International Journal of Pure and Applied Mathematics
Volume 118 No. 1 2018, 87-94
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v118i1.8

ijpam.eu

ON COMPARISON OF SOME PROPERTIES OF PROPOSED CONTINUOUS WAVELETS WITH EXISTING ONE

M.D. Rayeezuddin[1 §], Gaddam Narsimulu[2], B. Krishna Reddy[3]
[1,2,3]Department of Mathematics
Osmania University
Hyderabad, INDIA

Abstract: This paper exhibits some properties of new continuous wavelet family proposed in a recent time and compared them with the well-known existing continuous wavelets which help in choosing of these wavelets in any application of interest.

AMS Subject Classification: 42C40, 65T60
Key Words: continuous wavelets, time-bandwidth product, central frequency and Fourier factor

1. Introduction

The continuous wavelet transform (CWT) construct a time frequency representation of a signal that offers good time-frequency localization. CWT is very efficient in determining the damping ratio of oscillating signals and is also very resistant to the noise in the signal, see [1].

There are enumerable continuous wavelets available in the literature such as Gaussian wavelets, Morlet wavelet, Mexican Hat wavelet and so on. The choice of the mother wavelet is application dependent, one cannot take one class of the mother wavelet for all applications and expect good results. The proper choice of the mother wavelet function is important in different applications.

In the paper [4], the author has discussed few properties of mother wavelets which helps in deciding the mother wavelets in any application of interest.

Received: May 11, 2017
Revised: October 8, 2017
Published: February 16, 2018

§Correspondence author

In this paper, we discussed certain properties of the family of new continuous wavelets derived from the product of the Gaussian function and Cauchy-Lorentz distribution function and compared them with the well-known existing wavelets for their use in suitable applications, see [7].

2. Continuous Wavelets

2.1. Some of the Existing Continuous Wavelets

2.1.1. Gaussian Wavelets (gau)

This family is constructed starting from the Gaussian function $f(x) = C_p \mathrm{e}^{-x^2}$ and by taking the p^{th} derivative of $f(x)$. The integer p is the parameter of this family and in the previous formula, C_p (The normalization constant) is such that $\|f^p(x)\|^2 = 1$, where $f^p(x)$ is the p^{th} derivative of $f(x)$.

2.1.2. Morlet Wavelets (morl)

The real-valued Morlet wavelet is defined by the expression

$$\psi(x) = C\mathrm{e}^{-x^2}\cos(5x),$$

where C normalization constant.

2.1.3. Mexican Hat Wavelet (mexh)

This wavelet is proportional to the second derivative function of the Gaussian probability density function. The wavelet is a special case of a larger family of derivative of Gaussian (DOG) wavelets. It is also known as the Ricker wavelet and is defined by

$$\psi(x) = \mathrm{e}^{-x^2/2}(1 - x^2).$$

2.2. Family of New Continuous Wavelet

This family is constructed with the successive differentiation of the product of Gaussian function and Cauchy-Lorentz distribution function i.e.

$$\eta(x) = e^{-x^2}.\frac{1}{1+x^2}.$$

Now taking the k^{th} derivative of $\eta(x)$ produces a wavelet $\xi_k(x)$ of order k, $\forall\ k = 1, 2, ..., 8$

$$\xi_k(x) = \frac{1}{\sqrt{C_k}} \frac{d^k}{dx^k} (\eta(x)).$$

In the previous formula, C_k is such that $\|\xi_k(x)\|^2 = 1$, the normalization constant and $C_k = \int\limits_{-\infty}^{\infty} \left| \frac{d^k}{dx^k} [\xi_k(x)] \right|^2 dx$. The values of which are given in Table 1.

The following necessary conditions for the function $\xi_k(x)$ to be a wavelet can easily be verified.

- $\int\limits_{-\infty}^{\infty} |\xi_k(x)|^2 dx = 1 < \infty$, $\forall\ k = 1, 2, 3..., 8$.(Finite energy property)
- $C_{\xi_k} = \int\limits_{0}^{\infty} \frac{|\widehat{\xi_k}(\omega)|^2}{\omega} d\omega < \infty$ (Admissibility condition)
- $\int\limits_{-\infty}^{\infty} \xi_k(x)\, dx = 0$, $\forall\ k = 1, 2, 3..., 8$ (Zero average property).

Moreover the vanishing moments property $\int\limits_{-\infty}^{\infty} x^n \xi_k(x)\, dx = 0$, $\forall\, n = 1, 2, ..., k-1$ can also be verified.

