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ON LEFT AND RIGHT BASES OF A Γ-SEMIGROUP

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Abstract: In the line of I. Frabrici [I. Fabrici, One-sided bases of semigroups, Matematický casopis, 22(4), 1972, 286-290], the notions of left and right bases of a Γ-semigroup are introduced, and some examples are also presented. The structure of a Γ-semigroup containing right bases will be studied. Indeed, using a characterization of right bases, we prove that the right bases of a Γ-semigroup have the same cardinality. Moreover, the compliment of the union of all right bases of a Γ-semigroup, if it is non-empty, is a left ideal of the Γ-semigroup. Finally, a characterization when the compliment of the union of all right bases of a Γ-semigroup is maximal will be given.

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1. Preliminaries

Let S and Γ be any two non-empty sets. Then S is called a Γ-*semigroup* if there is a mapping from $S \times \Gamma \times S$ into S, written as $(a, \alpha, b) \mapsto a\alpha b$, such that

$$(x\alpha y)\beta z = x\alpha(y\beta z)$$

for all $x, y, z \in S$ and all $\alpha, \beta \in \Gamma$. This notion was first introduced as a generalization of semigroups in 1981 by M. K. Sen [5]. After that, results on

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semigroups have been extended to Γ-semigroups (see [11]-[14], [3]-[2], [6]-[8]).

In [15], a non-empty subset A of a semigroup S is called a *right base* of S if it satisfies the following two conditions:

(i) $S = A \cup SA$;

(ii) if B is a subset of A such that $S = B \cup SB$, then $B = A$.

Using this concept, I. Fabrici [4] studied the structure of semigroups containing right bases. The main purpose of this paper is to extend the results obtained by I. Fabrici to Γ-semigroups. Indeed, we define the concepts of left and right bases of a Γ-semigroup, and present some examples. Moreover, we study the structure of a Γ-semigroup containing right bases; it is observed that the results for a Γ-semigroup containing left bases are left-right dual. Indeed, using a characterization of right bases, we prove that the right bases of a Γ-semigroup have the same cardinality, and that the compliment of the union of all right bases of a Γ-semigroup, if it is non-empty, is a left ideal of the Γ-semigroup. Finally, a characterization when the compliment of the union of all right bases of a Γ-semigroup is a maximal left ideal will be given. We now recall some definitions and results used throughout the paper.

Let S be a Γ-semigroup. For A, B non-empty subsets of S, the set product $A\Gamma B$ is defined to be the set of all elements $a\gamma b$ in S where $a \in A$, $b \in B$ and $\gamma \in \Gamma$. That is

$$A\Gamma B := \{a\gamma b \mid a \in A, b \in B, \gamma \in \Gamma\}.$$

For $a \in S$, we write $B\Gamma a$ instead of $B\Gamma\{a\}$, and similarly for $a\Gamma B$. And we write $a \cup B\Gamma a$ for $\{a\} \cup B\Gamma a$.

A non-empty subset A of a Γ-semigroup S is called a Γ-*subsemigroup* (or simply a *subsemigroup*) of S if $A\Gamma A \subseteq A$.

Let S be a Γ-semigroup, and A a non-empty subset of S. Then A is called a Γ-*left-ideal* (or simply a *left ideal*) of S if $S\Gamma A \subseteq A$. Dually, A is called a Γ-*right-ideal* (or simply a *right ideal*) of S if $A\Gamma S \subseteq A$. A left or a right ideal A of S is said to be *proper* if $A \subset S$. The symbol $\subset$ stands for proper inclusion for sets. A proper left ideal L of S is said to be *maximal* if for any left ideal L' of S, $L \subseteq L'$ implies $L = L'$ or $L' = S$. It is observed that the following hold:

– the intersection of an indexed family of left ideals of S, if it is non-empty, is a left ideal of S;

– the union of an indexed family of left ideals of S is a left ideal of S;

– if A is a non-empty subset of S, then the intersection of all left ideals of S containing A, denoted by $(A)_L$, is a left ideal of S containing A, and it is of the form
$$(A)_L = A \cup S\Gamma A.$$
In particular, we write $(\{a\})_L$ by $(a)_L$, called the *principal left ideal* of S generated by a;

– for any $a \in S$, $S\Gamma a$ is a left ideal of S.

