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TOPOLOGIES ON FINITE SETS

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Abstract: In this paper, we have computed the repeated ratios of number of topologies on finite sets with the reference to the series A000798 [1]. The bound for number of topologies on finite sets is determined by many people [2, 3]. We have also obtained the sharper bounds for the number of topologies on the set with n elements.

AMS Subject Classification: 40A15, 54A99
Key Words: topology, repeated ratio, lower bound, upper bound, sequence

1. Introduction

The problem of finding topologies on finite sets is studied by many researchers[2, 3]. This problem is still open after decades. Let T_n denotes the number of topologies on the finite set with n elements. V. Krishnamurthy [3] has used the idea of n-basic numbers to calculate T_n and also calculated the basic numbers for $n = 2, 3$ and 4. He has stated that T_n is exactly equal to number of n-basic numbers for arbitrary n. He has given the bound for number of topologies as $T_n < 2^{n(n-1)}$. Later, G. H. Patil and M. S. Chaudhary [2] have determined the number of topologies T_n recursively and given the upper bound as $T_{n+1} \leq T_n(3T_n + 1)$. In this paper, rather than discussing the definition

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and method of computation of number of topologies, we are finding the repeated ratios with reference to the series A000798 [1]. We also observe that the sequence of repeated ratios oscillates around 1. After approximating the repeated ratios, we have obtained the lower and upper bounds which are more sharper than the bounds given by V. Krishnamurthy [3], G. H. Patil and M. S. Chaudhary [2]. This approximations can help us to estimate the interval for T_n for large n without calculating the actual topologies. Since, calculating the actual topologies for large n becomes a tedious job.

2. Repeated Ratios

We consider the table given by N.J.A. Sloane [1] for the series A000798. This table gives the number of topologies T_n corresponding to the finite set with n elements for $0 \leq n \leq 18$.

We are using the idea of finding the repeated ratios. The values of these ratios are calculated by using MATLAB R2015A. We define the ratios as below.

The first ratio a_n is given as

$$a_n = \frac{T_{n+1}}{T_n}$$

and the table of ratio a_n is calculated using values from Table 1 for $1 \leq n \leq 18$. The table of a_n is given as below:

We define the ratio b_n as

$$b_n = \frac{a_{n+1}}{a_n} = \frac{T_{n+2}T_n}{T_{n+1}^2}$$

and the table of ratio b_n is calculated using values from Table 2 for $1 \leq n \leq 17$. The table of b_n is given as below:

We define the ratio c_n as

$$c_n = \frac{b_{n+1}}{b_n} = \frac{T_{n+3}T_{n+1}^3}{T_{n+2}^3 T_n}$$

and the table of ratio c_n is calculated using values from Table 3 for $1 \leq n \leq 16$. The table of c_n is given as below:

We define the ratio d_n as

$$d_n = \frac{c_{n+1}}{c_n} = \frac{T_{n+4}T_{n+2}^6 T_n}{T_{n+3}^4 T_{n+1}^4}$$

Table 1: Number of Topologies

A000798

n	T_n
0	1
1	1
2	4
3	29
4	355
5	6942
6	209527
7	9535241
8	642779354
9	63260289423
10	8977053873043
11	1816846038736190
12	519355571065774021
13	207881393656668953041
14	115617051977054267807460
15	88736269118586244492485121
16	93411113411710039565210494095
17	134137950093337880672321868725846
18	261492535743634374805066126901117203

and the table of ratio d_n is calculated using values from Table 4 for $1 \leq n \leq 15$. The table of d_n is given as below:

We define the ratio e_n as

$$e_n = \frac{d_{n+1}}{d_n} = \frac{T_{n+5}T_{n+3}^{10}T_{n+1}^{5}}{T_{n+4}^{5}T_{n+2}^{10}T_n}$$

and the table of ratio e_n is calculated using values from Table 5 for $1 \leq n \leq 14$.

Table 2: Ratio a_n

n	1	2	3	4	5	6	7	8	9
a_n	4	7.25	12.2414	19.5549	30.1825	45.5084	67.4109	98.4168	141.907

n	10	11	12	13	14	15	16	17
a_n	202.388	285.856	400.268	556.168	767.502	1052.68	1436	1949.43

Table 3: Ratio b_n

n	1	2	3	4	5	6	7	8
b_n	1.8125	1.68847	1.59744	1.54347	1.50777	1.48128	1.45995	1.44189

n	9	10	11	12	13	14	15	16
b_n	1.4262	1.41242	1.40025	1.38949	1.37998	1.37157	1.36413	1.35755

Table 4: Ratio c_n

n	1	2	3	4	5	6	7	8
c_n	0.931568	0.946092	0.966214	0.976871	0.982431	0.985599	0.98763	0.989118

n	9	10	11	12	13	14	15
c_n	0.990332	0.991384	0.992319	0.993156	0.993906	0.994575	0.995173

The table of e_n is given as below:

We define the ratio f_n as

$$f_n = \frac{e_{n+1}}{e_n} = \frac{T_{n+6}T_{n+4}^{15}T_{n+2}^{15}T_n}{T_{n+5}^{6}T_{n+3}^{20}T_{n+1}^{6}}$$

and the table of ratio f_n is calculated using values from Table 6 for $1 \leq n \leq 13$.

