

STABILIZABILITY AND SOLVABILITY OF DELAY DIFFERENTIAL EQUATIONS USING BACKSTEPPING METHOD

Fadhel S. Fadhel¹, Saja F. Noaman^{2 §}

^{1,2}Department of Mathematics and Computer Applications
College of Science
Al-Nahrain University
Al-Rusaffa, Baghdad, IRAQ

Abstract: In this paper the method of backstepping for stabilizing and solving system of ordinary and partial differential equations will be applied for system of delay differential equations by transforming the nonlinear delay differential equations into system of ordinary differential equations by using the method of steps to solved delay differential equations. The basic idea of this approach to find Lyapunov function for stabilizing the system of different time steps which choose the definitions of the present approach.

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1. Introduction

Backstepping method was recently developed in the early of 1990s by Petar V. Kokotovic and others to stabilize a class of nonlinear cascade systems, [3].

This method is a particular approach for stabilizing dynamical systems which is a successful approach that may be used in the area of nonlinear control theory, [7]. Backstepping starts with the system equation or integrator, which is

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[§]Correspondence author

the starts from the control input as an initial step and reaches the control input at the last step, [10]. Also it is unlike any of the methods previously developed in literatures for controlling ordinary differential equations (ODEs) and partial differential equations (PDEs). It differs from optimal control methods in that way in which it sacrifices optimality, [7].

This method develops by introducing new variables into it in a form depending on the state variables, controlling parameters and stabilizing functions, which compensate for nonlinearities, exists in the system, [5].

It is remarkable that, the backstepping method is used in numerous applications, such as flight trajectory control, industrial automation system, electric machines, transportation and robotic system [5], application of a robust controller for stabilization and trajectory tracking of a quadrotor unmanned aerial vehicle perturbed by external disturbances is successfully demonstrated, [1].

Furthermore, backstepping method is a nonlinear control design method that provides an alternative to feedback linearization. The main idea of the backstepping method is to partition the entire system into numbers of subsystems. As a result, the states of the first subsystem acted as the control variables for the next subsystem. In backstepping approach, at first, the ideal input that can stimulate the desired output from the first subsystem is computed. Since the subsystems are connected in cascade, input to the first subsystem automatically comes from the output of the second subsystem. Ideal input to the second subsystem is computed in a similar manner to that of first subsystem. The desired inputs for all the subsystems are calculated in a sequential manner until the last subsystem is arrived. As a case in point, the desired input for the last subsystem gives the expression for the actual control input, and accordingly, the designers can implement a state feedback law for the entire nonlinear system, [9].

In this paper, the backstepping method will be used in cooperation with the method of steps to stabilize and solve certain type of systems of delay differential equations (DDEs) that will be developed and introduced in this work.

2. The Backstepping Method

The backstepping design, which may be now studied and generalized for a class of non-linear system of ODEs, for example as in the following parametric strict-

feedback form:

$$\left. \begin{aligned} \dot{x}_1 &= x_2 + f_1(x_1) \\ \dot{x}_2 &= x_3 + f_2(x_1, x_2) \\ &\vdots \\ \dot{x}_{n-1} &= x_n + f_{n-1}(x_1, x_2, \dots, x_{n-1}) \\ \dot{x}_n &= u + f_n(x_1, x_2, \dots, x_n) \end{aligned} \right\} \quad (1)$$

where $x_1, x_2, \dots, x_n$ are the system states and u is the control input. Now, the objective is to apply the backstepping method in order to design a state feedback control function which asymptotically stabilize the origin.

