

## CERTAIN CLASSES OF ANALYTIC FUNCTIONS INVOLVING MULTIPLIER TRANSFORMATIONS

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**Abstract:** In this paper, we define the classes  $T_j^\lambda(m, l, A, B)$ , using Janowski class and the multiplier transformations  $I(m, \lambda, l)f(z)$  and we study distortion bounds, extreme points and many more properties. We also establish some results concerning partial sums for functions belonging to these classes.

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**Key Words:** Janowski class, multiplier transformation, partial sums

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### 1. Introduction

Let  $\mathcal{A}(j)$  denote the class of functions of the form

$$f(z) = z + \sum_{k=j+1}^{\infty} a_k z^k, \quad (a_k \geq 0; j \in \mathbf{N})$$

which are analytic in the open unit disc  $\mathcal{U} = \{z : |z| < 1\}$ . Let  $\mathcal{T}(j)$  denote the class of functions of the form

$$f(z) = z - \sum_{k=j+1}^{\infty} a_k z^k, \quad (a_k \geq 0; j \in \mathbf{N}) \quad (1.1)$$

which are analytic in the open unit disc  $\mathcal{U} = \{z : |z| < 1\}$ .

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For  $-1 \leq A < B \leq 1$ , let  $\mathcal{P}(A, B)$  [3] denote the class of functions which are of the form

$$p(z) = \frac{1 + A\omega(z)}{1 + B\omega(z)},$$

where  $\omega$  is a bounded analytic function satisfying the conditions  $\omega(0) = 0$  and  $|\omega(z)| < 1$ .

For  $f \in \mathcal{A}(j)$ ,  $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ ,  $\lambda, l \geq 0$ , the operator [1]  $I(m, \lambda, l)f(z)$  is defined by

$$I(m, \lambda, l)f(z) = z - \sum_{k=j+1}^{\infty} \left( \frac{\lambda(k-1) + l + 1}{l + 1} \right)^m a_k z^k.$$

We say that a function  $f \in \mathcal{T}(j)$  is in the class  $\mathcal{T}_j^\lambda(m, l, A, B)$  if and only if

$$\frac{z[I(m, \lambda, l)f(z)]'}{I(m, \lambda, l)f(z)} \prec \frac{1 + Az}{1 + Bz}, \quad (m \in \mathbb{N}_0),$$

for  $-1 \leq A < B \leq 1$ ,  $l, \lambda > 0$  and for all  $z \in \mathcal{U}$ .

Note that  $T_j^0(1, 0, 2\alpha - 1, 1) = S_j^*(\alpha)$  introduced by Chatterjea [2] and  $T_j^1(1, 0, 2\alpha - 1, 1) = C_j(\alpha)$  studied by Srivastava [7].

We shall need the following Theorem of Ravikumar and Latha [4] to prove our results.

## 2. Main Results

**Theorem 2.1.** *A function  $f \in \mathcal{T}_j$  is in the class  $\mathcal{T}_j^\lambda(m, l, A, B)$  if and only if*

$$\sum_{k=j+1}^{\infty} \left( \frac{\lambda(k-1) + l + 1}{l + 1} \right)^m [k(B+1) - (A+1)] a_k \leq B - A \quad (2.1)$$

for  $m \in \mathbb{N}_0$ ,  $-1 \leq A < B \leq 1$ ,  $l, \lambda > 0$  and  $z \in \mathcal{U}$ .

**Corollary 2.2.** *If function  $f(z) \in \mathcal{T}_j$  is in the class  $\mathcal{T}_j^\lambda(m, l, A, B)$  then*

$$|a_k| \leq \frac{(B - A)}{\left( \frac{\lambda(k-1) + l + 1}{l + 1} \right)^m [k(B+1) - (A+1)]},$$

for some  $-1 \leq A < B \leq 1$ ,  $m \in \mathbb{N}_0$ ,  $l, \lambda > 0$  and  $z \in \mathcal{U}$ .