Here $\widehat{\xi_k}(\omega)$ is the fourier transform of $\xi_k(x)$ and

$$\begin{aligned}\widehat{\xi_k}(\omega) &= \int\limits_{-\infty}^{\infty} \xi_k(x)\, e^{-i\omega x} dx \\ &= \frac{\pi}{2}(i\omega)^k \left[\mathrm{e}^{1-\omega} \Big(\operatorname{erf}\big(\omega/2 - 1\big) + 1 \Big) - \mathrm{e}^{1+\omega} \Big(\operatorname{erf}\big(\omega/2 + 1\big) - 1 \Big) \right].\end{aligned}$$

Here $\operatorname{erf}(\alpha)$ is the Error function defined by $\operatorname{erf}(\alpha) = \frac{2}{\sqrt{\pi}} \int\limits_{0}^{\alpha} \mathrm{e}^{-\xi^2} d\xi$; the functions $\xi_k(x)$,$\forall\ k = 1, 2, 3..., 8$, forms a family of one dimensional continuous wavelets with effective support $[-5, 5]$.

k	C_k	C_{ξ_k}
1	1.601616580	3.016213576
2	9.112718044	10.92510487
3	95.97420331	87.16186635
4	1596.419729	1167.969380
5	38948.93124	23691.31307
6	1330506.195	687364.7346
7	61348820.77	27395877.23
8	3697995365	1449753504.2

Table 1: The values of normalization constant C_k and admissibility constant C_{ξ_k}

3. Properties

3.1. Support, Symmetry, and Regularity

The effective support of each member of the family is $[-5, 5]$ and are infinitely differentiable, hence the regularity is infinity. The waveform is symmetric when n even and antisymmetric when n odd.

3.2. Time-Bandwidth Product of the Wavelets

The time width, frequency width of the wavelet function $\xi_k(x)$ are defined as

$$\Delta_t^2 = \frac{\int\limits_{-\infty}^{\infty} x^2 \left|\xi_k(x)\right|^2 dx}{\int\limits_{-\infty}^{\infty} \left|\xi_k(x)\right|^2 dx}, \qquad \Delta_\omega^2 = \frac{\int\limits_{-\infty}^{\infty} \omega^2 \left|\widehat{\xi}_k(\omega)\right|^2 d\omega}{\int\limits_{-\infty}^{\infty} \left|\widehat{\xi}_k(\omega)\right|^2 d\omega}.$$

Thus $\Delta_t^2.\Delta_\omega^2$ defines the time-bandwidth product of the wavelet. Smaller the values of Δ_t^2 and Δ_ω^2 results in high time and frequency localization respectively[4].

The values of time width, frequency width and time-bandwidth product for each member of the new continuous wavelets along with continuous Morlet, Gaussian and Mexican Hat wavelets are computed and tabulated. As compared to the existing wavelets, the new continuous wavelets possess the lesser values of Δ_t^2 and hence can be employ in good time localization of spectral components of the signals such as examination of QRS complex in ECG signals.

The Power Spectral Density (PSD) which shows the strength of the variations(energy) as a function of frequency are also computed for each member of the family and are given in the following figure here P_{xx} denotes power of the wavelet in frequency domain.

Continuous Gaussian wavelets				New Continuous wavelets			
Order	Δ_t^2	Δ_ω^2	$\Delta_t^2.\Delta_\omega^2$	Order	Δ_t^2	Δ_ω^2	$\Delta_t^2.\Delta_\omega^2$
1	0.75000	3	2.25000	1	0.40255	5.68970	2.29038
2	0.58333	5	2.91667	2	0.25978	10.53190	2.73599
3	0.55000	7	3.85000	3	0.19865	16.63384	3.30429
4	0.53571	9	4.82143	4	0.15402	24.39768	3.75782
5	0.52778	11	5.80556	5	0.12049	34.16027	4.11593
6	0.52273	13	6.79545	6	0.09631	46.10938	4.44080
7	0.51923	15	7.78846	7	0.07945	60.27818	4.78889
8	0.51667	17	8.78333	8	0.06768	76.61182	5.18475

Continuous Morlet wavelets		
Δ_t^2	Δ_ω^2	$\Delta_t^2.\Delta_\omega^2$
0.24998	25.99991	6.49937
Continuous Mexican Hat wavelets		
Δ_t^2	Δ_ω^2	$\Delta_t^2.\Delta_\omega^2$
1.16667	2.50000	2.91667

Table 2: Time width, Frequency width and timebandwidth product of the wavelets

3.3. Conversion of Scale to Frequency

The continuous wavelet transform (CWT) converts the signal from time domain (one dimension) to scale-time domain (two dimension) which is not very easy to understand compared with the Fast Fourier Transform (FFT) result. The scale value can be converted into frequency (pseudo-frequency), the value of which depends on the central frequency of the applied wavelets and the scale value a and is given by

$$f_a = \frac{f_c}{a\Delta},$$

where

- a is a scale.
- Δ is the sampling period.
- f_c is the center frequency of a wavelet in Hz.
- f_a is the pseudo-frequency corresponding to the scale a, in Hz.

