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ROUGH CONVERGENCE OF BERNSTEIN FUZZY $I-$ CONVERGENT OF $\Gamma^{3I(F)}_{f(\Delta,p)}$ SPACE DEFINED BY ORLICZ FUNCTION

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Abstract: The aim of this paper is to introduce and study a new concept of the rough fuzzy ideal convergent triple entire sequences defined by Orlicz function and also some topological properties of the resulting sequence spaces of rough fuzzy numbers were examined.

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1. Introduction

The idea of rough convergence was first introduced by Phu [10-12] in finite dimensional normed spaces. He showed that the set LIM^r_x is bounded, closed and convex; and he introduced the notion of rough Cauchy sequence. He also investigated the relations between rough convergence and other convergence types and the dependence of LIM^r_x on the roughness of degree r.

Aytar [1] studied of rough statistical convergence and defined the set of rough statistical limit points of a sequence and obtained two statistical con-

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vergence criteria associated with this set and prove that this set is closed and convex. Also, Aytar [2] studied that the $r-$ limit set of the sequence is equal to intersection of these sets and that $r-$ core of the sequence is equal to the union of these sets. Dündar and C. Çakan [9] investigated of rough ideal convergence and defined the set of rough ideal limit points of a sequence The notion of $I-$ convergence of a triple sequence spaces which is based on the structure of the ideal I of subsets of $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$, where $\mathbb{N}$ is the set of all natural numbers, is a natural generalization of the notion of convergence and statistical convergence.

Let K be a subset of the set of positive integers $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ and let us denote the set $K_{ik\ell}=\{(m,n,k)\in K : m\geq i, n\leq j, k\leq \ell\}$. Then the natural density of K is given by

$$\delta(K)=\lim_{i,j,\ell\to\infty}\frac{|K_{ij\ell}|}{ij\ell},$$

where $|K_{ij\ell}|$ denotes the number of elements in $K_{ij\ell}$.

The Bernstein operator of order (r,s,t) is given by

$$B_{rst}(f,x)=\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t} f\left(\frac{mnk}{rst}\right)\binom{r}{m}\binom{s}{n}\binom{t}{k}x^{m+n+k}(1-x)^{(m-r)+(n-s)+(k-t)},$$

where f is a continuous (real or complex valued) function defined on $[0,1]$.

Throughout the paper, $\mathbb{R}$ denotes the real of three dimensional space with metric (X,d). Consider a triple sequence of Bernstein polynomials $(B_{mnk}(f,x))$ such that $(B_{mnk}(f,x))\in\mathbb{R}, m,n,k\in\mathbb{N}$.

Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials $(B_{mnk}(f,x))$ is said to be statistically convergent to $0\in\mathbb{R}$, written as $st-lim\ x=0$, provided that the set

$$K_\epsilon := \left\{(m,n,k)\in\mathbb{N}^3 : |B_{mnk}(f,x)-f(x)|\geq\epsilon\right\}$$

has natural density zero for any $\epsilon>0$. In this case, 0 is called the statistical limit of the triple sequence of Bernstein polynomials. i.e., $\delta(K_\epsilon)=0$. That is,

$$\lim_{rst\to\infty}\tfrac{1}{rst}\left|\{(m,n,k)\leq(r,s,t) : |B_{mnk}(f,x)-(f,x)|\geq\epsilon\}\right|=0.$$

In this case, we write $\delta-limB_{mnk}(f,x)=f(x)$ or $B_{mnk}(f,x)\rightarrow^{S_B} f(x)$.

Throughout the paper, $\mathbb{N}$ denotes the set of all positive integers , χ_A- the characteristic function of $A\subset\mathbb{N},\mathbb{R}$ the set of all real numbers. A subset A of $\mathbb{N}$ is said to have asymptotic density $d(A)$ if

$$d(A) = \lim_{ij\ell\to\infty} \frac{1}{ij\ell} \sum_{m=1}^{i} \sum_{n=1}^{j} \sum_{k=1}^{\ell} \chi_A(K).$$

A triple sequence (real or complex) can be defined as a function $x : \mathbb{N}\times\mathbb{N}\times\mathbb{N} \to \mathbb{R}(\mathbb{C})$, where $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequence was introduced and investigated at the initial by *Sahiner et al. [13,14]*, *Esi et al. [3-6]*, *Datta et al. [7]*,*Subramanian et al. [15]*, *Debnath et al. [8]* and many others.

