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ON A TOPOLOGY BETWEEN THE TOPOLOGIES τ_θ AND $\tau_{\theta\text{-}\mathcal{I}}$

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Abstract: We introduce a new class of sets, namely $\delta\theta$-$\mathcal{I}$-open sets, which form a topology finer than the topology τ_θ formed by the class of θ-open sets and coarser than the topology $\tau_{\theta\text{-}\mathcal{I}}$ formed by the class of θ-$\mathcal{I}$-open sets. Moreover, we investigated some interesting properties of this class of sets and its relationship with the classes of the θ-open, θ-$\mathcal{I}$-open and δ-$\mathcal{I}$-open sets.

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1. Introduction

The notions of θ-open and δ-open sets introduced by Veličko [11] have served as motivation for a considerable amount of topologists to study generalizations and/or modifications of classical topological concepts. In particular, we can mention the investigations carried out in papers [1], [10] and [12], where studied the notions of θ-$\mathcal{I}$-open and δ-$\mathcal{I}$-open sets using the theory of ideals in topological spaces due to Kuratowski [7]. Although the theory of ideals in topological spaces has been extensively studied, in recent years there have been some studies which have investigated the properties of new variants of the concept of local function, such as a-local function and δ-local function (see [2], [3] and [5]). Following this line of research, in this paper we use the concept of δ-local function to introduce a new class of sets, namely $\delta\theta$-$\mathcal{I}$-open sets, which form a topology which is between the topologies τ_θ and $\tau_{\theta\text{-}\mathcal{I}}$ formed by the classes of θ-open and θ-$\mathcal{I}$-open sets, respectively. Moreover, we investigate some interesting properties of this class of sets and its relationship with the classes of the θ-open, θ-$\mathcal{I}$-open and δ-$\mathcal{I}$-open sets.

2. Preliminaries

Throughout this paper, (X,τ) always means a topological space on which no separation axioms are assumed unless explicitly stated. If A is a subset of X, we denote the closure of A and the interior of A by $Cl(A)$ and $Int(A)$, respectively. A point $x \in X$ is called a *δ-cluster* (resp. *θ-cluster*) *point* of A if $Int(Cl(U)) \cap A \neq \emptyset$ (resp. $Cl(U) \cap A \neq \emptyset$) for each open set U containing x (see [11]). The set of all δ-cluster (resp. θ-cluster) points of A is called the *δ-closure* (resp. *θ-closure*) of A and is denoted by $\delta Cl(A)$ (resp. $Cl_\theta(A)$). A subset A of X is said to be *δ-closed* (resp. *θ-closed*) if $A = \delta Cl(A)$ (resp. $A = Cl_\theta(A)$). The complement of a δ-closed (resp. θ-closed) set is said to be a *δ-open* (resp. *θ-open*) set. Similarly, the θ-interior of a set A in X, written $Int_\theta(A)$, consists of those points x of X such that for some open set U containing x, $Cl(U) \subset A$. It is well known that a subset A of X is θ-open if and only if $A = Int_\theta(A)$. It follows from [11] that the collection of all δ-open (resp. θ-open) sets in a topological space (X,τ) forms a topology on X which is denoted by τ_δ (resp. τ_θ). From the definitions it follows that $\tau_\theta \subset \tau_\delta \subset \tau$. The topology τ_δ is called the *semi-regularization* of τ. Observe that δCl is the closure with respect to τ_δ, but Cl_θ is not the closure of A with respect to τ_θ.