Table 5: Ratio d_n

n	1	2	3	4	5	6	7
d_n	1.01559	1.02127	1.01103	1.00569	1.00322	1.00206	1.00151

n	8	9	10	11	12	13	14
d_n	1.00123	1.00106	1.00094	1.00084	1.00075	1.00067	1.0006

Table 6: Ratio e_n

n	1	2	3	4	5	6	7
e_n	1.00559	0.989975	0.994721	0.997546	0.99884	0.999447	0.999721

n	8	9	10	11	12	13
e_n	0.999835	0.999881	0.9999	0.999911	0.999919	0.999928

Table 7: Ratio f_n

n	1	2	3	4	5	6
f_n	0.984473	1.00479	1.00284	1.0013	1.00061	1.00027

n	7	8	9	10	11	12
f_n	1.00011	1.00005	1.00002	1.00001	1.00001	1.00001

The table of f_n is given as below:

We define the ratio g_n as

$$g_n = \frac{f_{n+1}}{f_n} = \frac{T_{n+7}T_{n+5}^{21}T_{n+3}^{35}T_{n+1}^{7}}{T_{n+6}^{7}T_{n+4}^{35}T_{n+2}^{21}T_n}$$

and the table of ratio g_n is calculated using values from Table 7 for $1 \leq n \leq 12$. The table of g_n is given as below:

Table 8: Ratio g_n

n	1	2	3	4	5	6
g_n	1.02064	0.998056	0.99846	0.999313	0.999665	0.999841

n	7	8	9	10	11
g_n	0.999931	0.999973	0.999991	0.999998	1

Table 9: Ratio h_n

n	1	2	3	4	5
h_n	0.977872	1.0004	1.00085	1.00035	1.00018

n	6	7	8	9	10
h_n	1.00009	1.00004	1.00002	1.00001	1

We define the ratio h_n as

$$h_n = \frac{g_{n+1}}{g_n} = \frac{T_{n+8}T_{n+6}^{28}T_{n+4}^{70}T_{n+2}^{28}T_n}{T_{n+7}^{8}T_{n+5}^{56}T_{n+3}^{56}T_{n+1}^{8}}$$

and the table of ratio h_n is calculated using values from Table 8 for $1 \leq n \leq 11$. The table of h_n is given as below:

We define the ratio i_n as

$$i_n = \frac{h_{n+1}}{h_n} = \frac{T_{n+9}T_{n+7}^{36}T_{n+5}^{126}T_{n+3}^{84}T_{n+1}^{9}}{T_{n+8}^{9}T_{n+6}^{84}T_{n+4}^{126}T_{n+2}^{36}T_n}$$

and the table of ratio i_n is calculated using values from Table 9 for $1 \leq n \leq 10$. The table of i_n is given as below:

We noted the nature of repeated ratios from above tables and observed that for all $n \geq 2$, $b_n > 1$ and $C_n < 1$; $d_n > 1$ and $e_n < 1$ and so on. One can notice that the sequence of ratios $\{b_n, c_n, \dots\}$ oscillates around 1. Moreover, as n becomes larger, the ratios become more closer to 1. We can use these ratios for approximating the values of T_n. After approximating these repeated ratios to 1, we observe that b_n gives the lower bound for T_{n+2}, c_n gives the upper bound for T_{n+3} and so on. In general, if the value of the repeated ratio is greater than

Table 10: Ratio i_n

n	1	2	3	4	5	6	7	8	9
i_n	1.02304	1.00045	0.999499	0.999823	0.999913	0.999953	0.999976	0.999989	0.999995

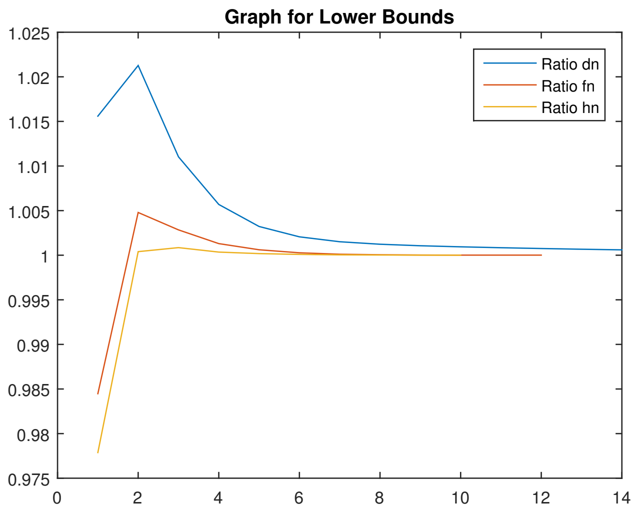

Figure 1: Lower Bounds.

