ONE MODULO THREE ROOT SQUARE MEAN LABELING
OF PATH RELATED GRAPHS

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Abstract: In this paper, we introduce a new labeling called one modulo three root square mean labeling. A graph G is said to be one modulo three root square mean graph if there is an injective function ϕ from the vertex set of G to the set {0, 1, 3, . . . , 3q-2, 3q} where q is the number of edges of G and ϕ induces a bijection ϕ* from the edge set of G to {1, 4, . . . , 3q-2} given by ϕ*(uv) = ⌈√[(ϕ(u))^2 + (ϕ(v))^2]/2⌉ or ⌊√[(ϕ(u))^2 + (ϕ(v))^2]/2⌋ and the function ϕ is called one modulo three root square mean labeling of G. Furthermore, we prove that some path related graphs are one modulo three root square mean graphs.

Key Words: one modulo three root square mean labeling, one modulo three root square mean graphs

1. Introduction

We begin with simple, finite, connected and undirected graph. For standard terminology and notations we follow Harary [1]. A graph labeling is an assignment of integers to the vertices or edges or both subject to certain condition(s).
If the domain of the mapping is the set of vertices (edges) then the labeling is called a vertex labeling (an edge labeling). Several types of graph labeling and a detailed survey is available in [2].

C. Jayasekaran and C. David Raj was introduced the concept of one modulo three harmonic mean labeling of graphs in [3]. Root square mean labeling was introduced by S. S. Sandhya, S. Somasundaram and S. Anusa in [4]. P. Jeyanthi and A. Maheswari was introduced the concept of one modulo three mean labeling of graphs in [5]. Also they proved one modulo three mean labeling of cycle related graphs in [6]. In this paper, we introduce a new labeling called one modulo three root square mean labeling and investigate one modulo three root square mean graphs.

We will provide a brief summary of definitions and other information’s which are necessary for our present investigation.

**Definition 1.1.** A graph $G$ is said to be one modulo three root square mean graph if there is an injective function $\phi$ from the vertex set of $G$ to the set $\{0, 1, 3, \ldots, 3q-2, 3q\}$ where $q$ is the number of edges of $G$ and $\phi$ induces a bijection $\phi^*$ from the edge set of $G$ to $\{1, 4, \ldots, 3q-2\}$ given by $\phi^*(uv) = \left\lceil \frac{\sqrt{\phi(u)^2 + \phi(v)^2}}{2} \right\rceil$ or $\left\lfloor \frac{\sqrt{\phi(u)^2 + \phi(v)^2}}{2} \right\rfloor$ and the function $\phi$ is called one modulo three root square mean labeling of $G$.

**Definition 1.2.** A walk of a graph $G$ is an alternating sequence of points and lines $v_0, x_1, v_1, x_2, \ldots v_n, x_n, v$ beginning and ending with points such that each line $x_i$ is incident with $v_{i-1}$ and $v_i$.

**Definition 1.3.** A walk is called a path if all its points are distinct. A path on vertices is denoted by.

**Definition 1.4.** The corona of two graphs $G_1$ and $G_2$ is the graph $G = G_1 \circ G_2$ formed from one copy of $G_1$ and $|V(G_1)|$ copies of $G_2$ where $i^{th}$ vertex of $G_1$ is adjacent to every vertices in the $i^{th}$ copy of $G_2$.

**Definition 1.5.** The graph $P_n \circ K_1$ is called a **comb**.

**Definition 1.6.** A star graph is a complete bigraph $K_{1,n}$.

**Definition 1.7.** A connected acyclic graph is called a tree.

**Definition 1.8.** Y–tree is a tree obtained by taking three paths of same length and identifying one end point of each path.

### 2. Main Results

**Theorem 2.1.** Any path $P_n$ is a one modulo three root square mean graph.

*Proof.* Let $P_n$ be the path $u_1u_2 \ldots u_n$. Then $V(P_n) = \{u_1, u_2, \ldots, u_n\}$ and $E(P_n) = \{u_iu_{i+1} / 1 \leq i \leq n-1\}$. Define a function $\phi : V(P_n) \to \{0, 1, 3,$
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Figure 1: $P_7$

$\ldots, 3q-2, 3q\}$ by $\phi(u_1) = 0$, $\phi(u_2) = 1$, $\phi(u_i) = 3(i-1)$, $3 \leq i \leq n$. Then $\phi$ induces a bijection $\phi^*: E(P_n) \to \{1, 4, \ldots, 3q-2\}$, where $\phi^*(u_iu_{i+1}) = 3i-2$, $1 \leq i \leq n-1$. Therefore, $\phi$ is a one modulo three root square mean labeling. Hence the path $P_n$ is a one modulo three root square mean graph.