An *ideal* $\mathcal{I}$ on a nonempty set X is a nonempty collection of subsets of X which satisfies the following two properties: (1) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$; (2) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. A topological space (X, τ) with an ideal $\mathcal{I}$ on X is called an *ideal topological space* and is denoted by $(X, \tau, \mathcal{I})$. Given an ideal topological space $(X, \tau, \mathcal{I})$, a set operator $(.)^\star : P(X) \to P(X)$, called the *local function* [7] of A with respect to τ and $\mathcal{I}$, is defined as follows: for $A \subset X$, $A^\star(\mathcal{I}, \tau) = \{x \in X : U \cap A \notin \mathcal{I}$ for every $U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau : x \in U\}$. When there is no chance for confusion, we will simply write $A^\star$ for $A^\star(\mathcal{I}, \tau)$. In general, $X^\star$ is a proper subset of X. The hypothesis $X = X^\star$ is equivalent to the hypothesis $\tau \cap \mathcal{I} = \emptyset$. According to [8], we call the ideals which satisfy this condition τ-*boundary* ideals. Note that $Cl^\star(A) = A \cup A^\star$ defines a Kuratowski closure for a topology $\tau^\star(\mathcal{I})$, finer than τ. A basis $\beta(\mathcal{I}, \tau)$ for $\tau^\star(\mathcal{I})$ can be described as follows: $\beta(\mathcal{I}, \tau) = \{V - J : V \in \tau$ and $J \in \mathcal{I}\}$ [6]. When there is no chance for confusion, we will simply write $\tau^\star$ for $\tau^\star(\mathcal{I})$. The elements of $\tau^\star$ are called $\tau^\star$-*open* and the complement of a $\tau^\star$-open is called $\tau^\star$-*closed*. It is well known that a subset A of an ideal topological space (X, τ, I) is $\tau^\star$-closed if and only if $A^\star \subset A$ [6]. A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be *R-$\mathcal{I}$-open* if $A = Int(Cl^\star(A))$ [12]. The complement of a R-$\mathcal{I}$-open set is called *R-$\mathcal{I}$-closed* and the collection of all R-$\mathcal{I}$-open sets in $(X, \tau, \mathcal{I})$ is denoted by $R\mathcal{I}O(X, \tau)$. A point x in an ideal topological space $(X, \tau, \mathcal{I})$ is called a *δ-$\mathcal{I}$-cluster* (resp. *θ-$\mathcal{I}$-cluster*) *point* of A if $Int(Cl^\star(U)) \cap A \neq \emptyset$ (resp. $Cl^\star(U) \cap A \neq \emptyset$) for each $U \in \tau(x)$. The set of all δ-$\mathcal{I}$-cluster (resp. θ-$\mathcal{I}$-cluster) points of A is called the *δ-$\mathcal{I}$-closure* (resp. *θ-$\mathcal{I}$-closure*) of A and is denoted by $\delta_{\mathcal{I}}Cl(A)$ (resp. $Cl^\star_\theta(A)$). A subset A is said to be *δ-$\mathcal{I}$-closed* [12] (resp. *θ-$\mathcal{I}$-closed* [1]) if $A = \delta_{\mathcal{I}}Cl(A)$ (resp. $A = Cl^\star_\theta(A)$). The complement of a δ-$\mathcal{I}$-closed (resp. θ-$\mathcal{I}$-closed) set is said to be a *δ-$\mathcal{I}$-open* (resp. *θ-$\mathcal{I}$-open*) set. It follows from [12, Theorem 2.1] (resp. [1, Theorem 1]) that the collection of all δ-$\mathcal{I}$-open (resp. θ-$\mathcal{I}$-open) sets in an ideal topological space $(X, \tau, \mathcal{I})$ forms a topology on X which is denoted by $\tau_{\delta\text{-}\mathcal{I}}$ (resp. $\tau_{\theta\text{-}\mathcal{I}}$). It is well-known, by [1, Corollary 4], that $\tau_\theta \subset \tau_{\theta\text{-}\mathcal{I}} \subset \tau_{\delta\text{-}\mathcal{I}} \subset \tau$, and by [12, Theorem 2.1], that $\tau_\theta \subset \tau_\delta \subset \tau_{\delta\text{-}\mathcal{I}} \subset \tau$. For a subset A of an ideal topological space $(X, \tau, \mathcal{I})$, the *δ-local function* of A with respect to $\mathcal{I}$ is defined in [5] as $A^{\delta\star}(\mathcal{I}, \tau) = \{x \in X : U \cap A \notin \mathcal{I}$ for every $U \in \tau_\delta(x)\}$, where $\tau_\delta(x) = \{U \in \tau_\delta : x \in U\}$. We will simply write $A^{\delta\star}$ for $A^{\delta\star}(\mathcal{I}, \tau)$. A Kuratowski closure operator $\delta Cl^\star(.)$ for a topology $\tau^{\delta\star}$, is defined by $\delta Cl^\star(A) = A \cup A^{\delta\star}$ [5]. The topology $\tau^{\delta\star}$ finer than τ_δ and $\beta(\mathcal{I}, \tau_\delta) = \{V - J : V \in \tau_\delta$ and $J \in \mathcal{I}\}$

is a basis for $\tau^{\delta\star}$. A point $x \in X$ is called a *$\delta_\star$-$\mathcal{I}$-cluster point* of A if $Int(\delta Cl^\star(U)) \cap A \neq \emptyset$ for each $U \in \tau(x)$. The set of all $\delta_\star$-$\mathcal{I}$-cluster points of A is called the *$\delta_\star$-$\mathcal{I}$-closure* of A and is denoted by $\delta_\star Cl(A)$. A subset A is said to be *$\delta_\star$-$\mathcal{I}$-closed* [5] if $A = \delta_\star Cl(A)$. The complement of a $\delta_\star$-$\mathcal{I}$-closed set is said to be a *$\delta_\star$-$\mathcal{I}$-open* set. If $\delta\tau^\star$ is the collection of all $\delta_\star$-$\mathcal{I}$-open sets of $(X, \tau, \mathcal{I})$, then $\delta\tau^\star$ is a topology on X such that $\tau_\delta \subset \delta\tau^\star \subset \tau$ (see [5, Theorem 12]).

Recall that a subset A of a space (X, τ) is called semiopen (resp. preopen) if $A \subset Cl(Int(A))$ (resp. $A \subset Int(Cl(A))$).

