refer to the articles [25, 26, 27]).

Convex polytopes are fundamental geometric objects. The beauty of their theory is nowadays complemented by their importance for many other mathematical subjects, ranging from integration theory, algebraic topology, and algebraic geometry to linear and combinatorial optimization (see [13]). Also people are paying attention in finding metric dimension and labeling of convex polytopes (see [1, 2, 21, 22]). From this motivational work, we take a step in finding the topological indices of line graph of subdivision of some convex polytopes.

The following lemma is helpful for computing the degree of a vertex of line graph.

**Lemma 1.1.** Let $G$ be a graph with $u, v \in V(G)$ and $e = uv \in E(G)$. Then:

$$d_e = d_u + d_v - 2.$$

**Lemma 1.2.** ([15]) Let $G$ be a graph of order $p$ and size $q$, then the line graph $L(G)$ of $G$ is a graph of order $q$ and size $\frac{1}{2}M_1(G) - q$. 

2. Line Graphs of Subdivision of Convex Polytopes $T_n$ and $S_n$

In this section we will discuss the combinatorial aspects of subdivision of convex polytopes $T_n$ and $S_n$ and line graphs of their subdivisions.

2.1. Convex Polytope $T_n$

The graph of convex polytope $T_n$ consists of 3-sided faces, 5-sided faces and $n$-sided face as defined in [2]. The convex polytope $T_n$ for $n = 8$ is shown in Figure 1.

![Figure 1: Convex Polytope $T_8$](image)

2.1.1. Subdivision of Convex Polytope $T_n$

We obtain the subdivision $S(T_n)$ by inserting additional vertex between each pair of adjacent vertices of $T_n$. $S(T_8)$ is shown in Figure 2. The $S(T_n)$ consists of $10n$ vertices out which $5n$ vertices are of degree 2, $2n$ vertices are of degree 3 and $n$ vertices are of degree 4. Using Lemma 1.2, we have $|E(S(T_n))| = 10n$. 

![Figure 2: Subdivision $S(T_8)$](image)
2.1.2. Line Graph of Subdivision of Convex Polytope $T_n$

The line graph of subdivision of $T_n$ consists of $10n$ vertices out which $6n$ vertices are of degree 3 and $4n$ vertices are of degree 4. Using Lemma 1.2, we have $|E(L(S(T_n))))| = 17n$. The $L(S(T_n))$ for $n = 8$ is shown in Figure 3.

2.2. Convex Polytope $S_n$

The convex polytope $S_n$ consists of 3-sided faces, 4-sided faces, 5-sided faces and $n$-sided face as defined in [21]. The convex polytope $S_n$ for $n = 8$ is shown in Figure 4.
2.2.1. Subdivision of Convex Polytope $S_n$

We obtain the subdivision $S(S_n)$ by inserting additional vertex between each pair of adjacent vertices of $S_n$. $S(S_8)$ is shown in Figure 5. The $S(S_n)$ consists of $13n$ vertices out which $8n$ vertices are of degree 2, $4n$ vertices are of degree 3 and $n$ vertices are of degree 4. Using Lemma 1.2, we have $|E(S(S_n))| = 16n$.

![Figure 5: Subdivision of $S_8$](image)

2.2.2. Line Graph of Subdivision of Convex Polytope $S_n$

The line graph of subdivision of $S_n$ consists of $16n$ vertices out which $12n$ vertices are of degree 3 and $4n$ vertices are of degree 4. Using Lemma 1.2, we have $|E(L(S(S_n))))| = 26n$. The $L(S(S_n))$ for $n = 8$ is shown in Figure 6.

![Figure 6: Line graph of subdivision of $S_8$](image)
2.3. The Edge Partitions of Line Graphs of Subdivision of Convex Polytopes $T_n$ and $S_n$ w.r.t Degree Sum

For a vertex $u \in V(G)$, let $S_u = \sum_{uv \in E(G)} d_v$ is the degree sum of $u$. For $uv \in E(G)$, $S_u$ and $S_v$ is the sum of degrees of all neighbors of vertex $u$ and $v$ in $G$ respectively. We partition $E(G)$ into subsets based on the degree sum of the end vertices of edges in $G$. The edge partition of $L(S(T_n))$ and $L(S(S_n))$ with respect to degree sum are shown in Tables 1 and 2 respectively.