The design procedure is modified and may be divided into the following steps similar to the systems discussed in [11]:

Step(1): First, we consider the stability of the first equation of system (1)

$$\dot{x}_1 = x_2 + f_1(x_1) \quad (2)$$

Define $z_1 = x_1$ and derive the dynamics of the new coordinate z_1 as:

$$\dot{z}_1 = \dot{x}_1 = x_2 + f_1(z_1) \quad (3)$$

Then $x_2 = \alpha_1(x_1)$ may be achieved only an error $z_2 = x_2 - \alpha_1(x_1)$, so that x_1 and x_2 are known explicit functions of z_2 and z_2 .

$$\dot{z}_1 = z_2 + \alpha_1(z_1) + f_1(z_1) \quad (4)$$

and the first control Lyapunov function may be defined by:

$$V_1 = z_1^T p_1 z_1 = \frac{1}{2} z_1^2 \quad (5)$$

The time-derivative of V_1 is:

$$\dot{V}_1 = \frac{\partial V_1}{\partial t} = \frac{\partial V_1}{\partial z_1} \frac{\partial z_1}{\partial t} = z_1 \dot{z}_1 = z_1(z_2 + \alpha_1(z_1) + f_1(z_1)) \quad (6)$$

Now, choose the virtual controller α_1 , which makes system (2) asymptotically stable to be as follows:

$$\alpha_1(z_1) = -z_1 - f_1(z_1) \quad (7)$$

Substituting (7) into (6), we obtain $\dot{V}_1$ is negative definite function on R^n . Thus by Lyapunov stability theory, the first equations of system (1) is asymptotically stable.

Step(2): In this step, the time derivative of the error dynamics $z_2 = x_2 - \alpha_1(z_1)$ is differentiated both sides to get:

$$\dot{z}_2 = \dot{x}_2 - \dot{\alpha}_1(z_1)$$

Therefore, rewrite the second equation of system (1) as:

$$\dot{z}_2 = x_3 + f_2(z_1, z_2 + \alpha_1(z_1)) - \dot{\alpha}_1(z_1) \quad (8)$$

Since $x_3 = \alpha_2(z_1, z_2)$ could not be achieved, there is an error function $z_3 = x_3 - \alpha_2(z_1, z_2)$, and hence eq.(8) may be rewritten as:

$$\dot{z}_2 = z_3 + \alpha_2(z_1, z_2) + f_2(z_1, z_2 + \alpha_1(z_1)) - \dot{\alpha}_1(z_1) \quad (9)$$

Now, consider the second Lyapunov function defined by:

$$\begin{aligned} V_2(z_1, z_2) &= V_1 + z_2^T p_2 z_2 \\ &= \frac{1}{2} z_1^2 + \frac{1}{2} z_2^2 \end{aligned} \quad (10)$$

and taking the time derivative of V_2 , yields to:

$$\begin{aligned} \dot{V}_2(z_1, z_2) &= \dot{V}_2(z_1, z_2) = z_1 \dot{z}_1 + z_2 \dot{z}_2 \\ &= -z_1^2 + z_1 z_2 + z_2 (z_3 + \alpha_2(z_1, z_2) + f_2(z_1, z_2 + \alpha_1(z_1)) \\ &\quad - \dot{\alpha}_1(z_1)) \end{aligned}$$

The first term was made negative (as in step 1), so may be chosen to make $\dot{V}_2$ negative, where:

$$\alpha_2(z_1, z_2) = -z_2 - z_1 - f_2(z_1, z_2 + \alpha_1(z_1)) + \dot{\alpha}_1(z_1) \quad (11)$$

Therefore, our system can be written using mathematical induction as:

$$\begin{aligned} \dot{z}_1 &= z_2 + \alpha_1(x_1) + f_1(z_1) \\ \dot{z}_2 &= z_3 + \alpha_2(z_1, z_2) + f_2(z_1, z_2 + \alpha_1(z_1)) - \dot{\alpha}_1(z_1) \\ &\vdots \\ \dot{z}_{n-1} &= z_n + \alpha_{n-1}(z_1, z_2, \dots, z_{n-1}) + f_{n-1}(z_1, z_2 + \alpha_1(z_1), \dots, z_{n-1} + \alpha_{n-2}(z_1, z_2, \dots, z_{n-2})) - \dot{\alpha}_{n-2}(z_1, z_2, \dots, z_{n-2}) \\ \dot{z}_n &= u + f_n(z_1, z_2 + \alpha_1(z_1), \dots, z_n + \alpha_{n-1}(z_1, z_2, \dots, z_{n-1})) - \dot{\alpha}_{n-1}(z_1, z_2, \dots, z_{n-1}) \end{aligned} \quad (12)$$