Lemma 2.1. *[9, Lemma 1.4]* If A is a semiopen subset of a space (X, τ), then $Cl(A) = \delta Cl(A)$.

Lemma 2.2. *[4, Proposition 3.6]* If A is a preopen subset of a space (X, τ), then $Cl(A) = Cl_\theta(A)$

3. Further Properties of $A^{\delta\star}(\mathcal{I}, \tau)$

In [5, Example 2] it was established that $A^\star$ and $A^{\delta\star}$ are independent, but this example is not true, since $\mathcal{I} = \{\emptyset, \{a, c\}\}$ is not an ideal on $X = \{a, b, c, d\}$. The next result gives the precise relationships between $A^\star$ and $A^{\delta\star}$.

Lemma 3.1. If A is any subset of an ideal topological space $(X, \tau, \mathcal{I})$, then the following properties hold:

(1) $A^\star \subset A^{\delta\star}$,

(2) $Cl^\star(A) \subset \delta Cl^\star(A) \subset \delta Cl(A)$.

Proof. (1) If $x \notin A^{\delta\star}$, then there exists $U \in \tau_\delta(x)$ such that $U \cap A \in \mathcal{I}$. Since $\tau_\delta(x) \subset \tau(x)$, $x \notin A^\star$. Therefore, $A^\star \subset A^{\delta\star}$.
(2) It follows from part (1) and [5, Theorem 1]. □

Remark 3.1. From Lemma 3.1 it follows that $\tau^{\delta\star} \subset \tau^\star$. Also, if $\mathcal{I}$ is τ-boundary, then $X = X^\star \subset X^{\delta\star}$, which implies that $X^{\delta\star} = X$. Hence, if $\mathcal{I}$ is an ideal τ-boundary then one of the equivalent conditions (1)-(4) of [5, Theorem 9] holds.

Theorem 3.1. If $\mathcal{N}$ is the ideal of all nowhere dense sets in a topological space (X, τ), then $A^{\delta\star}(\mathcal{N}, \tau) = A^\star(\mathcal{N}, \tau) = Cl(Int(Cl(A)))$ for each subset A of X.

Proof. Suppose that $x \notin A^\star(\mathcal{N},\tau) = Cl(Int(Cl(A)))$. By Lemma 2.1, $x \notin \delta Cl(Int(Cl(A)))$ and hence there exists $U \in \tau_\delta(x)$ such that $U \cap Int(Cl(A)) = \emptyset$. Thus,

$$\begin{aligned} U \cap A &\subset U \cap Cl(A) \\ &= U \cap [X - Int(Cl(A))] \cap Cl(A) \\ &= U \cap Fr(Cl(A)) \\ &\subset Fr(Cl(A)). \end{aligned}$$

Since the frontier of a closed set is nowhere dense, then $U \cap A \in \mathcal{N}$. Hence $x \notin A^{\delta\star}(\mathcal{N},\tau)$ and so $A^{\delta\star}(\mathcal{N},\tau) \subset A^\star(\mathcal{N},\tau) = Cl(Int(Cl(A)))$. By Lemma 3.1, we obtain $A^{\delta\star}(\mathcal{N},\tau) = A^\star(\mathcal{N},\tau) = Cl(Int(Cl(A)))$. □

4. $\delta\theta$-$\mathcal{I}$-Open Sets

Definition 4.1. Let $(X,\tau,\mathcal{I})$ be an ideal topological space and A be a subset of X.

(1) A point $x \in X$ is called a *$\delta\theta$-$\mathcal{I}$-cluster point* of A if $\delta Cl^\star(U) \cap A \neq \emptyset$ for every open set U of X containing x.

(2) The set of all $\delta\theta$-$\mathcal{I}$-cluster points of A is called the *$\delta\theta$-$\mathcal{I}$-closure* of A and is denoted by $\delta Cl^\star_\theta(A)$.

(3) A subset A is said to be *$\delta\theta$-$\mathcal{I}$-closed* if $\delta Cl^\star_\theta(A) = A$. The complement of a $\delta\theta$-$\mathcal{I}$-closed set is said to be *$\delta\theta$-$\mathcal{I}$-open.*

(4) A point $x \in X$ is called a *$\delta\theta$-$\mathcal{I}$-interior point* of a subset A if there exists an open set U such that $x \in U \subset \delta\mathrm{Cl}^\star(U) \subset A$.

(5) The set of all $\delta\theta$-$\mathcal{I}$-interior points of A is called the *$\delta\theta$-$\mathcal{I}$-interior* of A and is denoted by $\delta Int^\star_\theta(A)$.

Theorem 4.1. Let $(X,\tau,\mathcal{I})$ be an ideal topological space and A be a subset of X. Then, the following properties hold:

(1) $Cl(A) \subset \delta_{\mathcal{I}}Cl(A) \subset \delta_\star Cl(A) \subset \delta Cl^\star_\theta(A) \subset Cl_\theta(A)$,

(2) $Cl(A) \subset \delta_{\mathcal{I}}Cl(A) \subset Cl^\star_\theta(A) \subset \delta Cl^\star_\theta(A) \subset Cl_\theta(A)$.