**Example 2.2.** One modulo three root square mean labeling of $P_7$ is given in Figure 1.

**Theorem 2.3.** $nP_m$ is a one modulo three root square mean graph.

**Proof.** Let $v_{i,1}, v_{i,2}, \ldots, v_{i,m}$ be the $i^{th}$ copy of $P_m$ in $nP_m$, $1 \leq i \leq n$. Then $V = \{v_{i,j} / 1 \leq i \leq n, 1 \leq j \leq m\}$ is the vertex set and $E = \{v_{i,j}v_{i,j+1} / 1 \leq i \leq n, 1 \leq j \leq m-1\}$ is the edge set of $nP_m$. Define a function $\phi: V(nP_m) \to \{0, 1, 3, \ldots, 3q-2, 3q\}$ by $\phi(v_{1,1}) = 0$; $\phi(v_{1,2}) = 1$; $\phi(v_{1,j}) = 3(j-1)$, $3 \leq j \leq m-1$; $\phi(v_{i,j}) = 3[m(i-1)+j-i]$, $2 \leq i \leq n, 1 \leq j \leq m-1$; $\phi(v_{i,m}) = 3im-3i-2$, $1 \leq i \leq n$. Then $\phi$ induces a bijection $\phi^*: E(nP_m) \to \{1, 4, \ldots, 3q-2\}$, where $\phi^*(v_{i,j}v_{i,j+1}) = 3[m(i-1)+j-i]+1$, $1 \leq i \leq n, 1 \leq j \leq m-1$. Therefore, $\phi$ is a one modulo three root square mean labeling. Hence $nP_m$ is a one modulo three root square mean graph.

**Example 2.4.** One modulo three root square mean labeling of $4P_7$ is given in Figure 2.

**Theorem 2.5.** $Comb P_n \odot K_1$ is a one modulo three root square mean
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Figure 3: $P_6 \odot K_1$

Proof. Let $P_n$ be the path $u_1u_2\ldots u_n$. Let $v_i$ be the vertex adjacent to $u_i$, $1 \leq i \leq n$. The resultant graph is $P_n \odot K_1$. Here $V(P_n \odot K_1) = \{u_i, v_i/1 \leq i \leq n\}$ and $E(P_n \odot K_1) = \{u_i v_i, u_iu_{i+1}, u_n v_n/1 \leq i \leq n-1\}$. Then $P_n \odot K_1$ has $2n$ vertices and $2n-1$ edges. Define a function $\phi : V(P_n \odot K_1) \rightarrow \{0, 1, 3, \ldots, 3q-2, 3q\}$ by $\phi(u_1) = 0; \phi(u_i) = 6i-5, 2 \leq i \leq n; \phi(v_1) = 1, \phi(v_i) = 6i-6, 2 \leq i \leq n$. Then $\phi$ induces a bijection $\phi^* : E(P_n \odot K_1) \rightarrow \{1, 4, \ldots, 3q-2\}$, where $\phi^*(u_i u_{i+1}) = 6i-2, 1 \leq i \leq n-1; \phi^*(u_i v_i) = 6i-5, 1 \leq i \leq n$. Therefore, $\phi$ is a one modulo three root square mean labeling. Hence Comb $P_n \odot K_1$ is a one modulo three root square mean graph.

Example 2.6. One modulo three root square mean labeling of $P_6 \odot K_1$ is given in Figure 3.

Theorem 2.7. $P_n \odot \tilde{K}_2$ is a one modulo three root square mean graph.