Step(n): For the last step, the actual control input u will appear and it is at our disposal derivative of $z_n = x_n - \alpha_{n-1}(z_1, z_2, \dots, z_{n-1})$ and therefore it is possible to obtain:

$$\begin{aligned} \dot{z}_n = & u + f_n(z_1, z_2 + \alpha_1(z_1), \dots, z_n + \alpha_{n-1}(z_1, z_2, \dots, z_{n-1})) - \frac{\partial \alpha_{n-1}}{\partial z_1} f_1(z_1) \\ & - \frac{\partial \alpha_{n-1}}{\partial z_2} f_2(z_1, z_2 + \alpha_1(z_1)) - \dots - \frac{\partial \alpha_{n-1}}{\partial z_{n-1}} f_{n-1}(z_1, z_2 + \alpha_1(z_1), \dots, z_n + \\ & \alpha_{n-1}(z_1, z_2, \dots, z_{n-1})) \end{aligned} \quad (13)$$

A feedback law for u is now chosen to make the derivative of

$$V_n(z_1, z_2, \dots, z_n) = V_{n-1}(z_1, z_2, \dots, z_{n-1}) + z_n^T p_n z_n$$

Taking the time derivative of V_n by considering V_{n-1} in step $(n-1)$, we can get:

$$\dot{V}_n(z_1, z_2, \dots, z_n) = z_1 \dot{z}_1 + z_2 \dot{z}_2 + \dots + z_n \dot{z}_n \quad (14)$$

Finally, we are in a position to design a control function u by making $\dot{V}_n < 0$, as follows:

$$\begin{aligned} u = & -z_n - z_{n-1} - f_n(z_1, z_2 + \alpha_1(z_1), \dots, z_n + \alpha_{n-1}(z_1, z_2, \dots, z_{n-1})) + \\ & \dot{\alpha}_{n-1}(z_1, z_2, \dots, z_{n-1}) \end{aligned} \quad (15)$$

Substituting u given by eq.(15) back into eq.(13) will result in a non-linear system of ODEs that may be solved by the linearization method (if the original system is nonlinear) or any numerical method for solving systems of nonlinear ODEs.

3. General Strict-Feedback Delay System

For the rest of this paper, after study on the system discussed in [6], the designed nonlinear time delay systems of the form:

$$\left. \begin{aligned} \dot{x}_i(t) &= x_{i+1}(t) - \lambda_i x_i(t) + x_i(t)(1 - x_i(t - \tau)) \\ \dot{x}_n(t) &= u(t) - \lambda_n x_n(t) + x_n(t)(1 - x_n(t - \tau)) \end{aligned} \right\} \quad (16)$$

for all $i = 1, 2, \dots, n-1, t \geq t_0$; with the initial conditions:

$$x_i(t) = \varphi_{i0}(t), i = 1, 2, \dots, n \text{ for } t_0 - \tau \leq t \leq t_0$$

where $\tau \geq 0$ is the time delay, λ_i , $i = 1, 2, \dots, n$ are an known positive constants $x_1(t)$, $x_2(t), \dots, x_n(t)$ are the system state and $u(t)$ is the control function.

In order to proceed further, the best method for solving DDEs is the method of steps (see [4]). We have the initial conditions which are given for a time step interval with length equal to τ , and to find the solution for all $t \geq t_0$ divided into steps with length τ , which is also the same for each solution for the next time step for $t \geq t_0$, which are respectively $x_1(t), x_2(t), \dots$; over $[t_0, t_0 + \tau]$, $[t_0 + \tau, t_0 + 2\tau]$, ...

