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# HERMITE-HADAMARD TYPE INEQUALITY FOR PRODUCT OF CONVEX FUNCTIONS VIA SUGENO INTEGRALS

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**Abstract:** The aim of this paper is to obtain, Hermite-Hadamard type inequality for product of convex function using Sugeno integral which is based on $(\alpha, m)$-convex function. Some application of our results are also given.

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**Key Words:** Hermite-Hadamard type inequality for product of convex function, Sugeno integrals, $(\alpha, m)$-convex function

## 1. Introduction

In 1974, M. Sugeno [21] in his Ph.D. thesis, has introduced the theory of fuzzy measures and integrals. Ralescu and Admas in [7] generalized the fuzzy measure from $[0, 1]$ to $[0, \infty]$ for fuzzy integral.

Since then many authors [1], [11], [12], [2], [13], [16], [17], [8], [9], [18], [19] have generalized various inequalities to fuzzy integral. In this the authors have shown that many of the inequalities such as a Prekopa-Leindler type inequality [15], Stolarsky's inequality [3], and Convolution type inequality [14]. Due to the versatile operators in fuzzy measure and fuzzy integral it has application in different areas such as information fusion, electronic auctions, decision mak-

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ing, ect. In [10] author has produced a general Barnes-Godunova-Levin type inequality of the Sugeno integral for a concave function and generalizations concerning m-convex and $(\alpha, m)$-convex function.

Recently in [8], [18], [5], [6] authors have proved a Hermite-Hadamard type inequality for product of convex function using Sugeno integral.

Motivated by the above literature in this paper we obtain some Hermite-Hadamard type inequality for product of convex functions, using Sugeno integral.

## 2. Preliminary

In this section we give some basic definitions and properties of the fuzzy integral and $(\alpha, m)$-convex functions. For details [21], [24], [23].

Suppose that $\wp$ is a $\sigma$-algebra of subsets of $X$ and $\mu : \wp \longrightarrow [0, \infty)$ be a non-negative, extended real valued set function. We say that $\mu$ is a fuzzy measure if it satisfies:

1. $\mu(\phi) = 0$;
2. $E, F \in \wp$ and $E \subset F$ imply $\mu(E) \leq \mu(F)$;
3. $\{E_n\} \subset \wp, E_1 \subset E_2 \subset ...$, imply $\lim_{n \longrightarrow \infty} \mu(E_n) = \mu(\bigcup_{n=1}^{\infty} E_n)$;
4. $\{E_n\} \subset \wp, E_1 \supset E_2 \supset ..., \mu(E_1) < \infty$, imply $\lim_{n \longrightarrow \infty} \mu(E_n) = \mu(\bigcap_{n=1}^{\infty} E_n)$.

If $f$ is non-negative real-valued function defined on $X$, we denote the set $\{x \in X : f(x) \geq \alpha\} = \{x \in X : f \geq \alpha\}$ by $F_\alpha$ for $\alpha \geq 0$. Note that if $\alpha \leq \beta$ then $F_\beta \subset F_\alpha$.

Let $(X, \wp, \mu)$ be a fuzzy measure space, we denote $M^+$ the set of all non-negative measurable functions with respect to $\wp$.

**Definition 2.1.** (Sugeno [21]). Let $(X, \wp, \mu)$ be a fuzzy measure space, $f \in M^+$ and $A \in \wp$, the Sugeno integral (or fuzzy integral) of $f$ on $A$, with respect to the fuzzy measure $\mu$, is defined as

$$(s) \int_A f d\mu = \bigvee_{\alpha \geq 0} [\alpha \wedge \mu(A \cap F_\alpha)],$$

when $A = X$,

$$(s) \int_X f d\mu = \bigvee_{\alpha \geq 0} [\alpha \wedge \mu(F_\alpha)],$$

where $\vee$ and $\wedge$ denote the operations sup and inf on $[0, \infty)$, respectively.

The properties of fuzzy integral are well known and can be found in [25] as follows.

**Proposition 2.1.** *Let $(X, \wp, \mu)$ be fuzzy measure space, $A, B \in \wp$ and $f, g \in M^+$ then:*

1. $(s) \int_A f d\mu \leq \mu(A)$;
2. $(s) \int_A k d\mu = k \wedge \mu(A)$, $k$ *for non-negative constant;*
3. $(s) \int_A f d\mu \leq (s) \int_A g d\mu$, *for* $f \leq g$;
4. $\mu(A \cap \{f \geq \alpha\}) \geq \alpha \Longrightarrow (s) \int_A f d\mu \geq \alpha$;
5. $\mu(A \cap \{f \geq \alpha\}) \leq \alpha \Longrightarrow (s) \int_A f d\mu \leq \alpha$;
6. $(s) \int_A f d\mu > \alpha \Longleftrightarrow$ *there exists* $\gamma > \alpha$ *such that* $\mu(A \cap \{f \geq \gamma\}) > \alpha$;
7. $(s) \int_A f d\mu < \alpha \Longleftrightarrow$ *there exists* $\gamma < \alpha$ *such that* $\mu(A \cap \{f \geq \gamma\}) < \alpha$.

**Remark 2.1.** Consider the distribution function $F$ associated to $f$ on $A$, that is, $F(\alpha) = \mu(A \cap \{f \geq \alpha\})$.Then due to (4) and (5) of Proposition 2.1, we have $F(\alpha) = \alpha \Longrightarrow (s) \int_A f d\mu = \alpha$. Thus, from a numerical point of view, the fuzzy integral can be calculated solving the equation $F(\alpha) = \alpha$.

**Definition 2.2.** (Mihesan [23]). The function $f : [0, b] \longrightarrow R$ is said to be $(\alpha, m)$-convex, where $(\alpha, m) \in [0, 1]^2$, if for very $x, y \in [0, b]$ and $t \in [0, 1]$, satisfies

$$f(tx + m(1-t)y) \leq t^\alpha f(x) + m(1 - t^\alpha) f(y).$$

Note that for $(\alpha, m) \in \{(0, 0), (\alpha, 0), (1, 0), (1, m), (1, 1), (\alpha, 1)\}$ we can obtains the following classes of functions: increasing, $\alpha$- starshaped, starshaped, m-convex, convex and $\alpha$-convex .

