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ON SOME LINEAR COMBINATION OF TWO CONTIGUOUS HYPERGEOMETRIC FUNCTIONS

Imane Ghanimi
Department of Mathematics
Ibn Tofail University
Kenitra, MORROCCO

Abstract: The aim of this paper is to give, using some contiguous relations, the asymptotic behaviour of some linear combination of two symmetric contiguous hypergeometric functions, under some conditions of their parameters.

Key Words: special functions, contiguous hypergeometric functions

1. Introduction, Notations and Statement of the Main Results

Hypergeometric function belongs to an important class of special functions. They are very useful in many fields of physical and astronomical researches, in mathematical analysis and its application. They have significant properties: identities of special values, and transformation formulas. These properties have many intereting applications in combinatorial analsis and arithmetic geometric.

The hypergeometric series is defined by:

$$ {}_2F_1\left(a,b;c;z\right) = \Sigma_{n\geq 0}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!} $$

with $(a)_n := a(a+1)\cdots(a+n-1)$ for $n \geq 1$, $(a)_0 = 1$, where it is assumed that $c \neq 0, -1, -2, -3, ...$ so that no zero factors appear in the denominators of the terms of the series. The variable is z, and a,b and c are called the parameters of the function.

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The first systematic and thorough study of hypergeometric series was carried out by Gauss in his paper of 1820. Many people: Ernst Kummer, Jacobi, Riemann, E.W Barnes, RH Mellin gived different approches and interesting results about hypergeometric functions.

An hypergeometric function is called contiguous to the other $F(a, b, c, z)$ if one, two, or three of the parameters a, b, c are increased or diminished by an integer.

In his 1820 paper, Gauss first introduced the terminology "contiguous function". He definened two hypergeometric function to be contiguous if they have the same power series variable and if two of the parameters are pairwise equal and the third pair differs by -1. He found that every three contiguous hypergeometric functions are linearly related. Then he listed 15 such linear relations (which are now called "contiguous relations"), and he used them to derive basic formulas.

These contiguous hypergeometric functions have some interesting applications.

Indeed, contiguous relations gives an intetwining correspondance between Lie algebras and special functions.They are also very useful in the derivation of summation and transformation formulas for hypergeometric series.

Recently, many people studies contiguous hypergeometric functions and gives some new properties, consequences and interesting applications of them. See for instance [5], [6], [7], [8], [9], [10], [11].

in [1], the autors used well-known forms of the Gauss function to prove some simple identities relating shifted hypergeometric functions together with their derivatives. Then they combined these identities to obtain more general new identities.

Usually, while solving many interesting problems related to mathematic analysis, we are led to establish a precise asymptotic behavior of some functions. In this paper, using some contiguous relations of [1], we are able to give, under some conditions of their parameters, the asymptotic behaviour of linear combination of two symmetric contiguous hypergeometric functions.

Now we may state the main theorem of this paper:

Theorem 1.1. *For $(a-b), \alpha, \beta \in \mathbb{N}$ and $a, b, c \in \mathbb{C}$ with:*

$$Re(a+b+\alpha+\beta-c-1) > 0$$

we have

$$lim_{Z\to 1}(1-Z)^{-c+a+b+\alpha+\beta-1}((a)_\alpha (b)_\beta F_{21}(a+\alpha, b+\beta; c; z)$$

$$-(a)_\beta(b)_\alpha F_{21}(a+\beta, b+\alpha; c; z)) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\Gamma(a+b+\alpha+\beta-c-1)(a-b)(\alpha-\beta).$$

2. Proof of the Main Theorem

Proof. Denote by $F_{\alpha,\beta,\gamma}(Z)$ the hypergeometric function:

$$F_{21}(a+\alpha, b+\beta; c+\gamma; z) \text{ for } \alpha,\beta,\gamma \in \mathbb{N} \text{ and } a,b,c \in \mathbb{C}.$$

Then we have the following identitie: (see for instance [1] p. 407)

$$F_{\alpha,\beta,\gamma}(Z) = F_{\alpha-1,\beta,\gamma}(Z) + \frac{b+\beta}{c+\gamma} Z F_{\alpha,\beta+1,\gamma+1}(Z).$$

Then we can rewite:

$$F_{\alpha,\beta,0}(Z) = F_{\alpha-1,\beta,0}(Z) + \frac{b+\beta}{c} Z F_{\alpha,\beta+1,1}(Z),$$

and similarly

$$F_{\beta,\alpha,0}(Z) = F_{\beta-1,\alpha,0}(Z) + \frac{b+\alpha}{c} Z F_{\beta,\alpha+1,1}(Z),$$

$$(a)_\alpha(b)_\beta F_{\alpha,\beta,0}(Z) = (a)_\alpha(b)_\beta F_{\alpha-1,\beta,0}(Z) + (a)_\alpha(b)_{\beta+1}\frac{Z}{c} F_{\alpha,\beta+1,1}(Z).$$

Moreover

$$(a)_\beta(b)_\alpha F_{\beta,\alpha,0}(Z) = (a)_\beta(b)_\alpha F_{\beta-1,\alpha,0}(Z) + (a)_\beta(b)_{\alpha+1}\frac{Z}{c} F_{\beta,\alpha+1,1}(Z),$$

and

$$\begin{aligned}(a)_\alpha(b)_\beta F_{\alpha,\beta,0}(Z) &= (a)_\alpha(b)_\beta F_{\alpha-1,\beta,0}(Z)\\ &\quad + (a)_\alpha(b)_{\beta+1}\frac{Z}{c}[F_{\alpha-1,\beta+1,1}(Z) + \frac{b+\beta+1}{c+1} Z F_{\alpha,\beta+2,2}].\end{aligned}$$