The set of fuzzy real numbers is denoted by $f(x)(\mathbb{R})$, and d denotes the supremum metric on $f(X)\left(\mathbb{R}^3\right)$. Now let r be nonnegative real number. A triple sequence space of Bernstein polynomials of $(B_{mnk}(f, X))$ of fuzzy numbers is $r-$ convergent to a fuzzy number $f(X)$ and we write

$$B_{mnk}(f, X) \to^r f(X) \text{ as } m, n, k \to \infty,$$

provided that for every $\epsilon > 0$ there is an integer $m_\epsilon, n_\epsilon, k_\epsilon$ so that

$$d(B_{mnk}(f, X), f(X)) < r + \epsilon \text{ whenever } m \geq m_\epsilon, n \geq n_\epsilon, k \geq k_\epsilon.$$

The set

$$LIM^r B_{mnk}(f, X) := \left\{ f(X) \in f(X)\left(\mathbb{R}^3\right) : B_{mnk}(f, X) \to^r f(X), \text{ as } m, n, k \to \infty \right\}$$

is called the $r-$ limit set of the triple sequence space of Bernstein polynomials of $(B_{mnk}(f, X))$.

A triple sequence space of Bernstein polynomials of fuzzy numbers which is divergent can be convergent with a certain roughness degree. For instance, let us define

$$B_{mnk}(f, X) = \left\{ \begin{array}{ll} \eta(X), & \text{if } (m, n, k) \textit{ is odd integer}, \\ \mu(X), & \text{otherwise} \end{array} \right\},$$

where

$$\eta(X) = \left\{ \begin{array}{cc} X, & \text{if } X \in [0, 1], \\ -X + 2, & \text{if} X \in [1, 2], \\ 0, & \text{otherwise} \end{array} \right\}$$

and

$$\mu(X) = \left\{ \begin{array}{cc} X - 3, & \text{if } X \in [3, 4], \\ -X + 5, & \text{if} X \in [4, 5], \\ 0, & \text{otherwise} \end{array} \right\}.$$

Then we have where

$$LIM^r B_{mnk}(f,X) = \left\{ \begin{array}{cc} \phi, & \text{if} r < \frac{3}{2}, \\ [\mu - r_1, \eta + r_1], & \text{otherwise} \end{array} \right\},$$

where r_1 is nonnegative real number with
$[\mu - r_1, \eta + r_1] := \left\{ B_{mnk}(f,X) \in f(X)\left(\mathbb{R}^3\right) : \mu - r_1 \leq B_{mnk}(f,X) \leq \eta + r_1 \right\}.$

The ideal of rough convergence of a triple sequence space of Bernstein polynomials can be interpreted as follows:

Let $(B_{mnk}(f,Y))$ be a convergent triple sequence space of Bernstein polynomials of fuzzy numbers. Assume that $(B_{mnk}(f,Y))$ cannot be determined exactly for every $(m,n,k) \in \mathbb{N}^3$. That is, $(B_{mnk}(f,Y))$ cannot be calculated so we can use approximate value of $(B_{mnk}(f,Y))$ for simplicity of calculation. We only know that $(B_{mnk}(f,Y)) \in [\mu_{mnk}, \lambda_{mnk}]$, where $d(\mu_{mnk}, \lambda_{mnk}) \leq r$ for every $(m,n,k) \in \mathbb{N}^3$. The triple sequence space of Bernstein polynomials of $(B_{mnk}(f,X))$ satisfying $(B_{mnk}(f,X)) \in [\mu_{mnk}, \lambda_{mnk}]$, for all m,n,k. Then the triple sequence space of Bernstein polynomials of $(B_{mnk}(f,X))$ may not be convergent, but the inequality

$$\begin{aligned} & d(B_{mnk}(f,X), f(X)) \\ & \qquad \leq d(B_{mnk}(f,X), B_{mnk}(f,Y)) + d(B_{mnk}(f,Y), f(Y)) \\ & \qquad\qquad \leq r + d(B_{mnk}(f,Y), f(Y)) \end{aligned}$$

implies that the triple sequence space of Bernstein polynomials of $(B_{mnk}(f,X))$ is $r-$ convergent.

