SOME PROPERTIES OF CONTRA $\check{g}$ CONTINUOUS FUNCTIONS

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Abstract: In [4], Dontchev introduced and investigated a new notion of continuity called contra continuity. Later, Jafari and Noiri [3] introduced and investigated the concept of contra $\alpha$ continuous. The aim of this paper is to introduce and study the concept of contra $\check{g}$ continuous and the relationship with other contra functions and their characteristics are obtained.

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1. Introduction

In 1996, Dontchev [4] introduced the notion of contra continuous functions. In 2007, Caldas, Jafari, Noiri and Simoes [2] introduced a new class of functions called generalized contra continuous (contra $g$ - continuous) functions. They defined a function $f : X \rightarrow Y$ to be contra $g$ continuous if pre image every open subsets of $Y$ is $g$ closed in $X$. New types of contra generalized continuity such as contra $g^#p$ continuous function by Alli [7] and contra $gs$ continuous [5] and contra $\alpha^*$ continuous functions, almost contra $\alpha^*$ continuous function...
[12], contra πgr continuous, almost contra πgr continuous [6]. The purpose of this paper is to introduce some properties of notion of contra ĝ continuity via the concept ĝ closed set, ĝ continuity.

2. Preliminaries

Throughout this paper,\((X, \tau), (Y, \sigma)\) or \(X, Y\) represent non empty topological spaces on which no separation axioms are assumed unless otherwise mentioned. Let \(A\) be a subset of a space \(X\). The closure and interior of \(A\) are denoted by \(\text{Cl}(A)\) and \(\text{int}(A)\), respectively.

Definition: 2.1
1. A function \(f : (X, \tau) \rightarrow (y, \sigma)\) is called contra continuous [4] if \(f^{-1}(v)\) is closed set in \((X, \tau)\) for every open set \(v\) in \((Y, \sigma)\)

2. A function \(f : (X, \tau) \rightarrow (y, \sigma)\) is called contra \(g\) continuous [2] if \(f^{-1}(v)\) is \(g\)-closed set in \((X, \tau)\) for every open set \(v\) in \((Y, \sigma)\)

3. A function \(f : (X, \tau) \rightarrow (y, \sigma)\) is called contra \(\alpha\) continuous [3] if \(f^{-1}(v)\) is \(\alpha\)-closed set in \((X, \tau)\) for every open set \(v\) in \((Y, \sigma)\)

4. A function \(f : (X, \tau) \rightarrow (y, \sigma)\) is called contra semi-continuous [5] if \(f^{-1}(v)\) is semi-closed set in \((X, \tau)\) for every open set \(v\) in \((Y, \sigma)\)

5. A function \(f : (X, \tau) \rightarrow (y, \sigma)\) is called contra \(gs\) continuous (resp.contra \(sg\)-continuous) if the preimage of every open subset of \(Y\) is \(gs\) -closed set (resp.\(sg\) closed)\[5\] in \(X\)

6. A function \(f : (X, \tau) \rightarrow (y, \sigma)\) is called contra \(g\alpha\) continuous [7] if \(f^{-1}(v)\) is \(g\alpha\) -closed set in \((X, \tau)\) for every open set \(v\) in \((Y, \sigma)\)

7. A function \(f : (X, \tau) \rightarrow (y, \sigma)\) is called contra-pre-continuous [8] if \(f^{-1}(v)\) is pre-closed set in \((X, \tau)\) for every open set \(v\) in \((Y, \sigma)\)

8. A function \(f : (X, \tau) \rightarrow (y, \sigma)\) is called contra \(g^#p\) - continuous [7] if \(f^{-1}(v)\) is \(g^#p\) - closed set in \((X, \tau)\) for every open set \(v\) in \((Y, \sigma)\)

9. A function \(f : (X, \tau) \rightarrow (y, \sigma)\) is called contra \(gb\) - continuous [9] if \(f^{-1}(v)\)
is $gb$-closed set in $(X, \tau)$ for every open set $v$ in $(Y, \sigma)$

10. A function $f : (X, \tau) \rightarrow (y, \sigma)$ is called contra $\omega$-continuous [14] if $f^{-1}(v)$ is $\omega$-closed set in $(X, \tau)$ for every open set $v$ in $(Y, \sigma)$

11. A function $f : (X, \tau) \rightarrow (y, \sigma)$ is called contra $g\alpha$-continuous [7] if $f^{-1}(v)$ is $g\alpha$-closed set in $(X, \tau)$ for every open set $v$ in $(Y, \sigma)$

Definition: 2.2
Let $A$ be a subset of a topological space $(X, \tau)$. The set $\cap\{U \in \tau / A \subset U\}$ is called the kernel of $A$

Lemma 2.3 [3]
The following properties hold for subsets $A, B$ of a space $X$:
(1) $x \in ker(A)$ if and only if $A \cap F \neq \emptyset$, for any $F \in C(X, x)$
(2) $A \subset ker(A)$ and $A = ker(A)$ if $A$ is open in $X$
(3) if $A \subset B$, then $ker(A) \subset ker(B)$.

Theorem 2.4 [15]
Let $(X, \tau)$ be a topological spaces. Then
(1) Every closed set is $\check{g}$ closed.

(2) Every $\check{g}$-closed set is $\check{g}\alpha$ closed.

(3) Every $\check{g}$-closed set is $sg$ closed.

(4) Every $\check{g}$-closed set $g$ closed.

(5) Every $\check{g}$-closed set is $\alpha g$ closed.

(6) Every $\check{g}$-closed set is $gs$ closed.