Proof. From [1, Corollary 4] it follows that $Cl(A) \subset \delta_{\mathcal{I}}Cl(A) \subset Cl^{\star}_{\theta}(A)$.
(1) First, suppose that $x \notin Cl_{\theta}(A)$, then there exists $U \in \tau(x)$ such that $Cl(U) \cap A = \emptyset$. By Lemma 2.1, $\delta Cl(U) \cap A = Cl(U) \cap A = \emptyset$ and by Lemma 3.1, $\delta Cl^{\star}(U) \cap A \subset \delta Cl(U) \cap A = \emptyset$. Then, $\delta Cl^{\star}(U) \cap A = \emptyset$ and so $x \notin \delta Cl^{\star}_{\theta}(A)$. Hence $\delta Cl^{\star}_{\theta}(A) \subset Cl_{\theta}(A)$. Next, if $x \notin \delta Cl^{\star}_{\theta}(A)$, then there exists $U \in \tau(x)$ such that $\delta Cl^{\star}(U) \cap A = \emptyset$. Since $Int(\delta Cl^{\star}(U)) \cap A \subset \delta Cl^{\star}(U) \cap A = \emptyset$, then $Int(\delta Cl^{\star}(U)) \cap A = \emptyset$ and hence $x \notin \delta_{\star}Cl(A)$. Thus, $\delta_{\star}Cl(A) \subset \delta Cl^{\star}_{\theta}(A)$. Now, suppose that $x \notin \delta_{\star}Cl(A)$, then there exists $U \in \tau(x)$ such that $Int(\delta Cl^{\star}(U)) \cap A = \emptyset$. By Lemma 3.1, $Int(Cl^{\star}(U)) \cap A \subset Int(\delta Cl^{\star}(U)) \cap A = \emptyset$ and hence $Int(Cl^{\star}(U)) \cap A = \emptyset$, which implies that $x \notin \delta_{\mathcal{I}}Cl(A)$. Therefore $\delta_{\mathcal{I}}Cl(A) \subset \delta_{\star}Cl(A)$.
(2) We only need to prove that $Cl^{\star}_{\theta}(A) \subset \delta Cl^{\star}_{\theta}(A)$. Suppose that $x \notin \delta Cl^{\star}_{\theta}(A)$, then there exists $U \in \tau(x)$ such that $\delta Cl^{\star}(U) \cap A = \emptyset$. By Lemma 3.1, $Cl^{\star}(U) \cap A \subset \delta Cl^{\star}(U) \cap A = \emptyset$ and hence $Cl^{\star}(U) \cap A = \emptyset$. Thus, $x \notin Cl^{\star}_{\theta}(A)$ and $Cl^{\star}_{\theta}(A) \subset \delta Cl^{\star}_{\theta}(A)$. □

Remark 4.1. From Theorem 4.1 and preliminaries, we obtain the following diagram:

$$\begin{tikzcd}
 & \theta\text{-closed} \arrow[dl, "(2)"'] \arrow[d, "(6)"] \arrow[dr, "(8)"] & \\
\delta\theta\text{-}\mathcal{I}\text{-closed} \arrow[r, "(3)"] \arrow[d, "(1)"] \arrow[dr, "(4)"] & \delta_{\star}\text{-}\mathcal{I}\text{-closed} \arrow[d, "(7)"] & \delta\text{-closed} \arrow[l, "(9)"'] \arrow[dl, "(10)"'] \arrow[d, "(12)"] \\
\theta\text{-}\mathcal{I}\text{-closed} \arrow[r, "(5)"] & \delta\text{-}\mathcal{I}\text{-closed} \arrow[r, "(11)"] & \text{closed}
\end{tikzcd}$$

It is well known that none of the implications (5)-(12) reverses (see [1], [5], [11] and [12]). The following two examples shows that, in general, the implications (1)-(4) are not reversed.

Example 4.1. A θ-$\mathcal{I}$-closed set need not be $\delta\theta$-$\mathcal{I}$-closed and a $\delta\theta$-$\mathcal{I}$-closed set need not be θ-closed.
Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}\}$ and $\mathcal{I} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. If $A = \{c\}$ and $B = \{c, d\}$, then $Cl^{\star}_{\theta}(A) = \{c\}$, $\delta Cl^{\star}_{\theta}(A) = \{c, d\}$, $Cl_{\theta}(B) = X$ and $\delta Cl^{\star}_{\theta}(B) = \{c, d\}$. Therefore, $A = \{c\}$ is θ-$\mathcal{I}$-closed but is not $\delta\theta$-$\mathcal{I}$-closed and $B = \{c, d\}$ is $\delta\theta$-$\mathcal{I}$-closed but is not θ-closed.

Example 4.2. A $\delta_{\star}$-$\mathcal{I}$-closed set need not be $\delta\theta$-$\mathcal{I}$-closed.

Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$ and $\mathcal{I} = \{\emptyset, \{b\}\}$. If $A = \{d\}$, then $\delta_\star\mathrm{Cl}(A) = \{d\}$ and $\delta Cl^\star_\theta(A) = X$. Therefore, $A = \{d\}$ is $\delta_\star$-$\mathcal{I}$-closed but is not $\delta\theta$-$\mathcal{I}$-closed.

Proposition 4.1. Let $(X, \tau, \mathcal{I})$ be an ideal topological space and A be a subset of X. Then, the following properties hold:

(1) $\delta Int^\star_\theta(X - A) = X - \delta Cl^\star_\theta(A)$,

(2) A is $\delta\theta$-$\mathcal{I}$-open if and only if $A = \delta Int^\star_\theta(A)$.

Theorem 4.2. Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $\tau_{\delta\theta\text{-}\mathcal{I}} = \{A \subset X : A \text{ is a } \delta\theta\text{-}\mathcal{I}\text{-open set of } (X, \tau, \mathcal{I})\}$. Then, $\tau_{\delta\theta\text{-}\mathcal{I}}$ is a topology and $\tau_\theta \subset \tau_{\delta\theta\text{-}\mathcal{I}} \subset \tau_{\theta\text{-}\mathcal{I}} \subset \tau_{\delta\text{-}\mathcal{I}} \subset \tau$.

Proof. From Remark 4.1, it follows that $\tau_\theta \subset \tau_{\delta\theta\text{-}\mathcal{I}} \subset \tau_{\theta\text{-}\mathcal{I}} \subset \tau_{\delta\text{-}\mathcal{I}} \subset \tau$. Next, we show that $\tau_{\delta\theta\text{-}\mathcal{I}}$ is a topology.
(1) Clearly $\emptyset \in \tau_{\delta\theta\text{-}\mathcal{I}}$. If $x \in X$, then X is an open set such that $x \in X = \delta Cl^\star(X) \subset X$ and so, $x \in \delta Int^\star_\theta(X)$. Therefore, $X \in \tau_{\delta\theta\text{-}\mathcal{I}}$.
(2) Let $A = \bigcup_{\lambda\in\Lambda} A_\lambda$, with $A_\lambda \in \tau_{\delta\theta\text{-}\mathcal{I}}$ for each $\lambda \in \Lambda$. If $x \in A$, then $x \in A_\lambda$ for some $\lambda \in \Lambda$ and so, there exists $U \in \tau$ such that $x \in U \subset \delta Cl^\star(U) \subset A_\lambda \subset A$. Hence $x \in \delta Int^\star_\theta(A)$ and $A \in \tau_{\delta\theta\text{-}\mathcal{I}}$.
(3) Suppose $A, B \in \tau_{\delta\theta\text{-}\mathcal{I}}$. If $x \in A \cap B$, there exist $U, V \in \tau$ such that $x \in U \subset \delta Cl^\star(U) \subset A$ and $x \in V \subset \delta Cl^\star(V) \subset B$. Then $U \cap V$ is an open set such that $x \in U \cap V \subset \delta Cl^\star(U \cap V) \subset \delta Cl^\star(U) \cap \delta Cl^\star(V) \subset A \cap B$ and so $x \in \delta Int^\star_\theta(A \cap B)$. Hence $A \cap B \in \tau_{\delta\theta\text{-}\mathcal{I}}$. □

Corollary 4.1. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then, the following properties hold:

(1) $\emptyset$ and X are $\delta\theta$-$\mathcal{I}$-closed sets,

(2) Any intersection of $\delta\theta$-$\mathcal{I}$-closed sets is a $\delta\theta$-$\mathcal{I}$-closed set,

(3) Any finite union of $\delta\theta$-$\mathcal{I}$-closed sets is a $\delta\theta$-$\mathcal{I}$-closed set.

Theorem 4.3. A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is $\delta\theta$-$\mathcal{I}$-open if and only if for each $x \in A$, there exists a R-$\mathcal{I}$-open set V such that $x \in V \subset \delta Cl^\star(V) \subset A$.

Proof. Suppose that A is a $\delta\theta$-$\mathcal{I}$-open subset of $(X, \tau, \mathcal{I})$ and let x be any point of A. Then, $A = \delta Int^\star_\theta(A)$ and hence, there exists an open set U such that $x \in U \subset \delta Cl^\star(U) \subset A$. By Lemma 3.1, $x \in U = Int(U) \subset Int(Cl^\star(U)) \subset Cl^\star(Int(Cl^\star(U))) \subset \delta Cl^\star(Int(Cl^\star(U))) \subset A$ and so, the set $V = Int(Cl^\star(U))$ is a R-$\mathcal{I}$-open set V such that $x \in V \subset \delta Cl^\star(V) \subset A$. The converse is obvious. $\square$

Remark 4.2. In Example 4.1, we note that $\delta Cl^\star_\theta(\delta Cl^\star_\theta(A)) = \{c,d\} \neq \{c\} = A$, i.e., $\delta Cl^\star_\theta$ is not idempotent. Hence, $\delta Cl^\star_\theta$ is not a Kuratowski closure operator. However, it satisfies properties given in the following lemma.