A triple sequence $x = (x_{mnk})$ is called triple entire sequence if

$$|x_{mnk}|^{\frac{1}{m+n+k}} \to 0$$

as $m,n,k \to \infty$. The triple entire sequences will be denoted by Γ^3.

Definition 1. An Orlicz function ([see [16]) is a function $M : [0,\infty) \to [0,\infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by $M(x+y) \leq M(x) + M(y)$, then this function is called modulus function.

Lindenstrauss and Tzafriri (see [17]) used the idea of Orlicz function to construct Orlicz sequence space.

A sequence $g = (g_{mn})$ defined by

$$g_{mn}(v) = \sup\left\{ |v|\, u - (f_{mnk})(u) : u \geq 0 \right\}, m,n,k = 1,2,\cdots$$

is called the complementary function of a Musielak-Orlicz function f. For a given Musielak-Orlicz function f, (see [18]) the Musielak-Orlicz sequence space t_f is defined as follows

$$t_f = \left\{ x \in w^3 : I_f \left(|x_{mnk}| \right)^{1/m+n+k} \to 0 \, as \, m, n, k \to \infty \right\},$$

where I_f is a convex modular defined by

$$I_f (x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{mnk} \left(|x_{mnk}| \right)^{1/m+n+k}, x = (x_{mnk}) \in t_f.$$

We consider t_f equipped with the Luxemburg metric

$$d(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{mnk} \left(|x_{mnk}|^{1/m+n+k} \right)$$

A sequence $x = (x_{mnk})$is said to be triple analytic if

$$\sup_{m,n,k} |x_{mnk}|^{\frac{1}{m+n+k}} < \infty.$$

The vector space of all triple analytic sequences are usually denoted by Λ^3. A sequence $x = (x_{mnk})$ is called triple entire sequence if

$$|x_{mnk}|^{\frac{1}{m+n+k}} \to 0 \text{ as } m, n, k \to \infty.$$

The vector space of all triple entire sequences are usually denoted by Γ^3.

Let $w^3, \Gamma^3 (\Delta_{mnk}), \Lambda^3 (\Delta_{mnk})$ be denote the spaces of all, triple entire difference sequence space and triple analytic difference sequence space respectively and is defined as

$$\Delta_{mnk} = x_{mnk} - x_{m,n+1,k} - x_{m,n,k+1} + x_{m,n+1,k+1} \\ - x_{m+1,n,k} + x_{m+1,n+1,k} + x_{m+1,n,k+1} - x_{m+1,n+1,k+1},$$

and $\Delta^0 x_{mnk} = \langle x_{mnk} \rangle$.

In this paper, we first define the concept of rough convergence of Bernstein polynomials of fuzzy $I-$ convergent of triple sequence defined by Orlicz function and also discuss some topological properties of fuzzy numbers.

2. Definitions and Preliminaries

A fuzzy number X is a fuzzy subset of the real $\mathbb{R}^3$, which is normal fuzzy convex, upper semi-continuous, and the X^0 is bounded where

$$X^0 ;= cl \left\{ x \in \mathbb{R}^3 : X(x) > 0 \right\}$$

and cl is the closure operator. These properties imply that for each $\alpha \in (0,1]$, the $\alpha-$ level set X^α defined by

$$X^\alpha = \left\{ x \in \mathbb{R}^3 : X(x) \geq \alpha \right\} = \left[\underline{X}^\alpha, \overline{X}^\alpha \right]$$

is a non empty compact convex subset of $\mathbb{R}^3$.

The supremum metric d on the set $L\left(\mathbb{R}^3\right)$ is defined by

$$d(X,Y) = \sup_{\alpha \in [0,1]} \max \left(\left| \underline{X}^\alpha - \underline{Y}^\alpha \right|, \left| \overline{X}^\alpha - \overline{Y}^\alpha \right| \right).$$

Now, given $X, Y \in L\left(\mathbb{R}^3\right)$, we define $X \leq Y$ if $\underline{X}^\alpha \leq \underline{Y}^\alpha$ and $\overline{X}^\alpha \leq \overline{Y}^\alpha$ for each $\alpha \in [0,]$.