For the first time step $[t_0, t_0 + \tau]$, we have for all $i = 1, 2, \dots, n - 1$:

$$\left. \begin{aligned} \dot{x}_i(t) &= x_{i+1}(t) - \lambda_i x_i(t) + x_i(t)(1 - \varphi_{i0}(t - \tau)) \\ \dot{x}_n(t) &= u(t) - \lambda_n x_n(t) + x_n(t)(1 - x_{n0}(t - \tau)) \end{aligned} \right\} \quad (17)$$

Hence, the resulting system of differential equations will be solved by using the backstepping method, as it discussed previously. The solution of the obtained system of ODEs may be found by the linearization method (if the original system is nonlinear) or any numerical method for solving systems of nonlinear ODEs (see [2] and [8]). After solving the resulting system, then using the method of lagrange interpolation will be used for the discrete numerical solution to be interpolated and used as the initial solution to the next time step $\varphi_{i1}(t)$, i.e.,

$$\left. \begin{aligned} \dot{x}_i(t) &= x_{i+1}(t) - \lambda_i x_i(t) + x_i(t)(1 - x_i(t - \tau)) \\ \dot{x}_n(t) &= u(t) - \lambda_n x_n(t) + x_n(t)(1 - x_n(t - \tau)) \end{aligned} \right\} \quad (18)$$

for all $i = 1, 2, \dots, n - 1$; with the initial conditions:

$$x_i(t) = \varphi_{i1}(t), i = 1, 2, \dots, n; \text{ for } t_0 \leq t \leq t_0 + \tau$$

Again, applying the backstepping method to stabilize and solve the DDE for the next time step. Similarly, we repeat proceed to the next intervals.

4. Illustrative Examples

Two examples will be considered in this section. The first example is a system of 2×2 DDEs in which the details of the solution will be given for illustration purpose, while the the results of the second example, which is a system of 3×3 DDEs will be given directly without presenting the calculations.

Example 1. Consider the following 2×2 nonlinear system of time-delay:

$$\left. \begin{aligned} \dot{x}_1(t) &= x_2(t) - 2x_1(t) + x_1(t)(1 - x_1(t - 1)) \\ \dot{x}_2(t) &= u(t) - 3x_2(t) + x_2(t)(1 - x_2(t - 1)) \end{aligned} \right\}, t \geq 0 \quad (19)$$

with the initial conditions:

$$x_1(t) = \varphi_{10}(t) = 4t, x_2(t) = \varphi_{20}(t) = 4t + 4, \text{ for } -1 \leq t \leq 0$$

To find the solution in the first time-step $[0,1]$, apply the method of steps, we get:

$$\begin{aligned}\dot{x}_1(t) &= x_2(t) - 2x_1(t) + x_1(t)(1 - \varphi_{10}(t - 1)) \\ &= x_2(t) - 2x_1(t) + x_1(t)(5 - 4t) \\ \dot{x}_2(t) &= u(t) - 3x_2(t) + x_2(t)(1 - \varphi_{20}(t - 1)) \\ &= u(t) - 3x_2(t) + x_2(t)(1 - 4t)\end{aligned}$$

So the resulting system of ODEs with nonconstant coefficients has the form:

$$\left. \begin{aligned}\dot{x}_1(t) &= x_2(t) - 2x_1(t) + (5 - 4t)x_1(t) \\ \dot{x}_2(t) &= u(t) - 3x_2(t) + (1 - 4t)x_2(t)\end{aligned}\right\} \quad (20)$$

Now, the resulting system of differential equations (20) will be solved by backstepping method

Step(1): First, we consider the stability of the first equation of system (20), namely:

$$\dot{x}_1(t) = x_2(t) - 2x_1(t) + (5 - 4t)x_1(t)$$

Let $z_1(t) = x_1(t)$ and derive the dynamics of the new coordinate as:

$$\dot{z}_1(t) = \dot{x}_1(t) = x_2(t) - 2z_1(t) + (5 - 4t)z_1(t)$$

suppose $x_2(t) = \alpha_1(x_1)$, with error $z_2(t) = x_2(t) - \alpha_1(x_1)$ and since $x_2(t) = z_2(t) + \alpha_1$, then:

$$\dot{z}_1(t) = z_2(t) + \alpha_1 - 2z_1(t) + (5 - 4t)z_1(t)$$

Consider now the first control Lyapunov function:

$$V_1(t) = \frac{1}{2}z_1^2(t)$$

Differentiating V_1 with respect to time, will produce:

$$\begin{aligned}\dot{V}_1 &= \frac{\partial V_1}{\partial t} = \frac{\partial V_1}{\partial z_1} \frac{\partial z_1}{\partial t} \\ &= z_1(t)\dot{z}_1(t) \\ &= z_1(t)(z_2(t) + \alpha_1 - 2z_1(t) + (5 - 4t)z_1(t))\end{aligned}$$

$$= z_1(t)(\alpha_1 - 2z_1(t) + (5 - 4t)z_1(t)) + z_1(t)z_2(t)$$

Now, select the virtual control α_1 , which makes the derivative of V_1 negative definite, i.e., letting:

$$\alpha_1 = -k_1 z_1(t) + 2z_1(t) - (5 - 4t)z_1(t)$$

Then the derivative of the virtual control α_1 is given by:

$$\begin{aligned}\dot{\alpha}_1 &= \frac{\partial \alpha_1}{\partial x_1} \dot{x}_1(t) \\ &= (-k_1 - 3 + 4t)(x_2(t) - 2x_1(t) + (5 - 4t)x_1(t))\end{aligned}$$

where k_1 is a positive constant. Then the time derivative of V_1 becomes:

$$\dot{V}_1 = -k_1 z_1^2(t) + z_1(t)z_2(t)$$

Hence, in order to get $\dot{V}_1(z_1) < 0$, let $z_2(t) = 0$, then: $\dot{V}_1 = -k_1 z_1^2(t)$

Step(2): Now the error dynamics $z_2(t) = x_2(t) - \alpha_1$ may be differentiated both sides to get:

$$\begin{aligned}\dot{z}_2(t) &= \dot{x}_2(t) - \dot{\alpha}_1 \\ &= u(t) - 3x_2(t) + (1 - 4t)x_2(t) - \dot{\alpha}_1 \\ &= u(t) - 3x_2(t) + (1 - 4t)x_2(t) - (-k_1 - 3 + 4t)(x_2(t) \\ &\quad - 2x_1(t) + (5 - 4t)x_1(t))\end{aligned}$$

and one can see that the actual control input $u(t)$ will appeared.

The objective in this step is to design the actual control input $u(t)$, such that z_1 converge to zero. For this purpose, take the second Lyapunov function V_2 as:

$$V_2 = V_1 + \frac{1}{2}z_2^2(t)$$

Then the time derivative of which becomes:

$$\begin{aligned}\dot{V}_2 &= \dot{V}_1 + z_2(t)\dot{z}_2(t) \\ &= -k_1 z_1^2(t) + z_1(t)z_2(t) + z_2(t)(u(t) - 3x_2(t) + (1 - 4t)x_2(t) \\ &\quad - (-k_1 - 3 + 4t)(x_2(t) - 2x_1(t) + (5 - 4t)x_1(t)))\end{aligned}$$

Finally, we are in a position to design a control function u by making $\dot{V}_2 < 0$, as follows:

$$u(t) = -k_2 z_2(t) - z_1(t) + 3x_2(t) - (1 - 4t)x_2(t) + (-k_1 - 3 + 4t)$$

$$(x_2(t) - 2x_1(t) + (5 - 4t)x_1(t))$$

where k_2 is a positive constant. Then the derivative of the Lyapunov function V_2 is:

$$\dot{V}_2 = - \sum_{i=1}^2 k_i z_i^2(t) < 0$$

where:

$$\begin{aligned} z_1(t) &= x_1(t) \\ z_2(t) &= x_2(t) + k_1 x_1(t) - 2x_1(t) + (5 - 4t)x_1(t) \end{aligned}$$