Now we determine extreme points for the class  $\mathcal{T}_j^\lambda(m, l, A, B)$ .

**Theorem 2.3.** Let  $f(z) \in \mathcal{T}_j^\lambda(m, l, A, B)$ . Define  $f_1(z) = z$  and

$$f_k(z) = z - \frac{B - A}{\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m [k(B+1) - (A+1)]} z^k, \quad k \geq j + 1,$$

for some  $-1 \leq A < B \leq 1, m \in \mathbb{N}_0, l, \lambda > 0$  and  $z \in \mathcal{U}$ . Then  $f \in \mathcal{T}_j^\lambda(m, l, A, B)$  if and only if  $f$  can be expressed as  $f(z) = \sum_{k=j}^{\infty} \mu_k f_k(z)$ , where

$$\mu_k \geq 0 \text{ and } \sum_{k=j}^{\infty} \mu_k = 1.$$

*Proof.* If  $f(z) = \sum_{k=j}^{\infty} \mu_k f_k(z)$  with  $\sum_{k=j}^{\infty} \mu_k = 1, \mu_k \geq 0$ , then

$$\begin{aligned} & \sum_{k=j+1}^{\infty} \frac{\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m [k(B+1) - (A+1)]}{\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m [k(B+1) - (A+1)]} \mu_k (B - A) \\ &= \sum_{k=j+1}^{\infty} \mu_k (B - A) = (1 - \mu_j)(B - A) \\ &\leq (B - A). \end{aligned}$$

Hence  $f(z) \in \mathcal{T}_j^\lambda(m, l, A, B)$ .

Conversely, let  $f(z) = z - \sum_{k=j+1}^{\infty} a_k z^k \in \mathcal{T}_j^\lambda(m, l, A, B)$ , define

$$\mu_k = \frac{\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m [k(B+1) - (A+1)] |a_k|}{(B - A)}, \quad k \geq j + 1,$$

and define  $\mu_j = 1 - \sum_{k=j+1}^{\infty} \mu_k$ . From Theorem 2.1,  $\sum_{k=j+1}^{\infty} \mu_k \leq 1$  and hence

$$\mu_j \geq 0.$$

Since  $\mu_k f_k(z) = \mu_k f(z) + a_k z^k$ ,

$$\sum_{k=j+1}^{\infty} \mu_k f_k(z) = z - \sum_{k=j+1}^{\infty} a_k z^k = f(z). \quad \square$$

**Theorem 2.4.** *The class  $\mathcal{T}_j^\lambda(m, l, A, B)$  is closed under convex linear combination.*

*Proof.* Let  $f(z), g(z) \in \mathcal{T}_j^\lambda(m, l, A, B)$  and let

$$f(z) = z - \sum_{k=j+1}^{\infty} a_k z^k, \quad g(z) = z - \sum_{k=j+1}^{\infty} b_k z^k.$$

For  $\eta$  such that  $0 \leq \eta \leq 1$ , it suffices to show that the function defined by  $h(z) = (1 - \eta)f(z) + \eta g(z)$ ,  $z \in \mathcal{U}$  belongs to  $\mathcal{T}_j^\lambda(m, l, A, B)$ . Now

$$h(z) = z - \sum_{k=j+1}^{\infty} [(1 - \eta)a_k + \eta b_k] z^k.$$

Applying Theorem 2.1, to  $f(z), g(z) \in \mathcal{T}_j^\lambda(m, l, A, B)$ , we have

$$\begin{aligned} & \sum_{k=j+1}^{\infty} \left( \frac{\lambda(k-1) + l + 1}{l + 1} \right)^m [k(B+1) - (A+1)][(1 - \eta)a_k + \eta b_k] \\ &= (1 - \eta) \sum_{k=j+1}^{\infty} \left( \frac{\lambda(k-1) + l + 1}{l + 1} \right)^m [k(B+1) - (A+1)]a_k \\ &+ \eta \sum_{k=j+1}^{\infty} \left( \frac{\lambda(k-1) + l + 1}{l + 1} \right)^m [k(B+1) - (A+1)]b_k \\ &\leq (1 - \eta)(B - A) + \eta(B - A) = (B - A). \end{aligned}$$