We write $X \leq Y$ if $X \leq Y$ and there exists an $\alpha_0 \in [0,1]$ such that $\underline{X}^{\alpha_0} \leq \underline{Y}^{\alpha_0}$ or $\overline{X}^{\alpha_0} \leq \overline{Y}^{\alpha_0}$.

A subset E of $L\left(\mathbb{R}^3\right)$ is said to be bounded above if there exists a fuzzy number μ, called an upper bound of E, such that $X \leq \mu$ for every $X \in E$. μ is called the least upper bound of E if μ is an upper bound and $\mu \leq \mu^{'}$ for all upper bounds $\mu^{'}$.

A lower bound and the greatest lower bound are defined similarly. E is said to be bounded if it is both bounded above and below.

The notions of least upper bound and the greatest lower bound have been defined only for bounded sets of fuzzy numbers. If the set $E \subset L\left(\mathbb{R}^3\right)$ is bounded then its supremum and infimum exist.

The limit infimum and limit supremum of a triple sequence spaces (X_{mnk}) is defined by

$$\liminf_{mnk \to \infty} X_{mnk} := \inf \; A_X.$$

$$\limsup_{mnk \to \infty} X_{mnk} := \inf \; B_X.$$

where

$$A_X := \left\{ \mu \in L\left(\mathbb{R}^3\right) : The\ set\ \left\{ (m,n,k) \in \mathbb{N}^3 : X_{mnk} < \mu \right\} is\ infinite \right\},$$

$$B_X := \left\{ \mu \in L\left(\mathbb{R}^3\right) : The\ set\ \left\{ (m,n,k) \in \mathbb{N}^3 : X_{mnk} > \mu \right\} is\ infinite \right\}.$$

Now, given two fuzzy numbers $X, Y \in L\left(\mathbb{R}^3\right)$, we define their sum as $Z = X + Y$, where $\underline{Z}^\alpha := \underline{X}^\alpha + \underline{Y}^\alpha$ and $\overline{Z}^\alpha := \overline{X}^\alpha + \overline{Y}^\alpha$ for all $\alpha \in [0, 1]$.

To any real number $a \in \mathbb{R}^3$, we can assign a fuzzy number $a_1 \in L\left(\mathbb{R}^3\right)$, which is defied by

$$a_1(x) = \left\{ \begin{array}{ll} 1, & \text{if } x = a, \\ 0, & \text{otherwise} \end{array} \right\}.$$

An order interval in $L\left(\mathbb{R}^3\right)$ is defined by $[X, Y] := \left\{Z \in L\left(\mathbb{R}^3\right) : X \leq Z \leq Y\right\}$, where $X, Y \in L\left(\mathbb{R}^3\right)$.

A set E of fuzzy numbers is called convex if $\lambda\mu_1 + (1-\lambda)\mu_2 \in E$ for all $\lambda \in [0, 1]$ and $\mu_1, \mu_2 \in E$.

Definition 2. Let f be a continuous function of Orlicz defined on the closed interval $[0, 1]$. A rough triple sequence of Bernstein polynomials of $(B_{mnk}(f, X))$ of real numbers is said to be convergent to the fuzzy real number $B_{mnk}(f, X)$, if for every $\epsilon > 0$, there exists $m_0 = m_0(\epsilon), n_0 = n_0(\epsilon), k_0 = k_0(\epsilon) \in N$ such that $\overline{d}(B_{mnk}(f, X), f(X)) < r + \epsilon$ for all $m \geq m_0$, $n \geq n_0$, $k \geq k_0$.

Definition 3. Let f be a continuous function of Orlicz defined on the closed interval $[0, 1]$. A rough triple sequence of Bernstein polynomials of $(B_{mnk}(f, X))$ of real numbers is said to be $I-$convergent to the fuzzy number $B_{mnk}(f(X_0))$, if for all $\epsilon > 0$, the set

$$\left\{(m, n, k) \in N^3 : \overline{d}(B_{mnk}(f, X), f(X_0)) \geq r + \epsilon\right\} \in I_3.$$

We write $I_3 - lim B_{mnk}(f, X) = f(X_0)$.