Therefore:

$$\begin{aligned} u(t) &= -k_2 z_2(t) - x_1(t) + 3x_2(t) - (1 - 4t)x_2(t) + \\ &\quad (-k_1 - 3 + 4t)(x_2(t) - 2x_1(t) + (5 - 4t)x_1(t)) \end{aligned}$$

since $k_i > 0, i = 1, 2$ and for computation and comparison purpose let $k_1 = 2, k_2 = 1$, then:

$$\begin{aligned} u(t) &= -x_2(t) - (5 - 4t)x_1(t) - x_1(t) + 3x_2(t) - (1 - 4t)x_2(t) + \\ &\quad (-5 + 4t)(x_2(t) - 2x_1(t) + (5 - 4t)x_1(t)) \end{aligned}$$

Therefore, the resulting system of ODEs with nonconstant coefficients is given by:

$$\left. \begin{aligned} \dot{x}_1(t) &= x_2(t) - 2x_1(t) + (5 - 4t)x_1(t) \\ \dot{x}_2(t) &= -x_2(t) - x_1(t) - (5 - 4t)x_1(t) + (-5 + 4t)(x_2(t) - 2x_1(t) \\ &\quad + (5 - 4t)x_1(t)) \end{aligned} \right\} \quad (21)$$

with initial conditions:

$$x_1(0) = 0, x_2(0) = 4$$

Now system (21) may be solved using numerical methods, we get the following results presented in table (1) and their plots in Figure ??:

Table (1): The numerical solution of system (22) for the first time step [0,1]

t	$x_1(t)$	$x_2(t)$
0	0	4
0.2	0.574065	0.562151
0.4	0.771024	-0.823406
0.6	0.720506	-1.090066
0.8	0.544561	-0.915037
1	0.335975	-0.678299

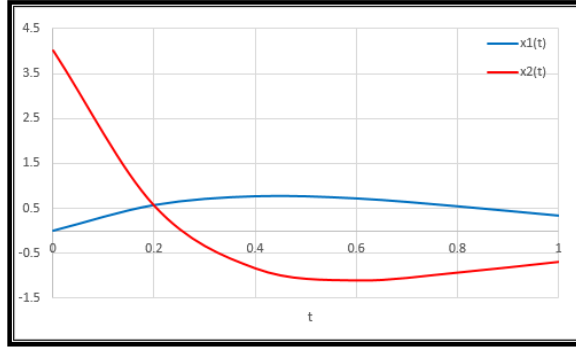


Figure 1: The numerical solution of $x_1(t)$ and $x_2(t)$.

Now, for the second time-step $[1, 2]$ we will use first the Lagrange interpolation polynomial of degree 5. In order to interpolate the discrete results of the first step and considered as the initial solution for the second time step. These polynomials are:

$$\begin{aligned}
 p_{15}(t) &= 4.016927t - 6.195145t^2 + 2.146849t^3 + 0.932057t^4 \\
 &\quad - 0.564714t^5 \\
 p_{25}(t) &= 4 - 24.154826t + 39.829053t^2 - 25.955286t^3 \\
 &\quad + 4.533984t^4 + 1.068776t^5
 \end{aligned}$$

Now, consider system (19) with the initial conditions:

$$\begin{aligned}
 x_1(t) &= \varphi_{11}(t) = 4.016927t - 6.195145t^2 + 2.146849t^3 \\
 &\quad + 0.932057t^4 - 0.564714t^5 \\
 x_2(t) &= \varphi_{21}(t) = 4 - 24.154826t + 39.829053t^2 - 25.955286t^3 \\
 &\quad + 4.533984t^4 + 1.068776t^5
 \end{aligned}$$

for $0 \leq t \leq 1$ and the solution in the second step interval $[1, 2]$ may be found similarly as in the first step by solving the system of equations:

$$\begin{aligned}
 \dot{x}_1(t) &= x_2(t) - 2x_1(t) + x_1(t)(1 - (4.016927(t-1) - \\
 &\quad 6.195145(t-1)^2 + 2.146849(t-1)^3 + 0.932057(t-1)^4 \\
 &\quad - 0.564714(t-1)^5)) \\
 \dot{x}_2(t) &= u(t) - 3x_2(t) + x_2(t)(1 - (4 - 24.154826(t-1)
 \end{aligned}$$