This implies that  $h(z) \in \mathcal{T}_j^\lambda(m, l, A, B)$ . □

**Theorem 2.5.** *Let for  $i = 1, 2, \dots, k$ ,  $f_i(z) = z - \sum_{k=j+1}^{\infty} a_{k,i} z^k \in$*

$\mathcal{T}_j^\lambda(m, l, A, B)$  and  $0 < \beta_i < 1$  such that  $\sum_{i=1}^k \beta_i = 1$ , then the function  $F(z)$

defined by  $F(z) = \sum_{i=1}^k \beta_i f_i(z)$  is also in  $\mathcal{T}_j^\lambda(m, l, A, B)$ .

*Proof.* For each  $i \in \{1, 2, 3, \dots, k\}$  we obtain

$$\sum_{k=j+1}^{\infty} \left( \frac{\lambda(k-1) + l + 1}{l + 1} \right)^m [k(B+1) - (A+1)] |a_{k,i}| < (B - A).$$

Since

$$\begin{aligned}
 F(z) &= \sum_{i=1}^k \beta_i \left( z - \sum_{k=j+1}^{\infty} a_{k,i} z^k \right) \\
 &= z - \sum_{k=j+1}^{\infty} \left( \sum_{i=1}^k \beta_i a_{k,i} \right) z^k. \\
 &\sum_{k=j+1}^{\infty} \left( \frac{\lambda(k-1) + l + 1}{l + 1} \right)^m [k(B+1) - (A+1)] \left[ \sum_{i=1}^k \beta_i a_{k,i} \right] \\
 &= \sum_{i=1}^k \beta_i \left[ \sum_{k=j+1}^{\infty} \left( \frac{\lambda(k-1) + l + 1}{l + 1} \right)^m [k(B+1) - (A+1)] \right] \\
 &< \sum_{i=1}^k \beta_i (B - A) < (B - A).
 \end{aligned}$$

Therefore  $F(z) \in \mathcal{T}_j^\lambda(m, l, A, B)$ . □

### 3. Partial Sums

Consider partial sums of functions in the class  $\mathcal{T}_j^\lambda(m, l, A, B)$  and obtain sharp lower bounds for the ratios of real part of  $f(z)$  to  $f_n(z)$  and  $f'(z)$  to  $f'_n(z)$ . In this paper, applying methods used by Silverman [5] and Silvia [6], we will investigate the ratio of a function  $f(z)$  of the form ( 1.1) to its sequence of partial sums

$$f_n(z) = z - \sum_{k=j+1}^n a_k z^k$$

when the coefficients are sufficiently small to satisfy conditions ( 2.1). More precisely, we will determine sharp lower bounds for

$$\Re \left\{ \frac{f(z)}{f_n(z)} \right\}, \Re \left\{ \frac{f_n(z)}{f(z)} \right\}, \Re \left\{ \frac{f'(z)}{f'_n(z)} \right\}, \text{ and } \Re \left\{ \frac{f'_n(z)}{f'(z)} \right\}.$$

In the sequel, we will make use of the result that

$$\Re \frac{(1 + \omega(z))}{(1 - \omega(z))} > 0 \quad (z \in \mathcal{U})$$

if and only if  $\omega(z) = \sum_{k=j+1}^{\infty} c_k z^k$  satisfies the inequality  $|\omega(z)| < |z|$ .