Definition 4. Let f be a continuous function of Orlicz defined on the closed interval $[0, 1]$. A rough triple sequence of Bernstein polynomials of $(B_{mnk}(f, X))$ of real numbers is said to be solid of $(B_{mnk}(f, Y)) \in E^F$ whenever $(B_{mnk}(f, X)) \in E^F$ and $|B_{mnk}(f, Y)| \leq |B_{mnk}(f, X)|$ for all $m, n, k \in \mathbb{N}^3$.

Let

$$K = \{(m_i, n_i, k_i) : i \in \mathbb{N}; m_1 < m_2 < m_3 \cdots, n_1 < n_2 < n_3 \cdots \\ \text{and } k_1 < k_2 < k_3 \cdots\} \subseteq \mathbb{N}^3$$

and E^F be a triple sequence space. A $K-$ step space of E^F is a sequence space

$$\lambda_K^E = \left\{(X_{m_i n_i k_i}) \in w^{3F} : (B_{mnk}(f, X)) \in E^F\right\}.$$

Let f be a continuous function of Orlicz defined on the closed interval $[0, 1]$. A triple sequence of Bernstein polynomials of $(B_{mnk}(f, X))$ of real numbers of a canonical pre-image of a sequence $(B_{m_i n_i k_i}(f,X)) \in E^F$ is a sequence $(B_{mnk}(f,Y))$ defined as follows:

$$B_{mnk}(f,Y) = \begin{cases} B_{mnk}(f,X), & \text{if } (m,n,k) \in K, \\ \overline{0}, & \text{otherwise.} \end{cases}$$

Let f be a continuous function of Orlicz defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials of $(B_{mnk}(f,X))$ of real numbers of a canonical pre-image of a step space λ_K^E is a set of canonical pre-images of all elements in λ_K^E.

Definition 5. Let f be a continuous function of Orlicz defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials of $(B_{mnk}(f,X))$ of real numbers is said to be monotone if E^F contains the canonical pre-image of all its step spaces.

Definition 6. Let f be a continuous function of Orlicz defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials of $(B_{mnk}(f,X))$ of real numbers is said to be symmetric if $\left(B_{\pi(m),\pi(n),\pi(k)}\right)(f,X) \in E^F$, whenever $(B_{mnk}(f,X)) \in E^F$, where π is a permutation of N^3

Definition 7. Let f be a continuous function of Orlicz defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials of $(B_{mnk}(f,X))$ of real numbers is said to be sequence algebra if $B_{mnk}(X \otimes Y \otimes Z) \in E^F$, whenever $(X_{mnk}), (Y_{mnk}), (Z_{mnk}) \in E^F$.

Definition 8. Let f be a continuous function of Orlicz defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials of $(B_{mnk}(f,X))$ of real numbers is said to be convergence free if $(B_{mnk}(f,Y)) \in E^F$, whenever (Z_{mnk}), $(B_{mnk}(f,X)) \in E^F$ and $B_{mnk}(f,X) = \overline{0}$ implies $B_{mnk}(f,Y) = \overline{0}$ implies $B_{mnk}(f,Z) = \overline{0}$.

Let f be a continuous function of Orlicz defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials of $(B_{mnk}(f,X))$ of real numbers and (p_{mnk}) be a triple sequence of analytic strictly positive real numbers such that $0 < p_{mnk} \leq \sup_{mnk} < \infty$.

We introduce the following sequence spaces:

$$\Gamma^{3I(F)}_{f(\Delta,p)} = \left\{(m,n,k) \in \mathbb{N}^3 : \left[\overline{d}\left(B_{mnk}(f,\Delta X)^{1/(m+n+k)}, f\left(\overline{0}\right)\right)\right]^{p_{mnk}} \geq r + \epsilon\right\} \in I_3,$$

for every $\epsilon > 0$.

$$\Lambda^{3F}_{f(\Delta,p)} = \left\{ X = (X_{mnk}) : \sup_{mnk} \left[\overline{d} \left(B_{mnk} \left(f, \Delta X\right)^{1/(m+n+k)}, f\left(\overline{0}\right) \right) \right]^{p_{mnk}} < \infty \right\}.$$

Also we write $\Lambda^{3I(F)}_{f(\Delta,p)} = \Gamma^{3I(F)}_{f(\Delta,p)} \bigcap \Lambda^{3F}_{f(\Delta,p)}$