Lemma 4.1. Let A and B be subsets of an ideal topological space $(X, \tau, \mathcal{I})$. Then, the following properties hold:

(1) $A \subset \delta Cl^\star_\theta(A)$,

(2) If $A \subset B$, then $\delta Cl^\star_\theta(A) \subset \delta Cl^\star_\theta(B)$,

(3) $\delta Cl^\star_\theta(A) = \bigcap\{Cl(W) : A \subset W \text{ and } W \in \tau^{\delta\star}\}$,

(4) $\delta Cl^\star_\theta(A)$ is closed,

(5) If $U \in \tau_\delta$, then $Cl(U) = \delta Cl^\star_\theta(U)$.

(6) $\delta Cl^\star_\theta(A \cup B) = \delta Cl^\star_\theta(A) \cup \delta Cl^\star_\theta(B)$,

Proof. The proofs of the parts (1) and (2) are obvious.
(3) Suppose that $x \notin \delta Cl^\star_\theta(A)$, then there exists $U \in \tau(x)$ such that $\delta Cl^\star(U) \cap A = \emptyset$. Put $V = X - \delta Cl^\star(U) \in \tau^{\delta\star}$, then $V \in \tau^{\delta\star}$, $A \subset V$ and $x \notin V$. Since $U \cap V = \emptyset$, we have $x \notin Cl(V)$ and hence $x \notin \bigcap\{Cl(W) : A \subset W \text{ and } W \in \tau^{\delta\star}\}$. Conversely, let $x \in \delta Cl^\star_\theta(A)$. Then, for each $U \in \tau(x)$ we have $\delta Cl^\star(U) \cap A \neq \emptyset$. Hence, if $V \in \tau^{\delta\star}$ and $x \in V$, then $V \cap \delta Cl^\star(U) \neq \emptyset$. By [5, Theorem 1], $V \cap \delta Cl^\star(U) = V \cap (U \cup U^{\delta\star}) = (V \cap U) \cup (V \cap U^{\delta\star}) \subset (V \cap U) \cup (V \cap U)^{\delta\star} = \delta Cl^\star(V \cap U)$, which implies that $\delta Cl^\star(V \cap U) \neq \emptyset$ and so $V \cap U \neq \emptyset$. Consequently, $x \in Cl(V)$ and hence $\delta Cl^\star_\theta(A) \subset \bigcap\{Cl(W) : A \subset W \text{ and } W \in \tau^{\delta\star}\}$. This proves the equality $\delta Cl^\star_\theta(A) = \bigcap\{Cl(W) : A \subset W \text{ and } W \in \tau^{\delta\star}\}$.
(4) It follows from (3).
(5) By Theorem 4.1, $Cl(U) \subset \delta Cl^\star_\theta(U)$. Suppose that $x \notin Cl(U)$, then there exists $V \in \tau(x)$ such that $V \cap U = \emptyset$. Therefore, $V \subset X - U$ and $\delta Cl^\star(V) \subset \delta Cl^\star(X - U) = X - U$. Thus, there exists $V \in \tau(x)$ such that $\delta Cl^\star(V) \cap U = \emptyset$ and hence $x \notin \delta Cl^\star_\theta(U)$.
(6) By (2), $\delta Cl^\star_\theta(A) \subset \delta Cl^\star_\theta(A \cup B)$ and $\delta Cl^\star_\theta(B) \subset \delta Cl^\star_\theta(A \cup B)$, which implies that $\delta Cl^\star_\theta(A) \cup \delta Cl^\star_\theta(B) \subset \delta Cl^\star_\theta(A \cup B)$. On the other hand, if $x \in \delta Cl^\star_\theta(A \cup B)$, then $\delta Cl^\star(U) \cap (A \cup B) \neq \emptyset$ for every $U \in \tau(x)$ and hence either $\delta Cl^\star(U) \cap A \neq \emptyset$ or $\delta Cl^\star(U) \cap B \neq \emptyset$ for every $U \in \tau(x)$. Thus, $x \in \delta Cl^\star_\theta(A)$ or $x \in \delta Cl^\star_\theta(B)$, which implies that $x \in \delta Cl^\star_\theta(A) \cup \delta Cl^\star_\theta(B)$ and hence $\delta Cl^\star_\theta(A \cup B) \subset \delta Cl^\star_\theta(A) \cup$

$\delta Cl^\star_\theta(B)$. This shows the equality $\delta Cl^\star_\theta(A \cup B) = \delta Cl^\star_\theta(A) \cup \delta Cl^\star_\theta(B)$. □

In the following results, we study some conditions on the ideal $\mathcal{I}$, in order to obtain the equality of $\tau_{\delta\theta\text{-}\mathcal{I}}$ with another topologies described above.