1 then the ratio gives the lower bound for T_n and if the value of the repeated ratio is less than 1 then the ratio gives the upper bound for T_n. We have plotted the graphs of repeated ratios which gives the upper bounds and lower bounds using MATLAB R2015A.

After testing these repeated ratios for finding lower and upper bounds, we observed that h_n gives better approximation to T_{n+8} as a lower bound and i_n gives better approximation for T_{n+9} as a upper bound. From the ratio h_n and i_n, we calculate the approximate values of T_{n+8} and T_{n+9} respectively for $n \geq 0$. Thus, we have

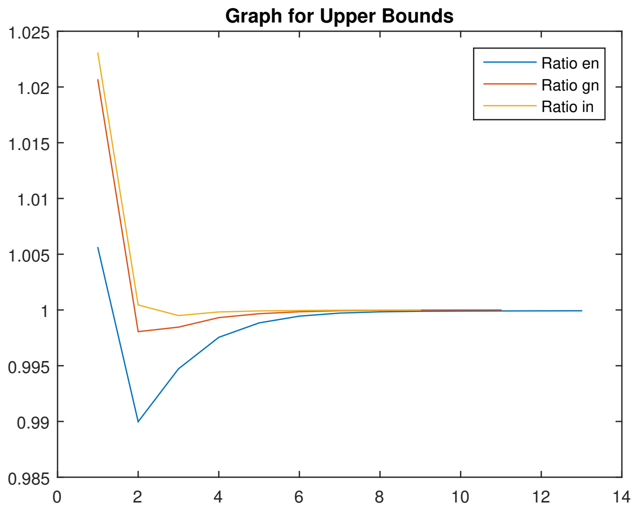

Figure 2: Upper bounds

$$x_{n+8} = \frac{T_{n+7}^{8} T_{n+5}^{56} T_{n+3}^{56} T_{n+1}^{8}}{T_{n+6}^{28} T_{n+4}^{70} T_{n+2}^{28} T_{n}}$$

which gives lower bound for the actual value of T_n and

$$y_{n+9} = \frac{T_{n+8}^{9} T_{n+6}^{84} T_{n+4}^{126} T_{n+2}^{36} T_{n}}{T_{n+7}^{36} T_{n+5}^{126} T_{n+3}^{84} T_{n+1}^{9}}$$

which gives upper bound for the actual value of T_n. i.e $x_n \leq T_n \leq y_n$. Calculation of x_{n+8} and y_{n+9} involves the numbers with almost 500 to 3000 digits in numerator and denominator. These calculations can not be handled with MATLAB. Hence, we have used MATHEMATICA to obtain the corresponding values of lower bound x_n and upper bound y_n.

Table 11 gives values of lower and upper bounds corresponding to T_n.

From above table, One can observe that the nature of bounds for first few terms is volatile. As n becomes larger, the bounds are more closer to T_n. Hence, we can give finest bound for T_n where n is large, by using above approximations. The following table gives the percentage error in lower and upper bounds.

One can observe from above table that the percentage error is very small. Hence, the bounds given in this paper are the finest bounds.

Table 11: Upper and lower bounds for T_n

Lower Bounds x_n	T_n	Upper Bounds y_n
8.9734×10^{12}	8977053873043	8.7748×10^{12}
1.181529×10^{15}	1816846038736190	1.18103×10^{15}
5.19172×10^{17}	5193555710657 74021	5.1961×10^{17}
2.07845×10^{20}	207881393656668953041	2.0791×10^{20}
1.15607×10^{23}	115617051977054267807460	1.1562×10^{23}
8.87325×10^{25}	88736269118586244492485121	8.87405×10^{25}
9.34094×10^{28}	93411113411710039565210494095	9.3413×10^{28}
1.34137×10^{32}	134137950093337880672321868725846	1.34139×10^{32}
2.61490×10^{35}	261492535743634374805066126901117203	2.61494×10^{35}

Table 12: Percentage error in lower and upper bounds

n	10	11	12	13	14	15	16	17	18
% Error in x_n	0.0404	0.085	0.0352	0.017	0.008	0.004	0.001	0.0006	0.0006
% Error in y_n	2.25	0.04	0.05	0.017	0.008	0.004	0.002	0.0011	0.0004

3. Conclusion

In this paper, we have investigated the bounds for the number of topologies on finite set. We have obtained the sharper lower and upper bounds for the number of topologies T_n on the set with n elements. These bounds can be used to approximate the number of topologies on the set with large number of elements.

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