2. Main Results

We begin this section with the definition of a right base of a Γ-semigroup as follows:

Definition 1. Let S be a Γ-semigroup. A non-empty subset A of S is called a *right base* of S if it satisfies the following two conditions:

(i) $S = A \cup S\Gamma A$, i.e. $S = (A)_L$;

(ii) if B is a subset of A such that $S = (B)_L$, then $B = A$.

For a left base of S is defined dually.

Example 1. Consider the Γ-semigroup $S = \{-i, i, 0\}$ with $\Gamma = \{\alpha, \beta, \gamma\}$ such that

α	$-i$	i	0
$-i$	i	$-i$	0
i	$-i$	i	0
0	0	0	0

β	$-i$	i	0
$-i$	$-i$	i	0
i	i	$-i$	0
0	0	0	0

γ	$-i$	i	0
$-i$	0	0	0
i	0	0	0
0	0	0	0

The right bases of S are $A_1 = \{-i\}$ and $A_2 = \{i\}$. And the left bases of S are the same as the right bases of S.

Example 2. Consider the Γ-semigroup $S = \{w, x, y, z\}$ with $\Gamma = \{\alpha, \beta, \gamma\}$ such that

α	w	x	y	z
w	w	x	w	w
x	w	x	w	w
y	w	x	w	w
z	w	x	w	w

β	w	x	y	z
w	w	x	w	w
x	w	x	w	w
y	w	x	y	y
z	w	x	z	z

γ	w	x	y	z
w	w	x	w	w
x	w	x	w	w
y	w	x	y	y
z	w	x	z	z

The right bases of S are $B_1 = \{x, y\}$ and $B_2 = \{x, z\}$. And the left base of S is $B_3 = \{y, z\}$.

Note that each of the examples above the cardinality of right bases are the same. However, it turns out that this is true in general, and we will prove in Theorem 4.

Firstly, the following lemma we need.

Lemma 1. *Let A be a right base of a Γ-semigroup S, and $a, b \in A$. If $a \in S\Gamma b$, then $a = b$.*

Proof. Assume that $a \in S\Gamma b$, and suppose that $a \neq b$. Let $B = A \setminus \{a\}$; then $B \subset A$. Since $a \neq b$, $b \in B$. To show that $(A)_L \subseteq (B)_L$, if suffices to show that $A \subseteq (B)_L$. Let $x \in A$. If $x \neq a$, then $x \in B$; hence $x \in (B)_L$. If $x = a$, then by assumption we have

$$x = a \in S\Gamma b \subseteq (B)_L.$$

Hence $A \subseteq (B)_L$, and $(A)_L \subseteq (B)_L$. By

$$S = (A)_L \subseteq (B)_L \subseteq S,$$

it follows that $(B)_L = S$. This is a contradiction. Therefore, $a = b$ as required. $\square$

Let S be a Γ-semigroup. Define a quasi-order on S by, for any $a, b \in S$,

$$a \leq_L b :\Leftrightarrow (a)_L \subseteq (b)_L.$$

We write $a <_L b$ if $a \leq_L b$ but $a \neq b$, i.e., $(a)_L \subset (b)_L$.

The following example shows that $\leq_L$ defined above is not, in general, a partial order.

Example 3. Consider the Γ-semigroup $S = \{a, b, c, d, e\}$ with $\Gamma = \{\alpha, \beta\}$ such that

α	a	b	c	d	e
a	a	a	a	a	a
b	a	e	d	c	b
c	a	d	b	e	c
d	a	c	e	b	d
e	a	b	c	d	e

β	a	b	c	d	e
a	a	a	a	a	a
b	a	d	b	e	c
c	a	b	c	d	e
d	a	e	d	c	b
e	a	c	e	b	d

We have $(b)_L \subseteq (c)_L$ (i.e., $b \leq_L c$) and $(c)_L \subseteq (b)_L$ (i.e., $c \leq_L b$). But $b \neq c$. Thus, $\leq_L$ is not a partial order on S.