Proposition 4.2. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If $\mathcal{I} = P(X)$, then $\tau_{\delta\theta\text{-}\mathcal{I}} = \tau_{\theta\text{-}\mathcal{I}} = \tau_{\delta\text{-}\mathcal{I}} = \tau$.

Proof. Suppose that $I = P(X)$. Then $\delta Cl^\star(S) = S \cup S^{\delta\star} = S \cup \emptyset = S$, for each subset S of X. Let A be any open set and $x \in A$, then there exists $U \in \tau(x)$ such that $\delta Cl^\star(U) = U \subset A$, which implies $x \in \delta Int^\star_\theta(A)$. Therefore, $A = \delta Int^\star_\theta(A)$ and so A is $\delta\theta$-$\mathcal{I}$-open. This shows that $\tau \subset \tau_{\delta\theta\text{-}\mathcal{I}}$. By Theorem 4.2, $\tau_{\delta\theta\text{-}\mathcal{I}} = \tau_{\theta\text{-}\mathcal{I}} = \tau_{\delta\text{-}\mathcal{I}} = \tau$. □

Proposition 4.3. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If $\mathcal{I}$ is τ-boundary, then $\tau_{\delta\theta\text{-}\mathcal{I}} = \tau_{\theta\text{-}\mathcal{I}} = \tau_\theta$.

Proof. Suppose that $\mathcal{I}$ is τ-boundary, then by [10, Theorem 3.6], we have $\tau_{\theta\text{-}\mathcal{I}} = \tau_\theta$. Now, by Theorem 4.2, $\tau_{\delta\theta\text{-}\mathcal{I}} = \tau_{\theta\text{-}\mathcal{I}} = \tau_\theta$. □

Corollary 4.2. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If $\mathcal{I} = \{\emptyset\}$ or $\mathcal{I} = \mathcal{N}$ the ideal of all nowhere dense subsets of (X, τ), then $\tau_{\delta\theta\text{-}\mathcal{I}} = \tau_{\theta\text{-}\mathcal{I}} = \tau_\theta$.

The following example shows that the converse of Propositions 4.2 and 4.3 and Corollary 4.2, in general, are not true.

Example 4.3. Let (X, τ) be any discrete space with at least two points and $\mathcal{I}$ be any ideal such that $\mathcal{I} \neq \{\emptyset\}$ and $X \notin \mathcal{I}$. Then $\tau_\theta = \tau_{\delta\theta\text{-}\mathcal{I}} = \tau_{\theta\text{-}\mathcal{I}} = \tau_{\delta\text{-}\mathcal{I}} = \tau$, but $\mathcal{I} \neq P(X)$ and $\mathcal{I}$ is not τ-boundary.

Definition 4.2. [12] An ideal topological space $(X, \tau, \mathcal{I})$ is said to be an *$A\mathcal{I}$-R space* if for each $x \in X$ and each R-$\mathcal{I}$-closed set F not containing x, there exist disjoint open sets U and V such that $x \in U$ and $F \subset V$.

Theorem 4.4. *[12, Theorem 4.3]* An ideal topological space $(X, \tau, \mathcal{I})$ is an $A\mathcal{I}$-R space if and only if for each $x \in X$ and each R-$\mathcal{I}$-open set V containing x, there exists an R-$\mathcal{I}$-open set U such that $x \in U \subset Cl^\star(U) \subset Cl(U) \subset V$.

Theorem 4.5. An ideal topological space $(X, \tau, \mathcal{I})$ is an $A\mathcal{I}$-R space if and only if for each $x \in X$ and each R-$\mathcal{I}$-open set V containing x, there exists an R-$\mathcal{I}$-open set U such that $x \in U \subset Cl^\star(U) \subset \delta Cl^\star(U) \subset \delta Cl(U) \subset V$.

Proof. Suppose that $(X, \tau, \mathcal{I})$ is an $A\mathcal{I}$-R space. Let x be any point of X and V be any R-$\mathcal{I}$-open set containing x. By Theorem 4.4, there exists an R-$\mathcal{I}$-open set U such that $x \in U \subset Cl^\star(U) \subset Cl(U) \subset V$ Since each R-$\mathcal{I}$-open set is open, from Lemmas 2.1 and 3.1, it follows that $x \in U \subset Cl^\star(U) \subset \delta Cl^\star(U) \subset \delta Cl(U) \subset V$. The converse is obvious by Theorem 4.4. □

Corollary 4.3. If $(X, \tau, \mathcal{I})$ is an $A\mathcal{I}$-R space, then each R-$\mathcal{I}$-open subset of X is a $\delta\theta$-$\mathcal{I}$-open set.