## 3. Hermite-Hadamard Type Inequality for Product of Convex Function

In [4], B. G. Pachpatte established two new Hermite-Hadamard type inequalities for products of convex functions as follows.

**Theorem 3.1.** *Let $f$, $g$ be real valued, nonnegative and convex function on $[a,b]$. Then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \le \frac{1}{3}M(a,b)+\frac{1}{6}N(a,b), \tag{1}$$

$$2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)g(x)dx + \frac{1}{6}M(a,b)+\frac{1}{3}N(a,b), \tag{2}$$

*where $M(a,b)=f(a)g(a)+f(b)g(b)$ and*
*$N(a,b)=f(a)g(b)+f(b)g(a)$.*

The following example show Hermite-Hadamard inequality for product of convex function via Sugeno integral based on $(\alpha,m)$-convex functions are not valid for (1).

**Example 3.1.** Consider $X=[0,1]$ and let $\mu$ be the Lebesgue measure on $X$. If we take $f(x)=\frac{1}{4}x^2$ and $g(x)=\frac{1}{2}x^2$, then $f(x)$, $g(x)$ are $(\frac{1}{2},\frac{1}{3})$-convex function.
$f(x)=\frac{1}{4}x^2=f(x.1+\frac{1}{3}(1-x).0)\le\sqrt{x}\frac{1}{4}+\frac{1}{3}(1-\sqrt{x}).0=\frac{1}{4}\sqrt{x}$.
$g(x)=\frac{1}{2}x^2=g(x.1+\frac{1}{3}(1-x).0)\le\sqrt{x}\frac{1}{2}+\frac{1}{3}(1-\sqrt{x}).0=\frac{1}{2}\sqrt{x}$.
For $x\in[0,1]$, from simple calculation we get

$$(s)\int_0^1 \frac{1}{8}x^4 d\mu = 0.0869.$$

on the other hand, $\frac{1}{3}M(a,b)+\frac{1}{6}N(a,b)=0.04166$.

This proves that right hand side (1) of Hermite-Hadamard type inequalities for product of convex function is not satisfied for Sugeno integral.

Now we give the Hermite-Hadamard type inequality for product of convex function via Sugeno integral based on $(\alpha,m)$-convex functions.

**Theorem 3.2.** *Let $f,g:[0,1]\longrightarrow[0,\infty)$ be $(\alpha,m)$-convex function. $(\alpha,m)\in(0,1)^2$, $f(0)\le f(1)$ and $g(0)\le g(1)$ and $\mu$ be the Lebesgue measure on $\mathbb{R}$ . Then*

$$(s)\int_0^1 f(x)g(x)d\mu \le min\{\beta,1\},$$

*where $\beta$ is given by*

$$1-\left(\frac{\beta-mg(0)}{g(1)-mg(0)}\right)^{\frac{1}{\alpha}}-\left(\frac{\beta-mf(0)}{f(1)-mf(0)}\right)^{\frac{1}{\alpha}}$$

$$+\left(\frac{\beta - mf(0)}{f(1) - mf(0)}\right)^{\frac{1}{\alpha}} . \left(\frac{\beta - mg(0)}{g(1) - mg(0)}\right)^{\frac{1}{\alpha}} . \tag{3}$$

**Proof.** As $f$, $g$ are $(\alpha, m)$- convex function for $x \in [0, 1]$, we have
$f(x) = f(x.1 + m(1 - x).0) \leq x^\alpha f(1) + m(1 - x^\alpha)f(0) = h_1(x)$.
$g(x) = g(x.1 + m(1 - x).0) \leq x^\alpha g(1) + m(1 - x^\alpha)g(0) = h_2(x)$.
By Proposition 2.1 of (3), we have

$$\begin{aligned}(s)\int_0^1 f(x)g(x)d\mu =&(s)\int_0^1 f(x.1 + m(1 - x).0). \\ &\qquad\qquad g(x.1 + m(1 - x).0)d\mu \\ \leq&(s)\int_0^1 (x^\alpha f(1) + m(1 - x^\alpha)f(0)). \\ &\qquad\qquad (x^\alpha g(1) + m(1 - x^\alpha)g(0))d\mu \\ =&(s)\int_0^1 h_1(x).h_2(x)d\mu. \end{aligned} \tag{4}$$

To calculate right hand side of (4), we consider the distribution function $F$ given by

$$\begin{aligned}F(\beta) =&\mu\big([0, 1] \cap \{h_1(x).h_2(x) \geq \beta\}\big) \\ =&\mu\big([0, 1] \cap \{h_1(x) \geq \beta\}\big).\mu\big([0, 1] \cap \{h_2(x) \geq \beta\}\big) \\ =&\mu\big([0, 1] \cap \{x|m(1 - x^\alpha)f(0) + x^\alpha f(1) \geq \beta\}\big). \\ &\quad \mu\big([0, 1] \cap \{x|m(1 - x^\alpha)g(0) + x^\alpha g(1) \geq \beta\}\big) \\ =&\mu\left([0, 1] \cap \left\{x|x^\alpha \geq \frac{\beta - mf(0)}{f(1) - mf(0)}\right\}\right). \\ &\quad \mu\left([0, 1] \cap \left\{x|x^\alpha \geq \frac{\beta - mg(0)}{g(1) - mg(0)}\right\}\right) \\ =&1 - \left(\frac{\beta - mf(0)}{f(1) - mf(0)}\right)^{\frac{1}{\alpha}} .1 - \left(\frac{\beta - mg(0)}{g(1) - mg(0)}\right)^{\frac{1}{\alpha}}, \end{aligned} \tag{5}$$

the required solution of (5) is given by (3).
By Proposition 2.1 of (1) and Remark 2.1, we have

$$(s)\int_0^1 f(x)g(x)d\mu \leq min\{\beta, 1\}. \qquad \square$$

Now we give application of Theorem 3.2 in the following example.