The following theorem characterizes when a non-empty subset of a Γ-semigroup is a right base of the Γ-semigroup.

Theorem 1. *A non-empty subset A of a Γ-semigroup S is a right base of S if and only if A satisfies the following two conditions:*

(1) *for any $x \in S$ there exists $a \in A$ such that $x \leq_L a$;*

(2) *for any two distinct elements $a, b \in A$ neither $a \leq_L b$ nor $b \leq_L a$.*

Proof. Assume first that A is a right base of S. Then $S = (A)_L$. Let $x \in S$; then $x \in A \cup S\Gamma A$. Then $x \in A$ or $x \in S\Gamma A$. If $x \in A$, then $x \leq_L x$. If $x \in S\Gamma A$, then $x = s\gamma a$ for some $s \in S, \gamma \in \Gamma$ and $a \in A$. By $x \in a \cup S\Gamma a$, it follows that $(x)_L \subseteq (a)_L$. Thus $x \leq_L a$. Hence the condition (1) holds. Let $a, b \in A$ be such that $a \neq b$. Suppose that $a \leq_L b$; then $(a)_L \subseteq (b)_L$. Since $a \neq b$, $a \in S\Gamma b$. By Lemma 1, $a = b$. This is a contradiction. The case $b \leq_L a$ can be proved similarly. Thus the condition (2) holds.

Conversely, assume that the conditions (1) and (2) hold. To show that $S = (A)_L$, let $x \in S$. By (1), there exists $a \in A$ such that $(x)_L \subseteq (a)_L$. Then

$$x \in (x)_L \subseteq (a)_L \subseteq (A)_L.$$

Thus $S \subseteq (A)_L$, and $S = (A)_L$. It remains to show that A is a minimal subset of S with the property: $S = (A)_L$. Let $B \subset A$ be such that $S = (B)_L$. Since $B \subset A$, there exists $a \in A \setminus B$. Since $a \in A \subseteq S = (B)_L$ and $a \notin B$, $a \in S\Gamma B$. Then $a = s\gamma b$ for some $s \in S, \gamma \in \Gamma$ and $b \in B$. It follows that $a \in b \cup S\Gamma b$. Hence $(a)_L \subseteq (b)_L$, i.e. $a \leq_L b$ where $a, b \in A$. This is a contradiction. Therefore, A is a right base of S. □

If a right base A of a Γ-semigroup S is a left ideal of S, then

$$S = A \cup S\Gamma A \subseteq A \cup A = A.$$

Hence $S = A$. The converse statement is obvious. Then we conclude that:

Theorem 2. *A right base A of a Γ-semigroup S is a left ideal of S if and only if $A = S$.*

Definition 2. A Γ-semigroup S is said to be *right singular* if $x\gamma y = y$ for all $x, y \in S$ and $\gamma \in \Gamma$.

In Example 1, it is observed that not every right base of a Γ-semigroup is a subsemigroup. This leads to prove the following.

Theorem 3. *A right base A of a Γ-semigroup S is a subsemigroup of S if and only if A is right singular.*

Proof. Assume that A is a subsemigroup of S. Let $a, b \in A$, and let $\gamma \in \Gamma$. By assumption, $a\gamma b \in A$. Then $a\gamma b = c$ for some $c \in A$. Since $c = a\gamma b \in S\Gamma b$, it follows by Lemma 1 that $c = b$. Thus $a\gamma b = b$. Therefore, A is right singular. The converse statement is clear. □