Proof. Let V be any R-$\mathcal{I}$-open subset of X. By Theorem 4.5, for each $x \in V$ there exists a R-$\mathcal{I}$-open set U such that $x \in U \subset \delta Cl^\star(U) \subset V$. Now, by Theorem 4.3, we have V is a $\delta\theta$-$\mathcal{I}$-open set. □

Corollary 4.4. *[1, Corollary 5]* If $(X, \tau, \mathcal{I})$ is an $A\mathcal{I}$-R space, then each R-$\mathcal{I}$-open subset of X is a θ-$\mathcal{I}$-open set.

Proof. It follows from Corollary 4.3 and the fact that each $\delta\theta$-$\mathcal{I}$-open set is θ-$\mathcal{I}$-open. □

Corollary 4.5. If $(X, \tau, \mathcal{I})$ is an $A\mathcal{I}$-R space, then $\tau_{\delta\theta\text{-}\mathcal{I}} = \tau_{\theta\text{-}\mathcal{I}} = \tau_{\delta\text{-}\mathcal{I}}$.

Proof. It follows from [12, Lemma 2.1] and Corollary 4.3. □

The following example shows that the converse of Corollaries 4.3, 4.4 and 4.5, in general, are not true.

Example 4.4. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{a, b\}\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then $\tau_{\delta\theta\text{-}\mathcal{I}} = \tau_{\theta\text{-}\mathcal{I}} = \tau_{\delta\text{-}\mathcal{I}} = R\mathcal{I}O(X, \tau) = \{\emptyset, X, \{a\}\}$. Observe that $(X, \tau, \mathcal{I})$ is not an $A\mathcal{I}$-R space, because the set $F = \{b, c\}$ and the point a can not be separated by disjoint open sets.

The following example shows that the notion of $\delta\theta$-$\mathcal{I}$-open set are not preserved under open functions.

Example 4.5. Consider $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{a, b\}\}$, $\sigma = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$, $\mathcal{I} = \{\emptyset, \{a\}\}$ and $\mathcal{J} = \{\emptyset, \{c\}\}$. The collection of $\delta\theta$-$\mathcal{I}$-open sets of (X, τ, I) is $\tau_{\delta\theta\text{-}\mathcal{I}} = \{\emptyset, X, \{a\}\}$ and the collection of $\delta\theta$-$\mathcal{J}$-open sets of (X, σ, J) is $\sigma_{\delta\theta\text{-}\mathcal{J}} = \{\{\emptyset\}, X\}$. The identity function $f : (X, \tau, I) \to (X, \sigma, J)$ is an open function and $f(\{a\}) = \{a\}$ is not a $\delta\theta$-$\mathcal{J}$-open set.

Now, if f is an open function and satisfies an additional condition we obtain that f preserves $\delta\theta$-$\mathcal{I}$-open sets.

Theorem 4.6. If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is an open function such that $f(A^{\delta\star}) = (f(A))^{\delta\star}$ for each $A \in \tau$, then f preserves $\delta\theta$-$\mathcal{I}$-open sets.

Proof. Suppose that V is a $\delta\theta$-$\mathcal{I}$-open set in X and let $y \in f(V)$. Then there exists $x \in V$ such that $y = f(x)$ and there exists an open set U_x in X such that $x \in U_x \subset \delta Cl^\star(U_x) \subset V$. Thus, $y = f(x) \in f(U_x) \subset f(\delta Cl^\star(U_x)) \subset f(V)$. Since $\delta Cl^\star(f(U_x)) = f(\delta Cl^\star(U_x))$, we have $y = f(x) \in f(U_x) \subset \delta Cl^\star(f(U_x)) \subset f(V)$. Now, the fact that f is open implies that $f(U_x) \in \sigma$, it follows that $y \in \delta Int_\theta^\star(f(V))$. Hence $f(V) \subset \delta Int_\theta^\star(f(V))$ and so $f(V)$ is a $\delta\theta$-$\mathcal{J}$-open set. □

Theorem 4.7. If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is a continuous function such that $f^{-1}(B^{\delta\star}) = (f^{-1}(B))^{\delta\star}$ for each $B \in \sigma$, then f inversely preserves $\delta\theta$-$\mathcal{I}$-open sets.

Proof. Suppose that W is a $\delta\theta$-$\mathcal{J}$-open set in Y and let $x \in f^{-1}(W)$. Then $f(x) \in W$ and there exists an open set V in Y such that $f(x) \in V \subset \delta Cl^\star(V) \subset W$. Therefore, $x \in f^{-1}(V) \subset f^{-1}(\delta Cl^\star(V)) \subset f^{-1}(W)$. Since $f^{-1}(\delta Cl^\star(V)) = \delta Cl^\star(f^{-1}(V))$, we have $x \in f^{-1}(V) \subset \delta Cl^\star(f^{-1}(V)) \subset f^{-1}(W)$ and as f is continuous, $f^{-1}(V) \in \tau$, it follows that $x \in \delta Int_\theta^\star(f^{-1}(W))$. Thus $f^{-1}(W) \subset \delta Int_\theta^\star(f^{-1}(W))$ and hence $f^{-1}(W)$ is a $\delta\theta$-$\mathcal{I}$-open set. □

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