**Example 3.2.** Consider $X = [0,1]$ and let $\mu$ be Lebesgue measure on $X$. If we take the function $f(x) = x^2$ and $g(x) = x^2$ then $f(x)$, $g(x)$ are $(\frac{1}{2}, \frac{1}{3})$-convex function. In fact,
$f(x) = x^2 = f(x.1 + (1-x).0) \leq \sqrt{x}.1 + \frac{1}{3}(1-\sqrt{x}).0 \leq \sqrt{x}$.
$g(x) = x^2 = g(x.1 + (1-x).0) \leq \sqrt{x}.1 + \frac{1}{3}(1-\sqrt{x}).0 \leq \sqrt{x}$.
For $x \in [0,1]$, by Theorem 3.2, we have
$(s)\int_0^1 x^4 d\mu \leq min\{\beta, 1\}$,
where $\beta$ satisfies the following equation
$\beta^4 - 4\beta^3 + 6\beta^2 - 5\beta + 1 = 0$.

**Theorem 3.3.** *Let $f, g : [0,1] \longrightarrow [0,\infty)$ be $(\alpha,m)$-convex function $(\alpha,m) \in (0,1]^2$, $f(0) > f(1)$ and $g(0) > g(1)$ and $\mu$ be the Lebesgue measure on $R$. If $f(0) > f(1)$ and $g(0) > g(1)$, then*
***Case(a).*** *If $\frac{f(1)}{f(0)} < \frac{g(1)}{g(0)}$, then*
***Case 1.*** *If $m \in \left(0, \frac{f(1)}{f(0)}\right)$, then $(s)\int_0^1 f(x)g(x)d\mu \leq min\{\beta, 1\}$,*
*where $\beta$ is given as in (3).*
***Case 2.*** *If $m = \frac{f(1)}{f(0)}$, then*

$$(s)\int_0^1 f(x)g(x)d\mu \leq min\left\{f(1)\left(1 - \left(\frac{\beta - mg(0)}{g(1) - mg(0)}\right)^{\frac{1}{\alpha}}\right), 1\right\}. \tag{6}$$

***Case3.*** *If $m \in \left(\frac{f(1)}{f(0)}, \frac{g(1)}{g(0)}\right)$, then $(s)\int_0^1 f(x)g(x)d\mu \leq min\{\beta, 1\}$,*
*where $\beta$ is given by*

$$\left(\frac{\beta - mf(0)}{f(1) - mf(0)}\right)^{\frac{1}{\alpha}} - \left(\frac{\beta - mf(0)}{f(1) - mf(0)}\right)^{\frac{1}{\alpha}} . \left(\frac{\beta - mg(0)}{g(1) - mg(0)}\right)^{\frac{1}{\alpha}}. \tag{7}$$

***Case 4.*** *If $m = \frac{g(1)}{g(0)}$, then*

$$(s)\int_0^1 f(x)g(x)d\mu \leq min\left\{\left(\frac{\beta - mf(0)}{f(1) - mf(0)}\right)^{\frac{1}{\alpha}} g(1), 1\right\}. \tag{8}$$

***Case(b).*** *If $\frac{f(1)}{f(0)} = \frac{g(1)}{g(0)}$, then*
***Case 1.*** *If $m \in \left(0, \frac{f(1)}{f(0)}\right)$, then $(s)\int_0^1 f(x)g(x)d\mu \leq min\{\beta, 1\}$,*
*where $\beta$ is given as in (3).*
***Case 2.*** *If $m = \frac{f(1)}{f(0)}$, then*

$$(s)\int_0^1 f(x)g(x)d\mu \leq min\{f(1)g(1), 1\}. \tag{9}$$

***Case 3.*** *If* $m \in \left(\frac{f(1)}{f(0)}, 1\right)$, *then* $(s)\int_0^1 f(x)g(x)d\mu \leq min\{\beta, 1\}$, $\beta$ *satisfies the following equation*

$$((f(1) - mf(0))(g(1) - mg(0)))\beta^\alpha + \beta mg(0) + \beta f(0) - \beta^2 - m^2 f(0)g(0) = 0. \quad (10)$$

**Case(c).** *If* $\frac{f(1)}{f(0)} > \frac{g(1)}{g(0)}$, *then*

***Case 1.*** *If* $m \in \left(0, \frac{g(1)}{g(0)}\right)$ *then* $(s)\int_0^1 f(x)g(x)d\mu \leq min\{\beta, 1\}$, *where* $\beta$ *is given as in* (3).

***Case 2.*** *If* $m = \frac{g(1)}{g(0)}$, *then*

$$(s)\int_0^1 f(x)g(x)d\mu \leq min\left\{\left(1 - \left(\frac{\beta - mf(0)}{f(1) - mf(0)}\right)^{\frac{1}{\alpha}}\right)g(1), 1\right\}. \quad (11)$$

***Case 3.*** *If* $m \in \left(\frac{g(1)}{g(0)}, \frac{f(1)}{f(0)}\right)$, *then* $(s)\int_0^1 f(x)g(x)d\mu \leq min\{\beta, 1\}$, *where* $\beta$ *is given by*

$$\left(\frac{\beta - mg(0)}{g(1) - mg(0)}\right)^{\frac{1}{\alpha}} - \left(\frac{\beta - mf(0)}{f(1) - mf(0)}\right)^{\frac{1}{\alpha}} . \left(\frac{\beta - mg(0)}{g(1) - mg(0)}\right)^{\frac{1}{\alpha}}. \quad (12)$$

***Case 4.*** *If* $m = \frac{f(1)}{f(0)}$, *then*

$$(s)\int_0^1 f(x)g(x)d\mu \leq min\left\{f(1). \left(\frac{\beta - mg(0)}{g(1) - mg(0)}\right)^{\frac{1}{\alpha}}, 1\right\}. \quad (13)$$

***Case 5.*** *If* $m \in \left(\frac{f(1)}{f(0)}, 1\right)$, *then* $(s)\int_0^1 f(x)g(x)d\mu \leq min\{\beta, 1\}$, *and* $\beta$ *satisfies the following equation*

$$((f(1) - mf(0))(g(1) - mg(0)))\beta^\alpha + \beta m(f(0) + g(0)) - \beta^2 - m^2 f(0)g(0) = 0. \quad (14)$$