Let S be a Γ-semigroup, and let $\alpha \in \Gamma$. Then $e \in S$ is said to be an α-*idempotent* of S if $e\alpha e = e$. The set of all α-idempotents is S will be denoted by $E_\alpha(S)$. Let

$$E(S) := \bigcup_{\alpha\in\Gamma} E_\alpha(S).$$

By Theorem 3 we have:

Corollary 1. *If a right base A of a Γ-semigroup S is a subsemigroup of S, then $E(S) \neq \emptyset$.*

Theorem 4. *The right bases of a Γ-semigroup S have the same cardinality.*

Proof. Let A and B be right bases of a Γ-semigroup S. Let $a \in A$. Since B is a right base of S, it follows by Theorem 1 (1) that there exists $b \in B$ such that $a \leq_L b$. Similarly, since A is a right base of S, there exists $a^* \in A$ such that $b \leq_L a^*$. Then $a \leq_L b \leq_L a^*$, and $a \leq_L a^*$. By Theorem 1 (2), $a = a^*$. Hence $(a)_L = (b)_L$. Now, define a mapping

$$\varphi : A \to B;\ \varphi(a) = b$$

for all $a \in A$. To show that φ is well-defined, let $a_1, a_2 \in A$ be such that $a_1 = a_2$, $\varphi(a_1) = b_1$ and $\varphi(a_2) = b_2$ for some $b_1, b_2 \in B$. Then $(a_1)_L = (b_1)_L$ and $(a_2)_L = (b_2)_L$. Since $a_1 = a_2$, $(a_1)_L = (a_2)_L$. Hence $(a_1)_L = (a_2)_L = (b_1)_L = (b_2)_L$, i.e., $b_1 \leq_L b_2$ and $b_2 \leq_L b_1$. By Theorem 1 (2), $b_1 = b_2$. Thus $\varphi(a_1) = \varphi(a_2)$. Therefore, φ is well-defined.

To show that φ is one-to-one, let $a_1, a_2 \in A$ be such that $\varphi(a_1) = \varphi(a_2)$. Then $\varphi(a_1) = \varphi(a_2) = b$ for some $b \in B$. We have $(a_1)_L = (a_2)_L = (b)_L$. Since $(a_1)_L = (a_2)_L$, $a_1 \leq_L a_2$ and $a_2 \leq_L a_1$. Since A is a right base of S and $a_1, a_2 \in A$, it follows that $a_1 = a_2$. Therefore, φ is one-to-one.

Finally, we will show that φ is onto. Let $b \in B$. To show that $\varphi(a) = b$ for some $a \in A$, if suffices to show $(a)_L = (b)_L$ for some $a \in A$. Since A is a right base of S, it follows by Theorem 1 (1) that there exists $a \in A$ such that $b \leq_L a$. Since B is a right base of S, it follows by Theorem 1 (1) that there exists $b^* \in B$ such that $a \leq_L b^*$. We have $b \leq_L a \leq_L b^*$, and $b \leq_L b^*$. By Theorem 1 (2), $b = b^*$. Hence $(a)_L = (b)_L$. Therefore, φ is onto. This completes the proof. □

Theorem 5. *Let A be a right base of a Γ-semigroup of S, and let $a \in A$. If $(a)_L = (b)_L$ for some $b \in S$ such that $a \neq b$, then b is an element of a right base of S which is distinct from A.*

Proof. Assume that $(a)_L = (b)_L$ for some $b \in S$ such that $a \neq b$. Setting

$$B := (A \setminus \{a\}) \cup \{b\};$$

then $B \neq A$. We will show that B is a right base of S using Theorem 1. Let $x \in S$. Since A is a right base of S, by Theorem 1 (1) there exists $c \in A$ such that $x \leq_L c$. If $c \neq a$, then $c \in B$. If $c = a$, then $(c)_L = (a)_L$. Since $(a)_L = (b)_L$, $(c)_L = (b)_L$. Thus,

$$(x)_L \subseteq (c)_L = (b)_L.$$

Hence $x \leq_L b$ and $b \in B$. Now, let $b_1, b_2 \in B$ be such that $b_1 \neq b_2$. Then are four cases to consider:

Case 1: $b_1 \neq b$ and $b_2 \neq b$. Then $b_1, b_2 \in A$. Since A is a right base of S, neither $b_1 \leq_L b_2$ nor $b_2 \leq_L b_1$.