**Theorem 3.1.** *If  $f(z) \in \mathcal{T}_j^\lambda(m, l, A, B)$ , then*

$$\Re \left\{ \frac{f(z)}{f_n(z)} \right\} > 1 - \frac{1}{c_{n+1}}, \quad (z \in \mathcal{U}) \quad (3.1)$$

where

$$c_k = \frac{\left( \frac{\lambda(k-1)+l+1}{l+1} \right)^m [k(B+1) - (A+1)]}{B-A}. \quad (3.2)$$

The estimates in ( 3.1) are sharp.

*Proof.* The function  $f(z) \in \mathcal{T}_j^\lambda(m, l, A, B)$ , if and only if  $\sum_{k=j+1}^{\infty} c_k a_k \leq 1$ .

It is easy to verify that  $c_{k+1} > c_k > 1$ .

Therefore we have

$$\sum_{k=j+1}^n a_k + c_{n+1} \sum_{k=n+1}^{\infty} a_k \leq \sum_{k=j+1}^{\infty} c_k a_k \leq 1. \quad (3.3)$$

By Setting

$$\begin{aligned} \omega(z) &= c_{n+1} \left\{ \frac{f(z)}{f_n(z)} - \left( 1 - \frac{1}{c_{n+1}} \right) \right\} \\ &= 1 - \frac{c_{n+1} \sum_{k=n+1}^{\infty} a_k z^{k-1}}{1 - \sum_{k=j+1}^n a_k z^{k-1}} \end{aligned}$$

and applying ( 3.3), we find that

$$\begin{aligned} \left| \frac{1 - \omega(z)}{1 + \omega(z)} \right| &\leq \frac{c_{n+1} \sum_{k=n+1}^{\infty} a_k}{2 - 2 \sum_{k=j+1}^n a_k - c_{n+1} \sum_{k=n+1}^{\infty} a_k} \\ &\leq 1, \quad z \in \mathcal{U}, \end{aligned}$$

which readily yields the assertion ( 3.3).

To see that

$$f(z) = z - \frac{z^{k+1}}{c_{k+1}}$$

gives sharp results, we observe that

$$\frac{f(z)}{f_n(z)} = 1 - \frac{z^k}{c_{k+1}}.$$

Letting  $z \rightarrow 1^-$ , we have

$$\frac{f(z)}{f_n(z)} = 1 - \frac{1}{c_{k+1}},$$

which shows that the bounds in is the best possible for each  $n \in \mathcal{N}$ .

□

**Theorem 3.2.** *If  $f(z) \in \mathcal{T}_j^\lambda(m, l, A, B)$ , then*

$$\Re \left\{ \frac{f_n(z)}{f(z)} \right\} > \frac{c_{n+1}}{1 + c_{n+1}}, \quad (z \in \mathcal{U}) \tag{3.4}$$

where  $c_n$  is defined in ( 3.2).The result is sharp for every  $n$  with extremal function given by ( 3.4).

*Proof.* By Setting

$$\begin{aligned} \omega(z) &= (1 + c_{n+1}) \left\{ \frac{f_n(z)}{f(z)} - \frac{c_{n+1}}{1 + c_{n+1}} \right\} \\ &= \frac{1 - \sum_{k=j+1}^n a_k z^{k-1} + c_{n+1} \sum_{k=n+1}^{\infty} a_k z^{k-1}}{1 - \sum_{k=j+1}^{\infty} a_k z^{k-1}}, \end{aligned}$$

and using ( 3.3) we find that

$$\begin{aligned} \left| \frac{1 - \omega(z)}{1 + \omega(z)} \right| &\leq \frac{(1 + c_{n+1}) \sum_{k=n+1}^{\infty} a_k}{2 - 2 \sum_{k=j+1}^n a_k - (1 - c_{n+1}) \sum_{k=n+1}^{\infty} a_k} \\ &\leq 1, \quad z \in \mathcal{N}. \end{aligned}$$

which gives ( 3.4), The bound in ( 3.4) is sharp for all  $n \in \mathcal{N}$  with the external function given by ( 3.2). □

**Theorem 3.3.** *If  $f(z) \in \mathcal{T}_j^\lambda(m, l, A, B)$ , then*

$$\Re \left\{ \frac{f'(z)}{f'_n(z)} \right\} > 1 - \frac{n+1}{c_{n+1}}, \quad (z \in \mathcal{U}) \quad (3.5)$$

where  $c_n$  is defined in ( 3.2).