**Proof.** As $f$, $g$ are $(\alpha, m)$-convex function for $x \in [0, 1]$. We have
$f(x) = f(x.1 + m(1 - x).0) \leq x^\alpha f(1) + m(1 - x^\alpha)f(0) = h_1(x)$.
$g(x) = g(x.1 + m(1 - x).0) \leq x^\alpha g(1) + m(1 - x^\alpha)g(0) = h_2(x)$.
By proposition 2.1 of (3), we have

$$(s)\int_0^1 f(x)g(x)d\mu = (s)\int_0^1 f(x.1 + m(1 - x).0).$$

$$\begin{aligned}
&\qquad\qquad g(x.1+m(1-x).0)d\mu \\
\leq&(s)\int_0^1 (x^\alpha f(1)+m(1-x^\alpha)f(0)). \\
&\qquad\qquad (x^\alpha g(1)+m(1-x^\alpha)g(0))d\mu \\
=&(s)\int_0^1 h_1(x).h_2(x)d\mu. \qquad (15)
\end{aligned}$$

In order to calculate the right hand side of (15), we consider the distribution function $F$ given by

$$\begin{aligned}
F(\beta) =&\mu\big([0,1]\cap\{h_1(x)h_2(x)\geq\beta\}\big) \\
=&\mu\big([0,1]\cap\{h_1(x)\geq\beta\}\big).\mu\big([0,1]\cap\{h_2(x)\geq\beta\}\big) \\
=&\mu\Bigg([0,1]\cap\Big\{x|m(1-x^\alpha)f(0)+x^\alpha f(1)\geq\beta\Big\}\Bigg). \\
&\mu\Bigg([0,1]\cap\Big\{x|m(1-x^\alpha)g(0)+x^\alpha g(1)\geq\beta\Big\}\Bigg). \qquad (16)
\end{aligned}$$

**Case (a).** If $\frac{f(1)}{f(0)} < \frac{g(1)}{g(0)}$, then

**Case 1.** If $m \in \left(0, \frac{f(1)}{f(0)}\right)$, then from (16), we have

$$\begin{aligned}
F(\beta) =& \mu\Bigg([0,1]\cap\Big\{x|mf(0)-x^\alpha mf(0)+x^\alpha f(1)\geq\beta\Big\}\Bigg). \\
&\mu\Bigg([0,1]\cap\Big\{x|mg(0)-x^\alpha mg(0)+x^\alpha g(1)\geq\beta\Big\}\Bigg) \\
=& \mu\Bigg([0,1]\cap\Bigg\{x|x\geq\left(\frac{\beta-mf(0)}{f(1)-mf(0)}\right)^{\frac{1}{\alpha}}\Bigg\}\Bigg). \\
&\mu\Bigg([0,1]\cap\Bigg\{x|x\geq\left(\frac{\beta-mg(0)}{g(1)-mg(0)}\right)^{\frac{1}{\alpha}}\Bigg\}\Bigg) \\
=& \mu\Bigg(\left(\frac{\beta-mf(0)}{f(1)-mf(0)}\right)^{\frac{1}{\alpha}},1\Bigg).\mu\Bigg(\left(\frac{\beta-mg(0)}{g(1)-mg(0)}\right)^{\frac{1}{\alpha}},1\Bigg) \\
=& \Bigg(1-\left(\frac{\beta-mf(0)}{f(1)-mf(0)}\right)^{\frac{1}{\alpha}}\Bigg). \\
&\Bigg(1-\left(\frac{\beta-mg(0)}{g(1)-mg(0)}\right)^{\frac{1}{\alpha}}\Bigg), \qquad (17)
\end{aligned}$$

and the required solution of equation (17) is given by (3).
By Proposition 2.1 of (1) and Remark 2.1, we have

$$(s)\int_0^1 f(x)g(x)d\mu \leq min\{\beta, 1\}.$$

**Case 2.** If $m = \frac{f(1)}{f(0)}$, then from (16), we have

$$\begin{aligned}
F(\beta) =&\mu\Big([0,1] \cap \Big\{x|f(1) \geq \beta\Big\}\Big).\\
&\mu\Big([0,1] \cap \Big\{x|x \geq \Big(\frac{\beta - mg(0)}{g(1) - mg(0)}\Big)^{\frac{1}{\alpha}}\Big\}\Big)\\
=&f(1).\mu\Big(\Big(\frac{\beta - mg(0)}{g(1) - mg(0)}\Big)^{\frac{1}{\alpha}}, 1\Big)\\
=&f(1).\Big(1 - \Big(\frac{\beta - mg(0)}{g(1) - mg(0)}\Big)^{\frac{1}{\alpha}}\Big),
\end{aligned} \tag{18}$$

and the solution of equation (18) is

$$f(1).\Big(1 - \Big(\frac{\beta - mg(0)}{g(1) - mg(0)}\Big)^{\frac{1}{\alpha}}\Big) = \beta.$$

By Proposition 2.1 of (1) and Remark 2.1, we have

$$(s)\int_0^1 f(x)g(x)d\mu \leq min\Big\{f(1).\Big(1 - \Big(\frac{\beta - mg(0)}{g(1) - mg(0)}\Big)^{\frac{1}{\alpha}}\Big), 1\Big\}.$$

**Case 3.** If $m \in \Big(\frac{f(1)}{f(0)}, \frac{g(1)}{g(0)}\Big)$, then from (16), we have

$$\begin{aligned}
F(\beta) =&\mu\Big([0,1] \cap \Big\{x|x \leq \Big(\frac{\beta - mf(0)}{f(1) - mf(0)}\Big)^{\frac{1}{\alpha}}\Big\}\Big).\\
&\mu\Big([0,1] \cap \Big\{x|x \geq \Big(\frac{\beta - mg(0)}{g(1) - mg(0)}\Big)^{\frac{1}{\alpha}}\Big\}\Big)\\
=&\mu\Big(0, \Big(\frac{\beta - mf(0)}{f(1) - mf(0)}\Big)^{\frac{1}{\alpha}}\Big).\mu\Big(\Big(\frac{\beta - mg(0)}{g(1) - mg(0)}\Big)^{\frac{1}{\alpha}}, 1\Big)\\
=&\Big(\frac{\beta - mf(0)}{f(1) - mf(0)}\Big)^{\frac{1}{\alpha}}.1 - \Big(\frac{\beta - mg(0)}{g(1) - mg(0)}\Big)^{\frac{1}{\alpha}},
\end{aligned}$$