Case 2: $b_1 \neq b$ and $b_2 = b$. Then $(b_2)_L = (b)_L$. If $b_1 \leq_L b_2$, then

$$(b_1)_L \subseteq (b_2)_L = (b)_L = (a)_L.$$

Thus $b_1 \leq_L a$ where $b_1, a \in A$. This is a contradiction. If $b_2 \leq_L b_1$, then

$$(a)_L = (b)_L = (b_2)_L \subseteq (b_1)_L.$$

Thus $a \leq_L b_1$ where $b_1, a \in A$. This is a contradiction.

Case 3: $b_1 = b$ and $b_2 \neq b$. Then $(b_1)_L = (b)_L$. If $b_1 \leq_L b_2$, then

$$(a)_L = (b)_L = (b_1)_L \subseteq (b_2)_L.$$

Thus $a \leq_L b_2$ where $b_2, a \in A$. This is a contradiction. If $b_2 \leq_L b_1$, then

$$(b_2)_L \subseteq (b_1)_L = (b)_L = (a)_L.$$

Thus $b_2 \leq_L a$ where $b_2, a \in A$. This is a contradiction.

Case 4: $b_1 = b$ and $b_2 = b$. This is impossible. □

Theorem 6. *Let $\mathcal{A}$ be the union of all right bases of a Γ-semigroup S. If $L = S \setminus \mathcal{A}$ is non-empty, then L is a left ideal of S.*

Proof. Assume that $L = S \setminus \mathcal{A}$ is non-empty. Let $x \in S, a \in L$, and $\gamma \in \Gamma$. Suppose that $x\gamma a \notin L$. Then $x\gamma a \in \mathcal{A}$. Thus $x\gamma a \in A$ for some a right base A of S. Let $x\gamma a = b$ for some $b \in A$. Then $b = x\gamma a \in S\Gamma a$. By $S\Gamma b \subseteq S\Gamma a$, it follows that $(b)_L \subseteq (a)_L$. By Theorem 5, $a \in \mathcal{A}$. This is a contradiction. Thus $(b)_L \subset (a)_L$. Since A is a right base of S, it follows by Theorem 1 (1) there exists $b_1 \in A$ such that $a \leq_L b_1$. Then $b <_L a \leq_L b_1$, and $b \leq_L b_1$ where $b, b_1 \in A$. This contradicts to the condition (2) of Theorem 1. Thus $x\gamma a \in L$, and the proof is completed. □

In Example 2, we have the of all right base of S are $B_1 = \{x, y\}$ and $B_2 = \{x, z\}$. Then $S \setminus \mathcal{A} = \{w\}$ is a left ideal of S, but it is not a maximal proper left ideal of S. In the following theorem we shall find conditions for which $S \setminus \mathcal{A}$ is a maximal proper left ideal of S.

Theorem 7. *Let $\mathcal{A}$ be the union of all right bases of a Γ-semigroup S such that $\mathcal{A} \neq \emptyset$. Then $S \setminus \mathcal{A}$ is a maximal proper left ideal of S if and only if $\mathcal{A} \neq S$ and $\mathcal{A} \subseteq (a)_L$ for all $a \in \mathcal{A}$.*

Proof. Let $L = S \setminus \mathcal{A}$ be a maximal proper left ideal of S. Then $\mathcal{A} \neq S$. Let $a \in \mathcal{A}$. Suppose that $\mathcal{A} \nsubseteq (a)_L$. Then there exists $x \in \mathcal{A}$ such that $x \notin (a)_L$. We have $(S \setminus \mathcal{A}) \cup (a)_L \subset S$. Then $(S \setminus \mathcal{A}) \cup (a)_L$ is a proper left ideal of S. This contradicts to the maximality of $S \setminus \mathcal{A}$. Therefore, $\mathcal{A} \subseteq (a)_L$.