*Proof.* By Setting

$$\begin{aligned} \omega(z) &= c_{n+1} \left\{ \frac{f'(z)}{f'_n(z)} - \left( 1 - \frac{n+1}{c_{n+1}} \right) \right\} \\ &= \frac{1 - \frac{c_{n+1}}{n+1} \sum_{k=n+1}^{\infty} ka_k z^{k-1} - \sum_{k=j+1}^{\infty} ka_k z^{k-1}}{1 - \sum_{k=j+1}^n ka_k z^{k-1}} \\ &= 1 - \frac{\frac{c_{n+1}}{n+1} \sum_{k=n+1}^{\infty} ka_k z^{k-1}}{1 - \sum_{k=j+1}^n ka_k z^{k-1}}. \end{aligned}$$

Now

$$\left| \frac{1 - \omega(z)}{1 + \omega(z)} \right| \leq \frac{\frac{c_{n+1}}{n+1} \sum_{k=n+1}^{\infty} ka_k}{2 - 2 \sum_{k=j+1}^n ka_k - \frac{c_{n+1}}{n+1} \sum_{k=n+1}^{\infty} ka_k} \leq 1$$

if and only if

$$2 \frac{c_{n+1}}{n+1} \sum_{k=n+1}^{\infty} ka_k \leq 2 - 2 \sum_{k=j+1}^n ka_k,$$

which is equivalent to

$$\sum_{k=j+1}^n ka_k + \frac{c_{n+1}}{n+1} \sum_{k=n+1}^{\infty} ka_k \leq 1$$

thus we obtain ( 3.5). The result is sharp with functions given by ( 3.2).  $\square$



**Theorem 3.4.** *If  $f(z) \in \mathcal{T}_j^\lambda(m, l, A, B)$ , then*

$$\Re \left\{ \frac{f'_n(z)}{f'(z)} \right\} > \frac{c_{n+1}}{nm + 1 + c_{n+1}}, \quad (z \in \mathcal{U}) \tag{3.6}$$

where  $c_n$  is defined in ( 3.2).The result is sharp for every  $n$  with extremal function given by ( 3.6).

*Proof.* By Setting

$$\begin{aligned} \omega(z) &= [(n + 1) + c_{n+1}] \left\{ \frac{f'_n(z)}{f'(z)} - \frac{c_{n+1}}{n + 1 + c_{n+1}} \right\} \\ &= 1 + \frac{(1 + \frac{c_{n+1}}{n+1}) \sum_{k=n+1}^{\infty} ka_k z^{k-1}}{1 - \sum_{k=j+1}^m ka_k z^{k-1}}. \end{aligned}$$

We deduce that that

$$\left| \frac{1 - \omega(z)}{1 + \omega(z)} \right| \leq \frac{(1 + \frac{c_{n+1}}{n+1}) \sum_{k=n+1}^{\infty} ka_k}{2 - 2 \sum_{k=j+1}^n ka_k - (1 + \frac{c_{n+1}}{n+1}) \sum_{k=n+1}^{\infty} ka_k} \leq 1,$$

if and only if

$$2(1 + \frac{c_{n+1}}{n+1}) \sum_{k=n+1}^{\infty} ka_k \leq 2 - 2 \sum_{k=j+1}^n ka_k,$$

which is equivalent to

$$\sum_{k=j+1}^n ka_k + (1 + \frac{c_{n+1}}{n+1}) \sum_{k=n+1}^{\infty} ka_k \leq 1.$$

which gives ( 3.6). The bound in ( 3.3) is sharp for all  $n \in \mathcal{N}$  with the external function given by ( 3.2). □

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