ESTIMATIONS OF THE SUPPORT OF SOLUTIONS FOR SOME CLASSES OF EVOLUTIONARY SYSTEMS

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Abstract: Nonlinear nonstationary systems used as an approximation to the well-known Bean model of the theory of superconductivity II are investigated. An identical system is considered but with convection, which plays the role of damping. The properties of finiteness of the support of the solutions of the Cauchy problem for nonlinear nonstationary systems in space are studied and estimations for these supports are obtained.

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1. Introduction

In this paper we study the Cauchy problem for the following degenerate quasilinear system of equations with respect to the unknown vector-function $\overline{H}(x,t) = (H_1(x,t), H_2(x,t), H_3(x,t))$:

\begin{align*}
\overline{H}_t + \nabla \times \left( |\nabla \times \overline{H}|^{p-2} \nabla \times \overline{H} \right) &= \overline{F}(x,t), \quad (x,t) \in Q, \\
\nabla \cdot \overline{H} &= 0, \quad (x,t) \in Q, \\
\overline{H}(x,0) &= \overline{H}_0(x), \quad x \in \mathbb{R}^3.
\end{align*}

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Here $Q = \mathbb{R}^3 \times (0, +\infty)$, $p = \text{const} \geq 2$, $\bar{F}(x,t)$ and $\bar{H}_0(x)$ - are the known vector-functions. The symbol $\nabla \times \bar{A}$ means a vector product of vectors $\nabla = \left( \frac{\partial}{\partial x_i} \right)$, $i = 1, 3$ and a vector-function $\bar{A}(x)$; the symbol $\nabla \cdot \bar{A}$ is the scalar product of the vectors $\nabla$ and $\bar{A}(x)$, $|\bar{A}|$ is the length of the vector $\bar{A}$; $\overline{H_t}$ means the derivative with time $t$.

The system of equations (1) - (3) is usually used as an approximation to the Bean model for type II superconductivity [1], [2]. In this case, the vector-function $\bar{H}(x,t)$ has the meaning of the magnetic field. It should be noted that according to the formulation itself, this model means the solution with a compact support. Therefore, it naturally raises the question of establishing the compactness of the support of solution (1) - (3) under the finite data $\bar{H}_0(x)$ and $\overline{F}(x,t)$.

Currently, such finiteness was established in the work [1] under a significant limitation: the unknown field $\bar{H}$ in the problem (1) - (3) is flat and depends only on two variables $x_1$ and $x_2$, i.e. $\bar{H} = (H_1(x_1, x_2, t), H_2(x_1, x_2, t), 0)$.

2. Main Results

For the problem (1) - (3) we introduce the concept of a weak solution: it is a vector-function $\overline{\Phi}(x,t) \in L^2_2(0, T; W^{1,p}_p(\mathbb{R}^3))\forall T > 0$, such that $\nabla \cdot \overline{\Phi}(x,t) = 0$ is almost everywhere in $Q$ and is the following integral identity is true

$$
\int_0^T \int_{\mathbb{R}^3} \left( -\bar{H} \cdot \overline{\Phi}_t + |\nabla \times \bar{H}|^{p-2} \right) \left( \nabla \times \bar{H} \right) \left( \nabla \times \overline{\Phi} \right) \, dx \, dt = 
$$

$$
= \int_{\mathbb{R}^3} \bar{H}_0(x) \cdot \overline{\Phi}(x,0) \, dx
$$

for any finite function with respect to a variable $x$ vector-function $\overline{\Phi}(x,t) \in W^{1}_2(0, T, W^{1,p}_p(\mathbb{R}^3))$, such that $\nabla \cdot \overline{\Phi}(x,t) = 0$, is almost everywhere and $\Phi(x,T) \equiv 0$ in $\mathbb{R}^3$.

From the results of the works [1], [2] it follows that the problem (1) - (3) has a unique weak solution $\overline{H}(x,t)$. Moreover, the equation (1) is fulfilled not only in the weak meaning, but it is also almost everywhere, i.e. a weak solution, in fact, is a strong solution [3], [4]. This solution $\overline{H}(x,t)$ is finite $\forall t > 0$, and also the support of the function $\overline{H}(x,t)$ has the property
\[
\text{supp}(\overline{H} (\cdot, t)) \subset \left\{ x \in \mathbb{R}^3 : |x| < \max \left\{ 4f(t), C (H_0, \Phi) \cdot t^2 \right\} \right\}
\]

where \( k = 5p - 6 \),

\[
f(t) = \sup_{0 \leq \tau \leq t} \inf_{r > 0} \left\{ \overline{F}(x, \tau) \equiv 0, \quad H_0(x) = 0, \quad |x| > r, \quad r > 0 \right\}
\]

- a bound size of the supports of the functions \( H_0(x) \) and \( F(x, t) \).