and the solution of above equation given as in (7).
By Proposition 2.1 of (1) and Remark 2.1, we have

$$(s)\int_0^1 f(x)g(x)d\mu \leq min\{\beta, 1\}.$$

**Case 4.** If $m = \frac{g(1)}{g(0)}$, then from (16), we have

$$\begin{aligned}F(\beta) =&\mu\Big([0,1]\cap\Big\{x|x\leq\Big(\frac{\beta-mf(0)}{f(1)-mf(0)}\Big)^{\frac{1}{\alpha}}\Big\}\Big).\\ &\mu\Big([0,1]\cap\Big\{x|g(1)\geq\beta\Big\}\Big)\\ =&\mu\Big(0,\Big(\frac{\beta-mf(0)}{f(1)-mf(0)}\Big)^{\frac{1}{\alpha}}\Big).g(1)\\ =&\Big(\frac{\beta-mf(0)}{f(1)-mf(0)}\Big)^{\frac{1}{\alpha}}.g(1),\end{aligned}$$

and the solution of above equation is

$$\Big(\frac{\beta-mf(0)}{f(1)-mf(0)}\Big)^{\frac{1}{\alpha}}.g(1)=\beta.$$

By Proposition 2.1 of (1) and Remark 2.1, we have

$$(s)\int_0^1 f(x)g(x)d\mu \leq min\Big\{\Big(\frac{\beta-mf(0)}{f(1)-mf(0)}\Big)^{\frac{1}{\alpha}}.g(1), 1\Big\}.$$

The proof of Case(b) and Case(c) can be given similarly, so we omit details. □

**Theorem 3.4.** *Let $f, g : [a, b] \longrightarrow [0,\infty)$ be $(\alpha, m)$-convex function, $(\alpha, m) \in (0, 1)^2$, $f(a) \leq f(b)$ and $g(a) \leq g(b)$ and $\mu$ be the Lebesgue measure on $R$. Then*

$$(s)\int_a^b f(x)g(x)d\mu \leq min\{\beta, b-a\}$$

*where $\beta$ is given by*

$$(b-ma)^2\Big[1-\Big(\frac{\beta-mg(a)}{g(b)-mg(a)}\Big)^{\frac{1}{\alpha}}-\Big(\frac{\beta-mf(a)}{f(b)-mf(a)}\Big)^{\frac{1}{\alpha}}$$

$$+\left(\frac{\beta - mf(a)}{f(b) - mf(a)}\right)^{\frac{1}{\alpha}} . \left(\frac{\beta - mg(a)}{g(b) - mg(a)}\right)^{\frac{1}{\alpha}}\Bigg]. \tag{19}$$

**Proof.** As $f$, $g$ is $(\alpha, m)$-convex function for $x \in [a, b]$, we have

$$\begin{aligned}
f(x) =& f\left(m\left(1 - \frac{x - ma}{b - ma}\right)a + \frac{x - ma}{b - ma}b\right)\\
\leq& m\left(1 - \left(\frac{x - ma}{b - ma}\right)^{\alpha}\right)f(a) + \left(\frac{x - ma}{b - ma}\right)^{\alpha} f(b)\\
=& h_1(x).\\
g(x) =& g\left(m\left(1 - \frac{x - ma}{b - ma}\right)a + \frac{x - ma}{b - ma}b\right)\\
\leq& m\left(1 - \left(\frac{x - ma}{b - ma}\right)^{\alpha}\right)g(a) + \left(\frac{x - ma}{b - ma}\right)^{\alpha} g(b)\\
=& h_2(x).
\end{aligned}$$

Then from of Proposition 2.1 of (3), we have

$$\begin{aligned}
(s)\int_a^b fgd\mu =& (s)\int_a^b f\left(m\left(1 - \frac{x - ma}{b - ma}\right)a + \frac{x - ma}{b - ma}b\right).\\
& g\left(m\left(1 - \frac{x - ma}{b - ma}\right)a + \frac{x - ma}{b - ma}b\right)\\
\leq& m\left(1 - \left(\frac{x - ma}{b - ma}\right)^{\alpha}\right)f(a) + \left(\frac{x - ma}{b - ma}\right)^{\alpha} f(b).\\
& m\left(1 - \left(\frac{x - ma}{b - ma}\right)^{\alpha}\right)g(a) + \left(\frac{x - ma}{b - ma}\right)^{\alpha} g(b)\\
=& \int_a^b h_1(x)h_2(x).
\end{aligned} \tag{20}$$

To calculate the right hand of the inequality (20), we consider the distribution function $F$ given by

$$\begin{aligned}
F(\beta) =& \mu([a, b] \cap \{h_1(x)h_2(x) \geq \beta\})\\
=& \mu([a, b] \cap \{h_1(x) \geq \beta\}).\mu([a, b] \cap \{h_2(x) \geq \beta\})\\
=& \mu\left([a, b] \cap \left\{x|m\left(1 - \left(\frac{x - ma}{b - ma}\right)^{\alpha}\right)f(a) + \left(\frac{x - ma}{b - ma}\right)^{\alpha} f(b) \geq \beta\right\}\right).\\
& \mu\left([a, b] \cap \left\{x|m\left(1 - \left(\frac{x - ma}{b - ma}\right)^{\alpha}\right)g(a) + \left(\frac{x - ma}{b - ma}\right)^{\alpha} g(b) \geq \beta\right\}\right)
\end{aligned}$$

$$=\mu\Bigg([a,b]\cap\Bigg\{x|x\geq(b-ma)\Bigg(\frac{\beta-mf(a)}{f(b)-mf(a)}\Bigg)^{\frac{1}{\alpha}}+ma\Bigg\}\Bigg).$$

$$\mu\Bigg([a,b]\cap\Bigg\{x|x\geq(b-ma)\Bigg(\frac{\beta-mg(a)}{g(b)-mg(a)}\Bigg)^{\frac{1}{\alpha}}+ma\Bigg\}\Bigg)$$