Proof. Let $P_n$ be the path $u_1u_2\ldots u_n$. Let $v_i, w_i$ be the vertices of $i^{th}$ copy of $\tilde{K}_2$. Join $v_i$ and $w_i$ with the vertex $u_i$, $1 \leq i \leq n$. The resultant graph is $P_n \odot \tilde{K}_2$ with $V(P_n \odot \tilde{K}_2) = \{u_i, v_i, w_i/1 \leq i \leq n\}$ and $E(P_n \odot \tilde{K}_2) = \{u_i v_i, u_i w_i, u_iu_{i+1}/1 \leq i \leq n, 1 \leq j \leq n-1\}$. Define a function $\phi : V(P_n \odot \tilde{K}_2) \rightarrow \{0, 1, 3, \ldots, 3q-2, 3q\}$ by $\phi(u_1) = 1; \phi(u_2) = 9; \phi(u_i) = 9i-6, 3 \leq i \leq n; \phi(v_1) = 0; \phi(v_2) = 10; \phi(v_i) = 9(i-1)-2, 3 \leq i \leq n; \phi(w_1) = 6; \phi(w_2) = 15; \phi(w_i) = 9i-5, 3 \leq i \leq n$. Then $\phi$ induces a bijection $\phi^* : E(P_n \odot \tilde{K}_2) \rightarrow \{1, 4, \ldots, 3q-2\}$, where $\phi^*(u_i u_{i+1}) = 9i-2, 1 \leq i \leq n-1; \phi^*(u_i v_i) = 9i-8, 1 \leq i \leq n; \phi^*(u_i w_i) = 9i-5, 1 \leq i \leq n$. Therefore, $\phi$ is a one modulo three root square mean labeling. Hence $P_n \odot \tilde{K}_2$ is a one modulo three root square mean graph.

Example 2.8. One modulo three root square mean labeling of $P_4 \odot \tilde{K}_2$ is given in Figure 4.

Theorem 2.9. A graph obtained by attaching $P_3$ at each vertex of $P_n$ is one modulo three root square mean graph.

Proof. Let $P_n$ be the path $u_1u_2\ldots u_n$. Let $x_i, v_i, w_i$ be the $i^{th}$ copy of $P_3, 1 \leq i \leq n$. Identify the vertex $u_i$ with $x_i$, $1 \leq i \leq n$. The resultant graph is $G$ with $V(G) = \{u_i, v_i, w_i/1 \leq i \leq n\}$ and $E(G) = \{u_i v_i, v_i w_i, v_n w_n, u_n v_n, u_i u_{i+1}/1 \leq
Then G has 3n vertices and 3n-1 edges. Define a function \( \phi : V(G) \to \{ 0, 1, 3, \ldots, 3q-2, 3q \} \) by \( \phi(u_i) = 3i+3, 1 \leq i \leq 2; \phi(u_i) = 9i-6, 3 \leq i \leq n; \phi(v_i) = 11i-10, 1 \leq i \leq 2; \phi(v_i) = 9i-5, 3 \leq i \leq n; \phi(w_i) = 13i-13, 1 \leq i \leq 2; \phi(w_i) = 9i-11, 3 \leq i \leq n. \) Then \( \phi \) induces a bijection \( \phi^* : E(G) \to \{ 1, 4, \ldots, 3q-2 \}, \) where \( \phi^*(u_iu_{i+1}) = 9i-2, 1 \leq i \leq n-1; \phi^*(u_1v_1) = 4; \phi^*(u_2v_2) = 10; \phi^*(u_iv_i) = 9i-5, 3 \leq i \leq n; \phi^*(v_1w_1) = 1; \phi^*(v_2w_2) = 13; \phi^*(v_iw_i) = 9i-8, 3 \leq i \leq n. \) Therefore, \( \phi \) is a one modulo three root square mean labeling. Hence G is a one modulo three root square mean graph.