Lemma 9. *Let f be a continuous function of Orlicz defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials of $(B_{mnk}(f,X))$ of real numbers is solid, then it is monotone. See [16], p. 53.*

3. Main Results

Proposition 10. *Let f be a continuous function of Orlicz defined on the closed interval $[0,1]$. A rough triple sequence of Bernstein polynomials of $(B_{mnk}(f,X))$ of real numbers then $\Lambda^{3I(F)}_{f(\Delta,p)}$ and $\Lambda^{3F}_{f(\Delta,p)}$ are linear spaces.*

Proof. It is easy. Therefore omit the proof. □

Proposition 11. *Let f be a continuous function of Orlicz defined on the closed interval $[0,1]$. A rough triple entire sequence of Bernstein polynomials of $(B_{mnk}(f,X))$ of real numbers then $\Gamma^{3I(F)}_{f(\Delta,p)} \subseteq \Lambda^{3F}_{f(\Delta,p)}$ and the inclusion are strict.*

Proof. The inclusion $\Gamma^{3I(F)}_{f(\Delta,p)} \subseteq \Lambda^{3F}_{f(\Delta,p)}$ is obvious. For establishing that the inclusion is proper, consider the following example. □

Example 12. We prove the result for the case $\Gamma^{3I(F)}_{f(\Delta,p)} \subseteq \Lambda^{3F}_{f(\Delta,p)}$, the other case similar.

Let $B_{mnk}(f,\Delta X) = f(\Delta X)$. Let the sequence $B_{mnk}((f,\Delta X), f(X))$ be defined by for $m > n > k$,

$$B_{mnk}\left(f, \Delta X(t), f(\Delta X)\right) = \begin{cases} (mt-m-1)^{(m+n+k)} (m-1)^{-(m+n+k)}, & \text{for } 1+\frac{1}{m} \le t \le 3, \\ (3-t)^{(m+n+k)}, & \text{for } 2 < t \le 4, \\ 0, & \text{otherwise.} \end{cases}$$

and for $m < n < k$

$$B_{mnk}\left(f, \Delta X(t), f(\Delta X)\right)$$

$$= \begin{cases} (mt-1)^{(m+n+k)} (m-1)^{-(m+n+k)}, & \text{for } \frac{1}{m} \leq t \leq 2, \\ (-t+2)^{(m+n+k)}, & \text{for } 1 \leq t \leq 3, \\ 0, & \text{otherwise.} \end{cases}$$

Then, $B_{mnk}(f, \Delta X(t), f(\Delta X)) \in \Lambda^{3F}_{f(\Delta,p)}$ but $B_{mnk}(f, \Delta X(t), f(\Delta X)) \notin \Gamma^{3I(F)}_{f(\Delta,p)}$.

Proposition 13. *Let f be a continuous function of Orlicz defined on the closed interval $[0,1]$. A rough triple sequence of Bernstein polynomials of $(B_{mnk}(f,X))$ of real numbers then the triple difference sequence of $\Lambda^{3I(F)}_{f(\Delta,p)}$ is not solid*

Proof. Let $(B_{mnk}(f,X), f(X)) = (1) \in \Lambda^{3I(F)}_{f(\Delta,p)}$. Let $\alpha_{mnk} = (-1)^{m+n+k}$, then $(B_{mnk}(f, \alpha_{mnk} X, f(X))) \notin \Lambda^{3I(F)}_{f(\Delta,p)}$. Hence $\Lambda^{3I(F)}_{f(\Delta,p)}$ is not solid. □

Corollary 14. *Let f be a continuous function of Orlicz defined on the closed interval $[0,1]$. A rough triple sequence of Bernstein polynomials of $(B_{mnk}(f,X))$ of real numbers then the triple difference sequence of $\Lambda^{3I(F)}_{f(\Delta,p)}$ is not monotone.*

Proof. By Lemma 2.8, it follows that the space $\Lambda^{3I(F)}_{f(\Delta,p)}$ is not monotone. □

Proposition 15. *Let f be a continuous function of Orlicz defined on the closed interval $[0,1]$. A rough triple entire sequence of Bernstein polynomials of $(B_{mnk}(f,X))$ of real numbers then the triple entire difference sequence space of $\Gamma^{3I(F)}_{f(\Delta,p)}$ is sequence algebra*