The idea is to associate with a given wavelet a purely periodic signal of frequency f_c or $f_c = \frac{1}{\lambda}$ where λ is the Fourier wavelength (Frequency fourier factor) and the relationship between the equivalent fourier period and the wavelet scale can be derived analytically for a particular wavelet function by substituting a cosine wave of a known frequency into wavelet transform definition and computing the scale a at which the wavelet power spectrum reaches its maximum [2][3] and it is found to be λ_k shown in the following table where $k = 1, 2, 3..., 8$.

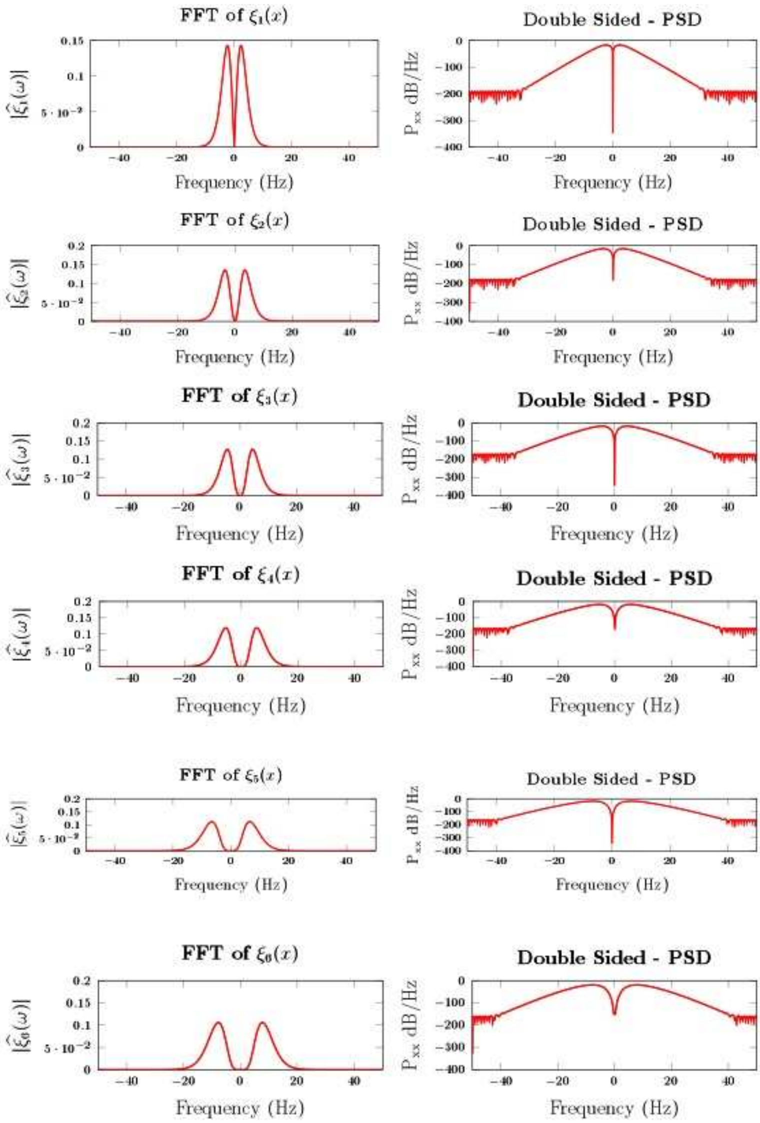

Figure 1: Power Spectral Density (PSD) of the wavelets

The central frequency based approximations of each member of the family of wavelets are given figure-2. In the figure the wavelet $\xi_k(x)$ is approximated with a purely periodic signal $[-A\sin(2\pi f_c x)$ (for odd k) and $A\cos(2\pi f_c x)$ (for even k)] with the corresponding central frequency f_c of the wavelets.