$$+39.829053(t-1)^2 - 25.955286(t-1)^3 \\ +4.533984(t-1)^4 + 1.068776(t-1)^5))$$

So the resulted linear system of ODEs is given by:

$$\left. \begin{aligned} \dot{x}_1(t) &= x_2(t) - 2x_1(t) + x_1(t)(11.86215 - 16.295966t + \\ &1.39621t^2 + 7.228519t^3 - 3.755627t^4 + 0.564714t^5) \\ \dot{x}_2(t) &= u(t) - 3x_2(t) + x_2(t)(-96.404373 + 194.470846t - \\ &134.211055t^2 + 33.403462t^3 + 0.809896t^4 - 1.068776t^5) \end{aligned} \right\} \quad (22)$$

Again, apply the backstepping method and carrying the same procedure followed in first-time step $[0,1]$ we get:

$$\left. \begin{aligned} \dot{x}_1(t) &= x_2(t) - 2x_1(t) + x_1(t)(11.86215 - 16.295966t + 1.39621t^2 \\ &+ 7.228519t^3 - 3.755627t^4 + 0.564714t^5) \\ \dot{x}_2(t) &= -x_2(t) - x_1(t) - x_1(t)(11.86215 - 16.295966t + 1.39621t^2 \\ &+ 7.228519t^3 - 3.755627t^4 + 0.564714t^5) + (-11.86215 + 16.295966t \\ &- 1.39621t^2 - 7.228519t^3 + 3.755627t^4 - 0.564714t^5)(x_2(t) - 2x_1(t) \\ &+ x_1(t)(11.86215 - 16.295966t + 1.39621t^2 + 7.228519t^3 - 3.755627t^4 \\ &+ 0.564714t^5)) \end{aligned} \right\} \quad (23)$$

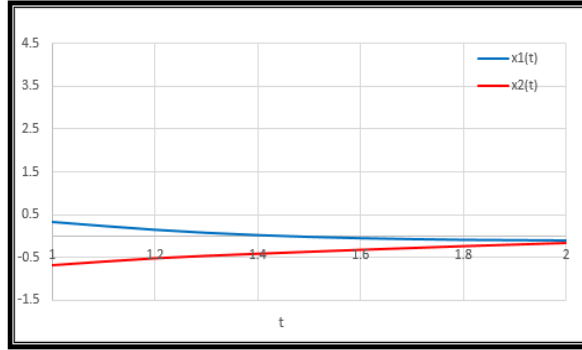
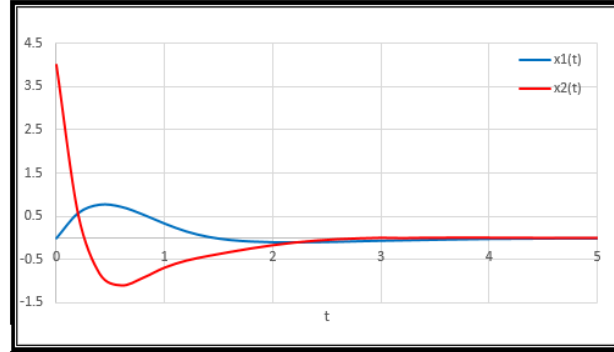
with initial condition $x_1(1) = -0.08992, x_2(1) = -0.161772$.

Now system (23) may be solved using numerical methods, we get the results in presented table (2) and their sketch in Figure 2, which shows the asymptotic stability of the solutions.

Table (2): The numerical solution of system (24) for the second time-step $[1,2]$

t	$x_1(t)$	$x_2(t)$
1	0.335974	-0.678299
1.2	0.15549	-0.518181
1.4	0.033036	-0.411472
1.6	-0.037934	-3.20106
1.8	-0.074372	-0.235863
2	-0.089943	-0.161769

The solution of the original system, of this example for five time step intervals, from the result, one may see the asymptotic stability of the obtained results which shows the very high agreement between the theoretical and numerical results of the followed approach in this paper. The results are collected

Figure 2: The numerical solution of $x_1(t)$ and $x_2(t)$.Figure 3: The numerical solution of $x_1(t)$ and $x_2(t)$.

for the first five time steps are presented in Figure 3, with the control function $u(t)$ is presented in Figure 4.