Proof. Let $(X_{mnk}), (Y_{mnk}) \in \Gamma^{3I(F)}_{f(\Delta,p)}$ and $0 < \epsilon < 1$. Then the result follows from the following inclusion relation:

$$\Big\{(m,n,k) \in \mathbb{N}^3$$
$$: \left[\overline{d}\left(B_{mnk}(f, (\Delta X_{mnk} \otimes \Delta Y_{mnk} \otimes \Delta Z_{mnk}))^{1/(m+n+k)}, f(\overline{0})\right)\right]^{p_{mnk}} < r + \epsilon\Big\}$$

$$\supseteq \left\{(m,n,k) \in \mathbb{N}^3 : \left[\overline{d}\left(B_{mnk}(f, \Delta X_{mnk})^{1/(m+n+k)}, f(\overline{0})\right)\right]^{p_{mnk}} < r + \epsilon\right\}$$
$$\bigcap \left\{(m,n,k) \in \mathbb{N}^3 : \left[\overline{d}\left(B_{mnk}(f, \Delta Y_{mnk})^{1/m+n+k}, f(\overline{0})\right)\right]^{p_{mnk}} < r + \epsilon\right\}$$
$$\bigcap \left\{(m,n,k) \in \mathbb{N}^3 : \left[\overline{d}\left(B_{mnk}(f, \Delta Z_{mnk})^{1/m+n+k}, f(\overline{0})\right)\right]^{p_{mnk}} < r + \epsilon\right\}.$$

Similarly we can prove the result for other cases. □

Proposition 16. *Let f be a continuous function of Orlicz defined on the closed interval $[0,1]$. A rough triple entire sequence of Bernstein polynomials of $(B_{mnk}(f,X))$ of real numbers then the triple entire difference sequence space of $\Gamma^{3I(F)}_{f(\Delta,p)}$ is complete metric space with respect to the metric ρ defined by*

$$\rho(X,Y,Z) = \sup_m \left[\overline{d}\left(B_{mnk}(f,(X_{m1k}Y_{m1k}Z_{m1k}))^{1/m+1+k}, f(\overline{0})\right)\right]^{p_{mnk}}$$
$$+ \sup_n \left[\overline{d}\left(B_{mnk}(f,(X_{1nk}Y_{1nk}Z_{1nk}))^{1/1+n+k}, f(\overline{0})\right)\right]^{p_{mnk}}$$
$$+ \sup_k \left[\overline{d}\left(B_{mnk}(f,(X_{mn1}Y_{mn1}Z_{mn1}))^{1/m+1+k}, f(\overline{0})\right)\right]^{p_{mnk}}$$
$$+ \sup_{mnk} \left[\overline{d}\left(B_{mnk}(f,(\Delta X\Delta Y\Delta Z))^{1/m+n+k}, f(\overline{0})\right)\right]^{p_{mnk}},$$

where $X = (X_{mnk}), Y = (Y_{mnk}), Z = (Z_{mnk}) \in \Gamma^{3I(F)}_{f(\Delta,p)}$,

$$\Delta X = x_{mnk} - x_{m,n+1,k} - x_{m,n,k+1} + x_{m,n+1,k+1} - x_{m+1,n,k} + x_{m+1,n+1,k} + x_{m+1,n,k+1} - x_{m+1,n+1,k+1}$$

and $\Delta^0 x_{mnk} = \langle x_{mnk}\rangle$. *for all* $m,n,k \in \mathbb{N}^3$.

Proposition 17. *Let f be a continuous function of Orlicz defined on the closed interval $[0,1]$. A rough triple sequence of Bernstein polynomials of $(B_{mnk}(f,X))$ of real numbers then the triple difference sequence space of $\Lambda^{3I(F)}_{f(\Delta,p)}$ is nowhere dense subsets of $\Lambda^{3F}_{f(\Delta,p)}$.*

Proof. By Proposition 3.1, the sequence space $\Lambda^{3I(F)}_{f(\Delta,p)}$ are proper subspace of $\Lambda^{3F}_{f(\Delta,p)}$. Hence by proposition 3.5 the result follows. □

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