Here $A = max|\xi_k(x)|$ is the maximum value of $|\xi_k(x)|$.

k	λ_k
1	$\frac{2\pi}{2.27498}$
2	$\frac{2\pi}{3.06434}$
3	$\frac{2\pi}{3.82490}$
4	$\frac{2\pi}{4.63553}$
5	$\frac{2\pi}{5.53537}$
6	$\frac{2\pi}{6.50526}$
7	$\frac{2\pi}{7.50045}$
8	$\frac{2\pi}{8.50002}$

Table 3: Frequency fourier factor of the wavelets

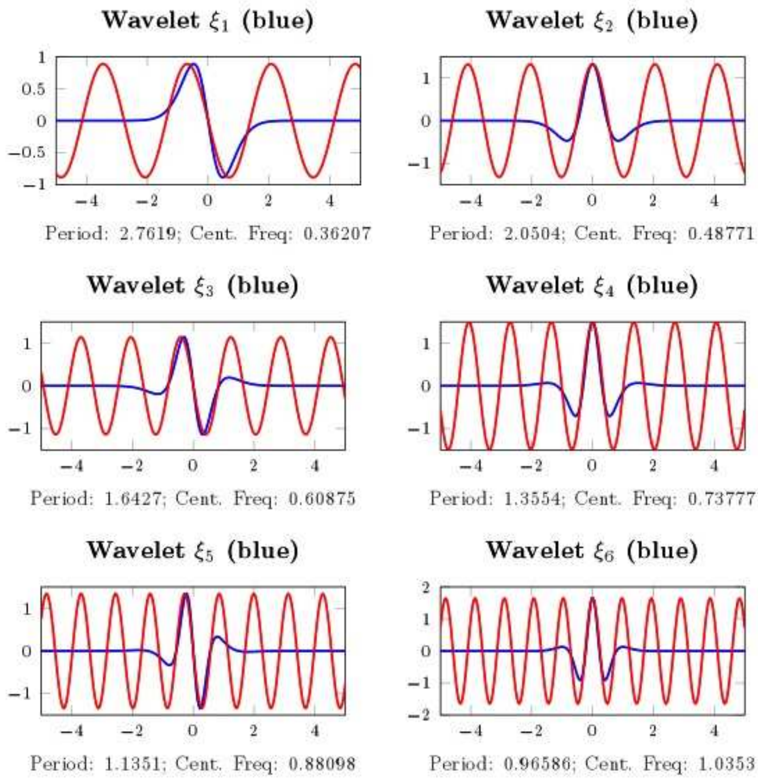

Figure 2: Central frequency based approximation of the new continuous wavelets

Conclusion

Some useful properties of the proposed wavelets such as symmetry, Regularity, support and time-bandwidth product are discussed. It is found that these new continuous wavelets possess lesser values of time width and time-bandwidth product as compared to the well-known existing wavelets and hence can be used in those applications which required good localization of spectral components in the signals. The plots of power spectral density for each member of the family are also computed.

References

[1] Slavič, Janko, Igor Simonovski, and Miha Boltežar, Damping identification using a continuous wavelet transform: application to real data, *Journal of Sound and Vibration*, **262**, No. 2 (2003), 291-307, **doi:** 10.1016/S0022-460X(02)01032-5.

[2] G. Zhao, D. Jiang, J.J. Diao, L. Qian, Application of wavelet time-frequency analysis on fault diagnosis for steam turbine, In: *5-th International Conference of Acoustical and Vibratory Surveillance Methods and Diagnostic Techniques*, CETIM, Senlis, France (2004).

[3] C. Torrence, G.P. Compo, A practical guide to wavelet analysis, *Bulletin of the American Meteorological Society*, **79**, No. 1 (1998), 61-78, **doi:** 10.1175/1520-0477(1998)079%3C0061:APGTWA%3E2.0.CO;2.

[4] N. Ahuja, S. Lertrattanapanich, N. Bose, Properties determining choice of mother wavelet, *IEE proceedings. Vision, Image and Signal Processing*, **152**, No. 5 (2005), 659-664, **doi:** 10.1049/ip-vis:20045034.

[5] S. Mallat, *A Wavelet Tour of Signal Processing: The Sparse Way*, Academic press, 2008.

[6] I. Daubechies, The wavelet transform, time-frequency localization and signal analysis, *IEEE Transactions on Information Theory*, **36**, No. 5 (1990), 961-1005, **doi:** 10.1109/18.57199.

[7] M. Rayeezuddin, B.K. Reddy, A new continuous wavelet family and its application to analysis of elementary periodic signals, *International Journal of Science and Research (IJSR)*, **4**, No. 10 (2015), 620-623.