**Example 2.10.** One modulo three root square mean labeling of G when \( n = 7 \) is given in Figure 5.

**Theorem 2.11.** \( P_n \odot \overline{K}_3 \) is a one modulo three root square mean graph.

**Proof.** Let \( u_1u_2 \ldots u_n \) be the path \( P_n \). Let \( v_i, x_i, y_i, z_i \) be the vertices of \( i^{th} \) copy of \( K_{1,3} \) with central vertex \( v_i \). Identify \( v_i \) with \( u_i, 1 \leq i \leq n. \) The resultant graph is \( G = P_n \odot \overline{K}_3 \) with \( V(G) = \{ u_i, x_i, y_i, z_i / 1 \leq i \leq n \} \) and \( E(G) = \{ u_ix_i, u_iy_i, u_iz_i, u_ju_{j+1} / 1 \leq i \leq n, 1 \leq j \leq n-1 \} \). Then G has 4n vertices and 4n-1 edges. Define a function \( \phi : V(P_n \odot \overline{K}_3) \to \{ 0, 1, 3, \ldots, 3q-2, 3q \} \) by \( \phi(u_1) = 1; \phi(u_2) = 15; \phi(u_i) = 12i-8, 3 \leq i \leq n; \phi(x_1) = 0; \phi(x_i) = 12i-14, 2 \leq i \leq n; \phi(y_1) = 6; \phi(y_2) = 16; \phi(y_i) = 12i-9, 3 \leq i \leq n; \phi(z_1) = 12i-3, 1 \leq i \leq n. \) Then \( \phi \) induces a bijection \( \phi^* : E(P_n \odot \overline{K}_3) \to \{ 1, 4, \ldots, 3q-2 \}, \) where \( \phi^*(u_iu_{i+1}) = 12i-2, 1 \leq i \leq n-1; \phi^*(u_1x_1) = 12i-11, 1 \leq i \leq n; \phi^*(u_1y_1) = \ldots \)
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Figure 6: $P_5 \circ K_3$

12i-8, 1 ≤ i ≤ n; $\phi^*(u_i z_i) = 12i-5, 1 ≤ i ≤ n$. Therefore, $\phi$ is a one modulo three root square mean labeling. Hence $P_n \circ K_3$ is a one modulo three root square mean graph.

**Example 2.12.** One modulo three root square mean labeling of $P_5 \circ K_3$ is given in Figure 6.

**Theorem 2.13.** A graph obtained by attaching the central vertex of $K_{1, 2}$ at each pendent vertex of a comb $P_n \circ K_1$ is a one modulo three root square mean graph.

**Proof.** Let $u_1 u_2 \ldots u_n$ be the path $P_n$ and let $v_i$ be a vertex adjacent to $u_i$, 1 ≤ i ≤ n. The resultant graph is $P_n \circ K_1$. Let $x_i, w_i, y_i$ be the vertices of $i$th copy of $K_{1, 2}$ with the central vertex $w_i$. Identify the vertex $w_i$ with $v_i$, 1 ≤ i ≤ n, we get the required graph $G$. Then $V(G) = \{u_i, v_i, y_i, x_i / 1 ≤ i ≤ n\}$ and $E(G) = \{u_i v_i, v_i x_i, v_i y_i, u_j u_{j+1} / 1 ≤ i ≤ n, 1 ≤ j ≤ n-1\}$ and hence $G$ has 4n vertices and 4n-1 edges. Define a function $\phi : V(G) \rightarrow \{0, 1, 3, \ldots, 3q-2, 3q\}$ by $\phi(u_1) = 6; \phi(u_2) = 13; \phi(u_i) = 12i-8, 3 ≤ i ≤ n; \phi(v_1) = 1; \phi(v_2) = 19; \phi(v_i) = 12i-9, 3 ≤ i ≤ n; \phi(x_1) = 0; \phi(x_2) = 4; \phi(x_i) = 12i-14, 3 ≤ i ≤ n; \phi(y_i) = 9i, 1 ≤ i ≤ 2; \phi(y_i) = 12i-3, 3 ≤ i ≤ n$. Then $\phi$ induces a bijection $\phi^* : E(G) \rightarrow \{1, 4, \ldots, 3q-2\}$, where $\phi^*(u_i u_{i+1}) = 12i-2, 1 ≤ i ≤ n-1; \phi^*(u_i v_i) = 12i-8, 1 ≤ i ≤ n; \phi^*(v_i x_i) = 12i-5, 1 ≤ i ≤ n; \phi^*(v_i y_i) = 12i-11, 1 ≤ i ≤ n$. Therefore, $\phi$ is a one modulo three root square mean labeling. Hence $G$ is a one modulo three root square mean graph.

**Example 2.14.** One modulo three root square mean labeling of $G$ when $n = 5$ is given in Figure 7.

**Remark 2.15.** If $G$ is one modulo three root square mean graph, then the edge must get label with 1 and so two adjacent vertices of $G$ must get the label 0 and 1.