$$=\mu\Bigg((b-ma)\Bigg(\frac{\beta-mf(a)}{f(b)-mf(a)}\Bigg)^{\frac{1}{\alpha}}+ma,b\Bigg).$$

$$\mu\Bigg((b-ma)\Bigg(\frac{\beta-mg(a)}{g(b)-mg(a)}\Bigg)^{\frac{1}{\alpha}}+ma,b\Bigg)$$

$$=b-ma-(b-ma)\Bigg(\frac{\beta-mf(a)}{f(b)-mf(a)}\Bigg)^{\frac{1}{\alpha}}.$$

$$b-ma-(b-ma)\Bigg(\frac{\beta-mg(a)}{g(b)-mg(a)}\Bigg)^{\frac{1}{\alpha}}, \tag{21}$$

and the required solution of (21) is given by (19).
By Proposition 2.1 of (1) and Remark 2.1, we have

$$(s)\int_a^b f(x)g(x)d\mu\leq min\{\beta,b-a\}.$$

Hence theorem. □

**Theorem 3.5.** *Let $f,g:[a,b]\longrightarrow[0,\infty)$ be $(\alpha,m)$-convex function, $(\alpha,m)\in(0,1)^2$, $f(a)>f(b)$ and $g(a)>g(b)$ and $\mu$ be the Lebesgue measure on $R$. If $f(a)>f(b)$ and $g(a)>g(b)$, then*
***Case (a).*** *If $\frac{f(b)}{f(a)}<\frac{g(b)}{g(a)}$, then*
***Case 1.*** *If $m\in\Big(0,\frac{f(b)}{f(a)}\Big)$, then*

$$(s)\int_a^b f(x)g(x)d\mu\leq min\{\beta,b-a\},$$

*where $\beta$ is given as in (19).*
***Case 2.*** *If $m=\frac{f(b)}{f(a)}$, then*

$$(s)\int_a^b f(x)g(x)d\mu\leq$$

$$min\Bigg\{f(b).(b-ma)\Bigg(1-\Bigg(\frac{\beta-mg(a)}{g(b)-mg(a)}\Bigg)^{\frac{1}{\alpha}}\Bigg),b-a\Bigg\}. \tag{22}$$

***Case 3.*** *If* $m \in \left(\frac{f(b)}{f(a)}, \frac{g(b)}{g(a)}\right)$, *then*

$$(s)\int_a^b f(x)g(x)d\mu \leq min\{\beta, b-a\},$$

*where* $\beta$ *is given by*

$$(b-ma)^2\left[\left(\frac{\beta - mf(a)}{f(b)-mf(a)}\right)^{\frac{1}{\alpha}} - \left(\frac{\beta - mf(a)}{f(b)-mf(a)}\right)^{\frac{1}{\alpha}}. \left(\frac{\beta - mg(a)}{g(b)-mg(a)}\right)^{\frac{1}{\alpha}}\right] + (ma-a)(b-ma)\left(1-\left(\frac{\beta - mg(a)}{g(b)-mg(a)}\right)^{\frac{1}{\alpha}}\right). \quad (23)$$

***Case 4.*** *If* $m = \frac{g(b)}{g(a)}$, *then*

$$(s)\int_a^b f(x)g(x)d\mu \leq min\left\{\left((b-ma)\left(\frac{\beta - mf(a)}{f(b)-mf(a)}\right)^{\frac{1}{\alpha}} + (ma-a)\right).g(b), b-a\right\}. \quad (24)$$

***Case 5.*** *If* $m \in \left(\frac{g(b)}{g(a)}, 1\right)$, *then*

$$(s)\int_a^b f(x)g(x)d\mu \leq min\{\beta, b-a\},$$

*where* $\beta$ *is given by*

$$(b-ma)^2\left(\frac{\beta - mf(a)}{f(b)-mf(a)}\right)^{\frac{1}{\alpha}}\left(\frac{\beta - mg(a)}{g(b)-mg(a)}\right)^{\frac{1}{\alpha}} + (ma-a)(b-ma) \left[\left(\frac{\beta - mf(a)}{f(b)-mf(a)}\right)^{\frac{1}{\alpha}} + \left(\frac{\beta - mg(a)}{g(b)-mg(a)}\right)^{\frac{1}{\alpha}}\right] + (ma-a)^2. \quad (25)$$

***Case (b).*** *If* $\frac{f(b)}{f(a)} = \frac{g(b)}{g(a)}$, *then*

***Case 1.*** *If* $m \in \left(0, \frac{f(b)}{f(a)}\right)$, *then*

$$(s)\int_a^b f(x)g(x)d\mu \leq min\{\beta, b-a\},$$

*where* $\beta$ *is given as in* (19).
***Case 2.*** *If* $m = \frac{f(b)}{f(a)}$, *then*

$$(s)\int_a^b f(x)g(x)d\mu \leq min\big\{f(b)g(b), b-a\big\}. \tag{26}$$

***Case 3.*** *If* $m \in \left(\frac{f(b)}{f(a)}, 1\right)$, *then*

$$(s)\int_a^b f(x)g(x)d\mu \leq min\{\beta, b-a\},$$

*where* $\beta$ *is as given by* (25).
***Case (c).*** *If* $\frac{f(b)}{f(a)} > \frac{g(b)}{g(a)}$, *then*
***Case 1.*** *If* $m \in \left(0, \frac{g(b)}{g(a)}\right)$, *then*

$$(s)\int_a^b f(x)g(x)d\mu \leq min\{\beta, b-a\},$$

*where* $\beta$ *is given as in* (19).
***Case 2.*** *If* $m = \frac{g(b)}{g(a)}$, *then*

$$(s)\int_a^b f(x)g(x)d\mu \leq$$
$$min\bigg\{(b-ma)\bigg(1-\bigg(\frac{\beta - mf(a)}{f(b)-mf(a)}\bigg)^{\frac{1}{\alpha}}\bigg).g(b), b-a\bigg\}. \tag{27}$$