Conversely, let $\mathcal{A} \neq S$ and $\mathcal{A} \subseteq (a)_L$ for all $a \in \mathcal{A}$. By Theorem 6, $S \setminus \mathcal{A}$ is a proper left ideal of S. Let L_1 be a left ideal of S such that $S \setminus \mathcal{A} \subseteq L_1 \subseteq S$. Suppose that $S \setminus \mathcal{A} \neq L_1$; then $S \setminus \mathcal{A} \subset L_1$. This implies that $L_1 \cap \mathcal{A} \neq \emptyset$. Let $a \in L_1 \cap \mathcal{A}$. Then $(a)_L \subseteq L_1$. Since $\mathcal{A} \subseteq (a)_L$ and $S \setminus \mathcal{A} \subset L_1$, we have

$$S = (S \setminus \mathcal{A}) \cup \mathcal{A} \subseteq L_1 \cup (a)_L \subseteq L_1 \subseteq S.$$

Thus $S = L_1$. Therefore, $S \setminus \mathcal{A}$ is a maximal proper left ideal of S. □

Theorem 8. *Let $\mathcal{A}$ be the union of all right bases of a Γ-semigroup S such that $\emptyset \neq \mathcal{A} \subset S$. If S contains a maximal left ideal of S containing every proper left ideal of S, denoted by L^*, then $S \setminus \mathcal{A} = L^*$ if and only if*

(1) $|A| = 1$ *for every right base A of S;*

(2) *one of the following conditions holds:*

  (2.1) $S\Gamma A = S$ *for every right base A of S;*

  (2.2) *S contains only one right base $A = \{a\}$ with $a \cup S\Gamma a = S$ but $a \notin S\Gamma a$.*

Proof. Assume that S contains a maximal left ideal of S containing every proper left ideal of S, say L^*. Let $S \setminus \mathcal{A} = L^*$. Thus $S \setminus \mathcal{A}$ is a maximal proper left ideal of S. By Theorem 7, $\mathcal{A} \subseteq (a)_L$ for all $a \in \mathcal{A}$. We claim that $S \setminus \mathcal{A} \subseteq (a)_L$ for all $a \in \mathcal{A}$. Suppose that $S \setminus \mathcal{A} \nsubseteq (a')_L$ for some $a' \in \mathcal{A}$. Then $(a')_L \subset S$, and $(a')_L$ is a proper left ideal of S. This implies $(a')_L \subseteq L^* = S \setminus \mathcal{A}$, and so $a' \in \mathcal{A}$. This is a contradiction. Hence $S \setminus \mathcal{A} \subseteq (a)_L$ for all $a \in \mathcal{A}$. By

$$S = (S \setminus \mathcal{A}) \cup \mathcal{A} \subseteq (a)_L \subseteq S.$$

for all $a \in \mathcal{A}$, it follows that $(a)_L = S$ for all $a \in \mathcal{A}$. Therefore, $\{a\}$ is a right base of S for all $a \in \mathcal{A}$. Let A be a right base of S, and let $a, b \in A$. Suppose that $a \neq b$. Since $A \subseteq \mathcal{A}$, $a \in \mathcal{A}$, and so $S = (a)_L$. Since $a \neq b$ and $b \in S, b \in S\Gamma a$. By Lemma 1, $a = b$. This is a contradiction. Thus $a = b$, and $|A| = 1$. To show that (2.1) or (2.2) holds, assume that (2.1) is false. Then there exists a right base $A = \{a\}$ of S such that $S\Gamma a \neq S$. If $a \in S\Gamma a$, then $S\Gamma a = a \cup S\Gamma a = S$, a contradiction. Thus $a \notin S\Gamma a$. Let $A_1 = \{a_1\}$ be a right base of S such that $S\Gamma a_1 \neq S$ and $a_1 \notin S\Gamma a_1$. Suppose that $A \neq A_1$. Since $a \in S = a_1 \cup S\Gamma a_1$ and $A \neq A_1$, $a \in S\Gamma a_1$. Since $A \subseteq \mathcal{A}$,

$$S \setminus \mathcal{A} \subseteq S \setminus A_1 = (a_1 \cup S\Gamma a_1) \setminus \{a_1\} = S\Gamma a_1 \subset S.$$