Using similar arguments

$$\begin{aligned}(a)_\beta(b)_\alpha F_{\beta,\alpha,0}(Z) &= (a)_\beta(b)_\alpha F_{\beta-1,\alpha,0}(Z)\\ &\quad + (a)_\beta(b)_{\alpha+1}\frac{Z}{c}[F_{\beta-1,\alpha+1,1}(Z) + \frac{b+\alpha+1}{c+1} Z F_{\beta,\alpha+2,2}]\end{aligned}$$

Therefore

$$(a)_\alpha(b)_\beta F_{\alpha,\beta,0}(Z) = (a)_\alpha(b)_\beta F_{\alpha-1,\beta,0}(Z) + \frac{Z}{c}(a)_\alpha(b)_{\beta+1}F_{\alpha-1,\beta+1,1}(Z) + \frac{Z^2}{c(c+1)}(a)_\alpha(b)_{\beta+2}F_{\alpha,\beta+2,2}(Z).$$

Similarly

$$(a)_\beta(b)_\alpha F_{\beta,\alpha,0}(Z) = (a)_\beta(b)_\alpha F_{\beta-1,\alpha,0}(Z) + \frac{Z}{c}(a)_\beta(b)_{\alpha+1}F_{\beta-1,\alpha+1,1}(Z) + \frac{Z^2}{c(c+1)}(a)_\beta(b)_{\alpha+2}F_{\beta,\alpha+2,2}(Z).$$

But we also have

$$F_{\alpha,\beta+2,2}(Z) = F_{\alpha-1,\beta+2,2}(Z) + \frac{b+\beta+2}{c+2}ZF_{\alpha,\beta+3,3}(Z).$$

Similarly

$$F_{\beta,\alpha+2,0}(Z) = F_{\beta-1,\alpha+2,2}(Z) + \frac{b+\alpha+2}{c+2}ZF_{\beta,\alpha+3,3}(Z).$$

Hence

$$(a)_\alpha(b)_\beta F_{\alpha,\beta,0}(Z) = (a)_\alpha(b)_\beta F_{\alpha-1,\beta,0}(Z) + \frac{Z}{c}(a)_\alpha(b)_{\beta+1}F_{\alpha-1,\beta+1,1}(Z) + \frac{Z^2}{c(c+1)}(a)_\alpha(b)_{\beta+2}F_{\alpha-1,\beta+2,2}(Z) + \frac{Z^3}{c(c+1)(c+2)}(a)_\alpha(b)_{\beta+3}F_{\alpha,\beta+3,3}(Z).$$

Analogously

$$(a)_\beta(b)_\alpha F_{\beta,\alpha,0}(Z) = (a)_\beta(b)_\alpha F_{\beta-1,\alpha,0}(Z) + \frac{Z}{c}(a)_\beta(b)_{\alpha+1}F_{\beta-1,\alpha+1,1}(Z) + \frac{Z^2}{c(c+1)}(a)_\beta(b)_{\alpha+2}F_{\beta-1,\alpha+2,2}(Z) + \frac{Z^3}{c(c+1)(c+2)}(a)_\beta(b)_{\alpha+3}F_{\beta,\alpha+3,3}(Z).$$

By iterating the process $(a-b)$ times we finally receive

$$(a)_\alpha(b)_\beta F_{\alpha,\beta,0}(Z) = \sum_{i=0}^{a-b-1}\frac{Z^i}{(c)_i}(a)_\alpha(b)_{\beta+i}F_{\alpha-1,\beta+i,i}(Z) + \frac{Z^{a-b}}{(c)_{a-b}}(a)_\alpha(b)_{\beta+a-b}F_{\alpha,\beta+a-b,a-b}(Z).$$

In the same way

$$(a)_\beta(b)_\alpha F_{\beta,\alpha,0}(Z) = \sum_{i=0}^{a-b-1} \frac{Z^i}{(c)_i}(a)_\beta(b)_{\alpha+i} F_{\beta-1,\alpha+i,i}(Z) + \frac{Z^{a-b}}{(c)_{a-b}}(a)_\beta(b)_{\alpha+a-b} F_{\beta,\alpha+a-b,a-b}(Z).$$

On the other hand

$$\begin{aligned}(a)_\alpha(b)_{\beta+a-b} F_{\alpha,\beta+a-b,a-b}(Z) &= (a)_\beta(b)_{\alpha+a-b} F_{\beta,\alpha+a-b,a-b}(Z) \\ &= \frac{\Gamma(a+\alpha)\Gamma(a+\beta)}{\Gamma(a)\Gamma(b)} F(a+\alpha, a+\beta, a-b, Z)\end{aligned}$$

Therefore for each i in $[0, a-b-1]$ we have

$$\frac{Z^i}{(c)_i}(a)_\alpha(b)_{\beta+i} F_{\alpha-1,\beta+i,i}(Z) \sim_1 (1-Z)^{c-a-b-\alpha-\beta+1}(a+\alpha-1)\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\Gamma(a+b+\alpha+\beta-c-1)$$

Hence

$$\frac{Z^i}{(c)_i}(a)_\beta(b)_{\alpha+i} F_{\beta-1,\alpha+i,i}(Z) \sim_1 (1-Z)^{c-a-b-\alpha-\beta+1}(a+\beta-1)\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\Gamma(a+b+\alpha+\beta-c-1)$$

The result has been proved. □

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