***Case 3.*** *If* $m \in \left(\frac{g(b)}{g(a)}, \frac{f(b)}{f(a)}\right)$, *then*

$$(s)\int_a^b f(x)g(x)d\mu \leq min\{\beta, b-a\},$$

*where* $\beta$ *is given by*

$$(b-ma)^2\bigg[\bigg(\frac{\beta - mg(a)}{g(b)-mg(a)}\bigg)^{\frac{1}{\alpha}} - \bigg(\frac{\beta - mf(a)}{f(b)-mf(a)}\bigg)^{\frac{1}{\alpha}}.$$
$$\bigg(\frac{\beta - mg(a)}{g(b)-mg(a)}\bigg)^{\frac{1}{\alpha}}\bigg] + (ma-a)(b-ma)\bigg(1-\bigg(\frac{\beta - mf(a)}{f(b)-mf(a)}\bigg)^{\frac{1}{\alpha}}\bigg). \tag{28}$$

***Case 4.*** *If* $m = \frac{f(b)}{f(a)}$, *then*

$$(s)\int_a^b f(x)g(x)d\mu \leq$$
$$min\Big\{f(b).\Big((b-ma)\Big(\frac{\beta - mg(a)}{g(b)-mg(a)}\Big)^{\frac{1}{\alpha}} + (ma-a)\Big), b-a\Big\}. \qquad (29)$$

***Case 5.*** *If* $m \in \left(\frac{f(b)}{f(a)}, 1\right)$, *then*

$$(s)\int_a^b f(x)g(x)d\mu \leq min\{\beta, b-a\},$$

*where* $\beta$ *is as given by* (25).

**Proof.** The proof of this theorem can be proved similar as in Theorem 3.4. □

As some special cases of $(\alpha, m)$-convex functions in Theorem 3.4 or 3.5, we have the following results.

**Remark 3.1.** Let $X = [a, b]$, $\alpha = 0$, $m = 0$ and $f$, $g$ be two increasing functions for all $x \in X$. If $\mu$ is a Lebesgue measure on $X$, then

$$(s)\int_a^b f(x)g(x)d\mu \leq min\{f(b)g(b), b-a\}. \qquad (30)$$

**Remark 3.2.** Let $X = [a, b]$, $\alpha = 1$, $m = 0$ and $f$, $g$ be two starshaped functions for all $x \in X$. If $\mu$ is a Lebesgue measure on $X$, then

$$(s)\int_a^b f(x)g(x)d\mu \leq min\{\beta, b-a\},$$

where $\beta$ satisfies the following equation

$$b^2\beta^2 - b^2\beta(f(b)+g(b)) - \beta f(b)g(b) + b^2 f(b)g(b) = 0. \qquad (31)$$

**Remark 3.3.** Let $X = [a, b]$, $\alpha = 1$, $m \in (0, 1)$ and $f$, $g$ be two $m$-convex functions for all $x \in X$. If $\mu$ is a Lebesgue measure on $X$, then
Case(i). If $f(a) \leq f(b)$ and $g(a) \leq g(b)$, then

$$(s)\int_a^b f(x)g(x)d\mu \leq min\{\beta, b-a\},$$

where $\beta$ satisfies the following equation

$$\begin{aligned}&(b-ma)^2\beta^2-(b-ma)^2\beta(f(b)+g(b))-\\&\quad\beta(f(b)-mf(a))(g(b)-mg(a))+(b-ma)^2f(b)g(b)=0.\end{aligned}\tag{32}$$

Case (ii). If $f(a)>f(b)$ and $g(a)>g(b)$, then
Case(a). If $\frac{f(b)}{f(a)}<\frac{g(b)}{g(a)}$, then
Case 1. If $m\in\left(0,\frac{f(b)}{f(a)}\right)$, then

$$(s)\int_a^b f(x)g(x)d\mu\le min\{\beta,b-a\},$$

where $\beta$ is satisfies the equation (32).
Case 2. If $m=\frac{f(b)}{f(a)}$, then

$$\begin{aligned}&(s)\int_a^b f(x)g(x)d\mu\le\\&\qquad min\left\{\frac{f(b)g(b)(b-ma)}{(g(b)-mg(a))+(b-ma)f(b)},b-a\right\}.\end{aligned}\tag{33}$$

Case 3. If $m\in\left(\frac{f(b)}{f(a)},\frac{g(b)}{g(a)}\right)$, then

$$(s)\int_a^b f(x)g(x)d\mu\le min\{\beta,b-a\},$$

where $\beta$ satisfies the following equation

$$\begin{aligned}&(b-ma)^2[\beta g(b)-mf(a)g(b)-\beta^2+\beta mf(a)]+\\&(ma-a)(b-ma)[f(b)g(b)-mf(a)g(b)-\beta f(b)+\beta mf(a)]\\&\qquad-\beta(f(b)-mf(a))(g(b)-mg(a))=0.\end{aligned}\tag{34}$$

Case 4. If $m=\frac{g(b)}{g(a)}$, then

$$\begin{aligned}&(s)\int_a^b f(x)g(x)d\mu\le\\&min\left\{\frac{(f(b)-mf(a))(ma-a)g(b)-(b-ma)mf(a)g(b)}{(f(b)-mf(a))-(b-ma)g(b)},b-a\right\}.\end{aligned}\tag{35}$$

Case 5. If $m \in \left(\frac{g(b)}{g(a)}, 1\right)$, then

$$(s)\int_a^b f(x)g(x)d\mu \leq min\{\beta, b-a\},$$

where $\beta$ satisfies the following equation

$$\begin{aligned}(b-ma)^2(\beta - mf(a))(\beta - mg(a)) + (ma-a)(b-ma)\\ [(g(b)-mg(a))(\beta - mf(a)) + (f(b)-mf(a))(\beta - mg(a))]\\ + (ma-a)^2(f(b)-mf(a))(g(b)-mg(a))\\ - \beta(f(b)-mf(a))(g(b)-mg(a)) = 0.\end{aligned} \quad (36)$$