Since $a \notin S\setminus\mathcal{A}$ and $a \in S\Gamma a_1$, $S\setminus\mathcal{A} \subset S\Gamma a_1$. This contradicts to the maximality $S \setminus \mathcal{A}$. Therefore (2.2) holds.

Conversely, assume that (1) and (2.1) hold. We will show that $S \setminus \mathcal{A} = L^*$. Since $S \setminus \mathcal{A} \neq \emptyset$, it follows by Theorem 6 that $S \setminus \mathcal{A}$ is a proper left ideal of S. Let L be a left ideal of S such that $S \setminus \mathcal{A} \subseteq L \subseteq S$. Suppose that $S \setminus \mathcal{A} \neq L$. Then $S\setminus\mathcal{A} \subset L$, and $\mathcal{A}\cap L \neq \emptyset$. Let $a \in \mathcal{A}\cap L$. Then $a \in A$ for some a right base A of S. By (1), $A = \{a\}$. By (2.1), $S\Gamma a = S$. Since $a \in L$, $S\Gamma a \subseteq S\Gamma L \subseteq L$. Hence

$$S = S\Gamma a \subseteq S\Gamma L \subseteq L \subseteq S.$$

Thus $L = S$. It follows that $S \setminus \mathcal{A}$ is a maximal proper left ideal of S. Let L_1 be a proper left ideal of S. Suppose that $L_1 \nsubseteq S \setminus \mathcal{A}$. Then there exists $a \in L_1$ and $a \notin S \setminus \mathcal{A}$. We have

$$S\Gamma a \subseteq S\Gamma L_1 \subseteq L_1 \subset S.$$

Since $a \notin S \setminus \mathcal{A}$, $a \in \mathcal{A}$, and so $a \in A$ for some a right base A of S. By (1), $A = \{a\}$. By (2.1), $S\Gamma a = S$. This is a contradiction. Hence $L_1 \subseteq S \setminus \mathcal{A}$. Thus $S \setminus \mathcal{A} = L^*$.

Finally, we assume that (1) and (2.2) hold. By (1), every right base of S has only one element. By (2.2), S contains only one right base $A = \{a\}$ with

$a \cup S\Gamma a = S$ but $a \notin S\Gamma a$. We have $\{a\} = \mathcal{A}$. Thus $S \setminus \{a\} = S \setminus \mathcal{A} \neq \emptyset$. By Theorem 6, $S \setminus \mathcal{A}$ is a proper left ideal of S. Let L be a left ideal of S such that $S \setminus \mathcal{A} \subseteq L \subseteq S$. Suppose that $S \setminus \mathcal{A} \neq L$; then $S \setminus \mathcal{A} \subset L$. Hence $\mathcal{A} \cap L \neq \emptyset$. Let $a \in \mathcal{A} \cap L$. Then

$$S = a \cup S\Gamma a \subseteq L \subseteq S.$$

It follows that $S \setminus \mathcal{A}$ is a maximal proper left ideal of S. Let L_1 be a proper left ideal of S. If $L_1 \nsubseteq S \setminus \mathcal{A}$, then there exists $a \in L_1$ such that $a \notin S \setminus \mathcal{A}$; hence

$$S = a \cup S\Gamma a \subseteq L_1 \subseteq S.$$

This is a contradiction. Thus $L_1 \subseteq S \setminus \mathcal{A}$. Therefore $S \setminus \mathcal{A} = L^*$. □

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