Example(2): Consider the following 3×3 nonlinear time-delay system:

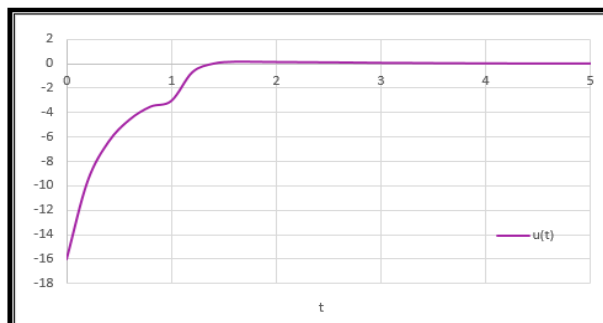
$$\left. \begin{aligned} \dot{x}_1(t) &= x_2(t) - 3x_1(t) + x_1(t)(1 - x_1(t-1)) \\ \dot{x}_2(t) &= x_3(t) - 3x_2(t) + x_2(t)(1 - x_2(t-1)) \\ \dot{x}_3(t) &= u(t) - 3x_3(t) + x_3(t)(1 - x_3(t-1)) \end{aligned} \right\} \quad (24)$$

with initial conditions:

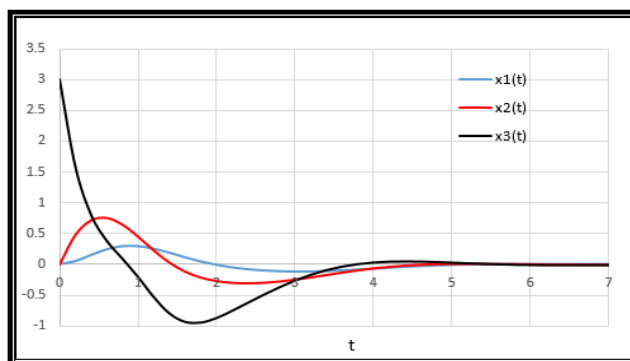
$$x_1(t) = \varphi_{10}(t) = t, x_2(t) = \varphi_{20}(t) = 3t, x_3(t) = \varphi_{30}(t) = 3t + 3,$$

for

$$-1 \leq t \leq 0$$

Figure 4: The control function $u(t)$.

and carrying the save procedure followed in example (1), we get the following results for $x_1(t), x_2(t), x_3(t)$ and $u(t)$, which are presented in Figures 5 and 6.

Figure 5: The numerical solution of $x_1(t), x_2(t)$ and $x_3(t)$ of example (2).

5. Conclusions

In the present paper, the backstepping method have been used for stabilizing and solving certain type of a system of DDEs, in which the obtained results that are presented in tables (1) and (2) shows the high efficiency of the approach for solving such type of problems, and that k_i values have an impact on the speed and stability of the solution, especially in the selection of these values in order

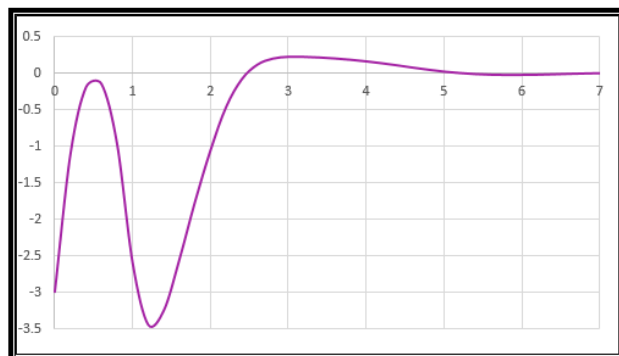


Figure 6: The control function $u(t)$.

to simplify the solution and thus reduce the time steps to make the system is asymptotic stability as in Figs. (3) and (5) where shows the asymptotic stability of the systems as the time step interval increases. Therefore, the backstepping approach may be considered as a reliable approach for solving and stabilizing problems of differential equations.

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