**Theorem 2.16.** Star graph $K_{1, n}$ is a one modulo three root square mean graph if and only if $n ≤ 3$.

**Proof.** Let $G$ be the star graph $K_{1, n}$. Let $V(G) = \{u, u_i / 1 ≤ i ≤ n\}$ and $E(G) = \{uu_i / 1 ≤ i ≤ n\}$. $K_{1, 1}$ is same as $P_2$ and $K_{1, 2}$ is $P_3$. Hence by
Theorem 2.1, \( K_{1,1} \) and \( K_{1,2} \) are one modulo three root square mean graph. One modulo three root square mean labeling for \( K_{1,3} \) is shown in Figure 8.

Suppose \( K_{1,n} \) is a one modulo three root square mean labeling for \( n > 3 \). Then there is a function \( \phi : V(G) \rightarrow \{0, 1, 3, \ldots, 3q-2, 3q\} \). Let \( u \) be the central vertex of \( K_{1,n} \). By remark 2.15, two adjacent vertices are labeled by 0 and 1. Then \( u \) must be labeled with either 0 or 1. The number of edges in \( K_{1,n} \) is \( q = n > 3 \). Here we consider two cases.

Case 1. \( \phi(u) = 0 \)

Then clearly there is no edge with label \( 3q-2 \) (\( q = n > 3 \)), since the labels of the edges \( uu_i \) are less than or equal to \( \left\lceil \sqrt{\frac{0^2 + (3q)^2}{2}} \right\rceil \) or \( \left\lfloor \sqrt{\frac{0^2 + (3q)^2}{2}} \right\rfloor \). That is the labels of the edges \( uu_i \) are less than or equal to 2q. But 3q-2 is greater than or equal to 2q. Hence \( K_{1,n} \) is not a one modulo three root square mean graph.

Case 2. \( \phi(u) = 1 \)

Then clearly there is no edge with label \( 3q-2 \) (\( q = n > 3 \)), since the labels of the edges \( uu_i \) are less than or equal to \( \left\lceil \sqrt{\frac{1^2 + (3q)^2}{2}} \right\rceil \) or \( \left\lfloor \sqrt{\frac{1^2 + (3q)^2}{2}} \right\rfloor \). Hence
$K_{1,n}$ is not a one modulo three root square mean graph. Thus in the above two cases $K_{1,n}$ is not a one modulo three root square mean graph for $n > 3$. Therefore star graph $K_{1,n}$ is a one modulo three root square mean graph if and only if $n \leq 3$.

**Theorem 2.17.** Y-tree is a one modulo three root square mean graph.

**Proof.** Let $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ and $z_1, z_2, \ldots, z_n$ be the three paths of length $n$ each. Identify $x_1$, $y_1$, and $z_1$ and label it as $v$. The resultant graph is $G$ (Y-tree). Here $V(G) = \{v, x_i, y_i, z_i / 2 \leq i \leq n\}$ and $E(G) = \{vx_2, vy_2, vz_2, x_ix_{i+1}, y_iy_{i+1}, z_iz_{i+1} / 2 \leq i \leq n-1\}$. Then $G$ has $3n-2$ vertices and $3n-3$ edges. Define a function $\phi : V(G) \to \{0, 1, 3, \ldots, 3q-2, 3q\}$ by $\phi(v)= 0$, $\phi(x_2) = 1$, $\phi(x_i) = 9i-14$, $4 \leq i \leq n$ and $i$ is even, $\phi(x_i) = 9i-12$, $3 \leq i \leq n$ and $i$ is odd; $\phi(y_2) = 10$, $\phi(y_i) = 9i-6$, $3 \leq i \leq n$; $\phi(z_2) = 6$, $\phi(z_i) = 9i-9$, $3 \leq i \leq n$. Then $\phi$ induces a bijection $\phi^* : E(G) \to \{1, 4, \ldots, 3q-2\}$, where $\phi^*(vx_2) = 1$, $\phi^*(x_ix_{i+1}) = 9i-8$, $2 \leq i \leq n-1$; $\phi^*(vy_2) = 7$, $\phi^*(y_iy_{i+1}) = 9i-2$, $2 \leq i \leq n-1$; $\phi^*(vz_2) = 4$, $\phi^*(z_iz_{i+1}) = 9i-5$, $2 \leq i \leq n-1$. Therefore, $\phi$ is a one modulo three root square mean labeling. Hence Y-tree is a one modulo three root square mean graph.

**Example 2.18.** One modulo three root square mean labeling of $G$ when $n = 7$ is given in Figure 9.

**References**


Figure 9