Further we consider the Cauchy problem for a degenerate quasilinear system of equations of the Bean model type with convection:

\[
\begin{align*}
\overline{H}_t + \nabla \times \left( |\nabla \times \overline{H}|^{p-2} \nabla \times \overline{H} \right) & + \\
+ \frac{2\alpha x}{1 + |x|^2} \times \left( |\nabla \times \overline{H}|^{p-2} \nabla \times \overline{H} \right) & = 0,
\end{align*}
\]

(4)

\((x, t) \in Q_T = \mathbb{R}^3 \times (0, T), \alpha, T > 0, p > 2,\)

\[
\nabla \cdot \overline{H} = 0, \quad (x, t) \in Q_T,
\]

(5)

\[
\overline{H}(x, 0) = H_0(x), \quad x \in \mathbb{R}^3.
\]

(6)

and the Cauchy problem for a degenerate parabolic equation:

\[
\begin{align*}
u_t = \nabla \cdot \left( |\nabla u|^{p-2} \nabla u \right) & + p\nabla \theta \cdot \nabla u |\nabla u|^{p-2},
\end{align*}
\]

(7)

\((x, t) \in Q_T = \mathbb{R}^n \times (0, T), \quad n1, \quad p > 2,\)

\[
u(x, 0) = u_0(x), \quad x \in \mathbb{R}^n.
\]

(8)

Here \( \theta(x) = |x|^{\gamma}, \quad \gamma > 1 \). Equation (7) is nonstationary equation with a \( p \)-Laplacian with a convectional component.

Suppose \( H_0(x) \) is a measurable finite vector-function, that satisfies the following conditions:

\[
\nabla \cdot H_0 = 0, \quad \left( 1 + |x|^2 \right)^\alpha |\nabla \times H_0(x)| \in L_p (\mathbb{R}^3),
\]

\[
|\nabla \times \left( 1 + |x|^2 \right)^\alpha |\nabla \times H_0|^{p-2} \nabla \times H_0| \in L_2 (\mathbb{R}^3).
\]
Let us denote by $W_{p,\alpha}^1(\mathbb{R}^3)$ the Sobolev weight space with a final norm

$$||f||_{W_{p,\alpha}^1(\mathbb{R}^3)} = \left( \int_{\mathbb{R}^3} \left(1 + |x|^2\right)^\alpha \left(|f|^p + |\nabla f|^p\right) dx \right)^{\frac{1}{p}}.$$

Let us call the vector-function $\overline{H}(x,t)$ as a weak solution of the problem (4) - (6), if $\overline{H}(x,t) \in L^2(0,T; W_{p,\alpha}^1(\mathbb{R}^3)) \forall T > 0$, $\nabla \cdot \overline{H}(x,t) = 0$ is almost everywhere in $Q_T$ and the following integral identity is true

$$\int_0^T \int_{\mathbb{R}^3} \left(1 + |x|^2\right)^\alpha \left(-\overline{H} \cdot \overline{\Phi}_t + |\nabla \times \overline{H}|^{p-2} \right) (\nabla \times \overline{H}) (\nabla \times \overline{\Phi}) \, dx dt =$$

$$= \int_{\mathbb{R}^3} \left(1 + |x|^2\right)^\alpha \overline{H}_0(x) \cdot \overline{\Phi}(x,0) \, dx$$

for any finite with respect to a variable $x$ function $\overline{\Phi}(x,t) \in W^1_2(0,T; W_{p,\alpha}^1(\mathbb{R}^3))$ such that $\nabla \cdot \overline{\Phi}(x,t) = 0$ almost everywhere and $\overline{\Phi}(x,T) = 0$ in $\mathbb{R}^3$. As for the problem (7), (8), let us call its weak solution the function $u(x) \in L^2(0,T; W_{p,\omega(x)}^1(\mathbb{R}^n)) \forall T > 0$ such that the integral identity is realized

$$\int_0^T \int_{\mathbb{R}^n} \omega(x) \left(-u \varphi_t + |\nabla u|^{p-2} \nabla u \nabla \varphi\right) \, dx dt = \int_{\mathbb{R}^n} \omega(x) u_0(x) \varphi(x,0) \, dx$$

for any finite with respect to a variable $x$ function $\varphi(x,t) \in W^1_2(0,T; W^1_p(\mathbb{R}^n))$ and $\varphi(x,T) \equiv 0$ in $\mathbb{R}^n$.

Here $W_{p,\omega(x)}^1(\mathbb{R}^n)$ is the space of measurable functions having generalized derivatives in the Sobolev meaning for which the following norm is finite

$$||f||_{W_{p,\omega(x)}^1(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} \omega(x) |f|^p \, dx \right)^{\frac{1}{p}} + \left( \int_{\mathbb{R}^n} \omega(x) |\nabla f|^p \, dx \right)^{\frac{1}{p}},$$

$$\omega(x) = \exp(p\theta(x)).$$
For the problems (4) - (6) and (7), (8) there are respectively unique weak solutions \( H(x, t) \) and \( u(x, t) \) which in fact are strong solutions. These solutions in \( Q_\infty \) are finite \( \forall t > 0 \) and the following estimations of a size of the supports of the solution are held:

\[
\text{supp}(H(x, t)) \subset B_{R_H(t)} = \left\{ x \in \mathbb{R}^3 : |x| < R_H(t) \right\},
\]

\[
\text{supp}(u(x, t)) \subset B_{R_u(t)} = \left\{ x \in \mathbb{R}^n : |x| < R_u(t) \right\},
\]

where

\[
R_H(t) = 4R_0 + C \left( \int_{\mathbb{R}^3} \left( 1 + |x|^2 \right)^\alpha |H_0|^2 \, dx \right)^{\frac{p-2}{2}} t^k
\]

\[
R_u(t) = (\ln(t))^\frac{1}{\gamma}, \quad k = (3 + 2\alpha)(p - 2) + 2p,
\]

\[
\text{supp}(H_0) \in B_{R_0} = \left\{ x \in \mathbb{R}^3 : |x| < R_0 \right\},
\]

\[
\text{supp}(u_0) \in B_{R_0} = \left\{ x \in \mathbb{R}^n : |x| < R_0 \right\}.
\]

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References