Case (b). If $\frac{f(b)}{f(a)} = \frac{g(b)}{g(a)}$, then
Case 1. If $m \in \left(0, \frac{f(b)}{f(a)}\right)$, then

$$(s)\int_a^b f(x)g(x)d\mu \leq min\{\beta, b-a\},$$

where $\beta$ satisfies the equation (32).
Case 2. If $m = \frac{f(b)}{f(a)}$, then

$$(s)\int_a^b f(x)g(x)d\mu \leq min\{f(b)g(b), b-a\}. \quad (37)$$

Case 3. If $m \in (\frac{f(b)}{f(a)}, 1)$, then

$$(s)\int_a^b f(x)g(x)d\mu \leq min\{\beta, b-a\},$$

where $\beta$ satisfies the equation (36).
Case(c). If $\frac{f(b)}{f(a)} > \frac{g(b)}{g(a)}$, then
Case 1. If $m \in \left(0, \frac{g(b)}{g(a)}\right)$, then

$$(s)\int_a^b f(x)g(x)d\mu \leq min\{\beta, b-a\},$$

where $\beta$ satisfies the equation (32).
Case 2. If $m = \frac{g(b)}{g(a)}$, then

$$(s)\int_a^b f(x)g(x)d\mu \leq$$
$$min\left\{\frac{(f(b)-mf(a))(b-ma)g(b)+(b-ma)mf(a)g(b)}{(f(b)-mf(a))+(b-ma)g(b)}, b-a\right\}. \quad (38)$$

Case 3. If $m \in \left(\frac{g(b)}{g(a)}, \frac{f(b)}{f(a)}\right)$, then

$$(s)\int_a^b f(x)g(x)d\mu \leq min\{\beta, b-a\},$$

where $\beta$ satisfies the following equation

$$(b-ma)^2[f(b)\beta+\beta mg(a)-mg(a)f(b)-\beta^2]+(ma-a)(b-ma)$$
$$[f(b)g(b)+\beta mg(a)-mg(a)f(b)-\beta g(b)]$$
$$-\beta(f(b)-mf(a))(g(b)-mg(a))=0. \quad (39)$$

Case 4. If $m = \frac{f(b)}{f(a)}$, then

$$(s)\int_a^b f(x)g(x)d\mu \leq$$
$$min\left\{\frac{(ma-a)f(b)(g(b)-mg(a))-(b-ma)mg(a)f(b)}{(g(b)-mg(a))-f(b)(b-ma)}, b-a\right\}. \quad (40)$$

Case 5. If $m \in \left(\frac{f(b)}{f(a)}, 1\right)$, then

$$(s)\int_a^b f(x)g(x)d\mu \leq min\{\beta, b-a\},$$

where $\beta$ satisfies the equation (36).

**Remark 3.4.** Let $X = [a,b]$, $\alpha = 1$, $m = 1$ and $f$, $g$ be two convex functions for all $x \in X$. If $\mu$ is a Lebesgue measure on $X$, then
Case(i). If $f(a) < f(b)$ and $g(a) < g(b)$, then

$$(s)\int_a^b f(x)g(x)d\mu \leq min\{\beta, b-a\},$$

where $\beta$ satisfies the following equation

$$(b-a)^2[\beta^2 + f(b)g(b) - \beta f(b) - \beta g(b)] \\ - \beta(f(b)-f(a))(g(b)-g(a)) = 0. \tag{41}$$

Case(ii). If $f(a) = f(b)$ and $g(a) = g(b)$, then

$$(s)\int_a^b f(x)g(x)d\mu \leq min\{f(a)g(a), b-a\}. \tag{42}$$

Case(iii). If $f(a) > f(b)$ and $g(a) > g(b)$, then

$$(s)\int_a^b f(x)g(x)d\mu \leq min\{\beta, b-a\},$$

where $\beta$ satisfies the following equation

$$(b-a)^2(\beta - f(a))(\beta - g(a)) - \beta(f(b)-f(a))(g(b)-g(a)) = 0. \tag{43}$$

**Remark 3.5.** Let $X = [a, b]$, $\alpha \in (0, 1)$, $m = 0$ and $f$, $g$ be two $\alpha$-starshaped functions for all $x \in X$. If $\mu$ is a Lebesgue measure on $X$, then

$$(s)\int_a^b f(x)g(x)d\mu \leq min\{\beta, b-a\},$$

where $\beta$ satisfies the following equation

$$b^2\beta^{\frac{2}{\alpha}} + b^2 f^{\frac{1}{\alpha}}(b)g^{\frac{1}{\alpha}}(b) - b^2\beta^{\frac{1}{\alpha}}[f^{\frac{1}{\alpha}}(b) + g^{\frac{1}{\alpha}}(b)] \\ - \beta f^{\frac{1}{\alpha}}(b).g^{\frac{1}{\alpha}}(b) = 0. \tag{44}$$

**Remark 3.6.** Let $X = [a, b]$, $\alpha \in (0, 1)$, $m = 1$ and $f$, $g$ be two $\alpha$-convex functions for all $x \in X$. If $\mu$ is a Lebesgue measure on $X$, then
Case (i). If $f(a) < f(b)$ and $g(a) < g(b)$, then

$$(s)\int_a^b f(x)g(x)d\mu \leq min\{\beta, b-a\},$$

where $\beta$ is given by

$$(b-a)^2\left[1 - \left(\frac{\beta - g(a)}{g(b)-g(a)}\right)^{\frac{1}{\alpha}} - \left(\frac{\beta - f(a)}{f(b)-f(a)}\right)^{\frac{1}{\alpha}} \right. \\ \left. + \left(\frac{\beta - f(a)}{f(b)-f(a)}\right)^{\frac{1}{\alpha}} . \left(\frac{\beta - g(a)}{g(b)-g(a)}\right)^{\frac{1}{\alpha}}\right]. \tag{45}$$

Case (ii). If $f(a) = f(b)$ and $g(a) = g(b)$, then

$$(s)\int_a^b f(x)g(x)d\mu \leq min\{f(a)g(a), b-a\}. \tag{46}$$

Case(iii). If $f(a) > f(b)$ and $g(a) > g(b)$, then

$$(s)\int_a^b f(x)g(x)d\mu \leq min\{\beta, b-a\},$$

where $\beta$ satisfies the following equation

$$(b-a)^{2\alpha}(\beta - f(a))(\beta - g(a)) - \beta^{\alpha}(f(b) - f(a))(g(b) - g(a)) = 0. \tag{47}$$

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