<table>
<thead>
<tr>
<th>$(S_u, S_v)$</th>
<th>Number of edges</th>
<th>$(S_u, S_v)$</th>
<th>Number of edges</th>
</tr>
</thead>
<tbody>
<tr>
<td>(9, 9)</td>
<td>5n</td>
<td>(15, 16)</td>
<td>4n</td>
</tr>
<tr>
<td>(9, 10)</td>
<td>2n</td>
<td>(15, 15)</td>
<td>n</td>
</tr>
<tr>
<td>(10, 10)</td>
<td>$n$</td>
<td>(16, 16)</td>
<td>2n</td>
</tr>
<tr>
<td>(10, 15)</td>
<td>$2n$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 1: The edge partition of $L(S(T_n))$ w.r.t degree sum

<table>
<thead>
<tr>
<th>$(S_u, S_v)$</th>
<th>Number of edges</th>
<th>$(S_u, S_v)$</th>
<th>Number of edges</th>
</tr>
</thead>
<tbody>
<tr>
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<td>$4n$</td>
<td>(10, 15)</td>
<td>$2n$</td>
</tr>
<tr>
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<td>$6n$</td>
<td>(11, 14)</td>
<td>$2n$</td>
</tr>
<tr>
<td>(9, 11)</td>
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<td>(14, 14)</td>
<td>$2n$</td>
</tr>
<tr>
<td>(10, 10)</td>
<td>$n$</td>
<td>(15, 16)</td>
<td>$4n$</td>
</tr>
<tr>
<td>(10, 14)</td>
<td>$2n$</td>
<td>(16, 16)</td>
<td>$2n$</td>
</tr>
</tbody>
</table>

Table 2: The edge partition of $L(S(S_n))$ w.r.t degree sum

3. Topological Indices of Line Graph of Subdivision of Convex Polytopes $T_n$ and $S_n$

In this section we will compute $ABC_4(G)$ index and $GA_5(G)$ index of the line graph of subdivision of convex polytopes $T_n$ and $S_n$.

3.1. Fourth Atom-Bond Connectivity Index

M. Ghorbani et al. in [9, 10, 11] proposed Fourth Atom-Bond connectivity index as:

$$ABC_4(G) = \sum_{uv \in E(G)} \sqrt{\frac{S_u + S_v - 2}{S_u S_v}}.$$  (1)
3.2. Fifth Geometric-Arithmetic Index

This index was introduced by Graovac et al. in [12] as:

\[ GA_5(G) = \sum_{uv \in E(G)} \frac{2\sqrt{S_u S_v}}{S_u + S_v}. \] (2)

**Theorem 3.1.** Let \( L(S(T_n)) \) and \( L(S(S_n)) \) are the line graphs of subdivision of convex polytopes \( T_n \) and \( S_n \) respectively then:

\[
\begin{align*}
ABC_4(L(S(T_n))) &= \frac{20}{9} n + \frac{1}{15} n\sqrt{170} + \frac{3}{10} n\sqrt{2} + \frac{1}{15} n\sqrt{138} + \frac{2}{15} n\sqrt{7} \\
&\quad + \frac{1}{15} n\sqrt{435} + \frac{1}{8} n\sqrt{30}, \\
ABC_4(L(S(S_n))) &= \frac{16}{9} n + \frac{1}{5} n\sqrt{170} + \frac{1}{35} n\sqrt{770} + \frac{1}{7} n\sqrt{26} + \frac{1}{77} n\sqrt{3542} \\
&\quad + \frac{1}{11} n\sqrt{22} + \frac{3}{10} n\sqrt{2} + \frac{1}{15} n\sqrt{138} + \frac{1}{15} n\sqrt{435} + \frac{1}{8} n\sqrt{30}.
\end{align*}
\]

**Proof.** The Fourth Atom–Bond Connectivity Index can be obtained by using Formula (1) and using edge partitions as shown in Tables 1 and 2 respectively. \( \square \)

**Theorem 3.2.** Let \( L(S(T_n)) \) and \( L(S(S_n)) \) are the line graphs of subdivision of convex polytopes \( T_n \) and \( S_n \) respectively then:

\[
\begin{align*}
GA_5(L(S(T_n))) &= 9n + \frac{12}{19} n\sqrt{10} + \frac{4}{5} n\sqrt{6} + \frac{32}{31} n\sqrt{15}. \\
GA_5(L(S(S_n))) &= 9n + \frac{36}{19} n\sqrt{10} + \frac{1}{3} n\sqrt{35} + \frac{4}{25} n\sqrt{154} \\
&\quad + \frac{3}{10} n\sqrt{11} + \frac{4}{5} n\sqrt{6} + \frac{32}{31} n\sqrt{15}.
\end{align*}
\]

**Proof.** The Fifth Geometric–Arithmetic Index can be obtained by using Formula (2) and using edge partitions as shown in Tables 1 and 2 respectively. \( \square \)

**References**