(7) Every $\check{g}$-closed set is $gsp$ closed.
3. Contra ĝ-Continuous Functions

Definition: 3.1
A function $f : (X, \tau) \to (Y, \sigma)$ is called contra ĝ continuous function if $f^{-1}(O)$ is ĝ closed in $(X, \tau)$ for every open set $O$ in $(Y, \sigma)$.

Example: 3.2
Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, \{b\}, X\}$ and $\sigma = \{\phi, \{a\}, X\}$ we have, $\tilde{G}(C(X)) = \{\phi, \{a, c\}, X\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity map. Then $f$ is called contra ĝ continuous.

Theorem: 3.3
Every contra continuous is a contra ĝ continuous function.

Proof:
Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Let $O$ be an open set in $(Y, \sigma)$. Since, $f$ is contra continuous, then, $f^{-1}(O)$ is closed in $(X, \tau)$. [Since, every closed set is ĝ closed], then, $f^{-1}(O)$ is ĝ-closed in $(X, \tau)$.
Therefore, $f$ is contra ĝ continuous.

Examples: 3.4
Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, \{b\}, X\}$ and $\sigma = \{\phi, \{a\}, Y\}$ we have, $\tilde{G}(C(X)) = \{\phi, \{a, b\}, X\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity map. Then $f$ is called contra ĝ continuous but not contra ĝ continuous, since $f^{-1}\{c\} = \{c\}$ is not contra ĝ continuous.

REMARKS: 3.5
(i)Every contra ĝ continuous is contra ĝ$\alpha$ continuous. But converse of the above need not be true.

Examples: 3.5.1
Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, \{a\}, X\}$ and $\sigma = \{\phi, \{a, b\}, Y\}$ we have, $\tilde{G}(C(X)) = \{\phi, \{b, c\}, X\}$ and $\tilde{G}_\alpha C(X) = \{\phi, \{b\}, \{c\}, \{b, c\}, X\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity map. Then $f$ is called contra ĝ$\alpha$ continuous but not contra ĝ continuous, since $f^{-1}\{c\} = \{c\}$ is not contra ĝ continuous.

(ii)Every contra ĝ continuous is contra $sg$ continuous. But converse of the above need not be true.

Examples: 3.5.2
Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, \{a\}, X\}$ and $\sigma = \{\phi, \{a, b\}, Y\}$ we have,
\( \hat{G}(C(X)) = \{\phi, \{b, c\}, X\} \) and \( SGC(X) = \{\phi, \{b\}, \{c\}, \{b, c\}, X\} \). Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be the identity map. Then \( f \) is called contra \( sg \) continuous but not contra \( \hat{g} \) continuous, since \( f^{-1}\{c\} = \{c\} \) is not contra \( \hat{g} \) continuous

(iii) Every contra \( \hat{g} \) continuous is contra \( g \) continuous. But converse of the above need not be true.

Examples: 3.5.3
Let \( X = Y = \{a, b, c\}, \tau = \{\phi, \{a\}, X\} \) and \( \sigma = \{\phi, \{a, b\}, Y\} \) we have, \( \hat{G}(C(X)) = \{\phi, \{b, c\}, X\} \) and \( GC(X) = \{\phi, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\} \).

Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be the identity map. Then \( f \) is called contra \( g \) continuous but not contra \( \hat{g} \) continuous, since \( f^{-1}\{c\} = \{c\} \) is not contra \( \hat{g} \) continuous

(iv) Every contra \( \hat{g} \) continuous is contra \( \alpha g \) continuous. But converse of the above need not be true.

Examples: 3.5.4
Let \( X = Y = \{a, b, c\}, \tau = \{\phi, \{a\}, X\} \) and \( \sigma = \{\phi, \{a, b\}, Y\} \) we have, \( \hat{G}(C(X)) = \{\phi, \{b, c\}, X\} \) and \( G\alpha C(X) = \{\phi, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\} \).

Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be the identity map. Then \( f \) is called contra \( \alpha g \) continuous but not contra \( \hat{g} \) continuous, since \( f^{-1}\{c\} = \{c\} \) is not contra \( \hat{g} \) continuous

(v) Every contra \( \hat{g} \) continuous is contra \( gs \) continuous. But converse of the above need not be true.

Example: 3.5.5
Let \( X = Y = \{a, b, c\}, \tau = \{\phi, \{a\}, X\} \) and \( \sigma = \{\phi, \{a, b\}, Y\} \) we have, \( \hat{G}(C(X)) = \{\phi, \{b, c\}, X\} \) and \( GSC(X) = \{\phi, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\} \).

Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be the identity map. Then \( f \) is called contra \( gs \) continuous but not contra \( \hat{g} \) continuous, since \( f^{-1}\{c\} = \{c\} \) is not contra \( \hat{g} \) continuous

(vi) Every contra \( \hat{g} \) continuous is contra \( gsp \) continuous. But converse of the above need not be true.

Examples: 3.5.6
Let \( X = Y = \{a, b, c\}, \tau = \{\phi, \{b\}, X\} \) and \( \sigma = \{\phi, \{a, b\}, Y\} \) we have, \( \hat{G}(C(X)) = \{\phi, \{a, c\}, X\} \) and \( GSPC(X) = \{\phi, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\} \).

Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be the identity map. Then \( f \) is called contra \( gsp \) continuous but not contra \( \hat{g} \) continuous, since \( f^{-1}\{c\} = \{c\} \) is not contra \( \hat{g} \) continuous.

Theorem: 3.6
For a function \( f : (X, \tau) \rightarrow (y, \sigma) \) the following continuous are equivalent

1. \( f \) is a contra \( \hat{g} \) continuous.
(2) for every closed subset F of Y, \( f^{-1}(F) \in \tilde{g} O(X, \tau) \).

(3) for each \( x \in X \), and each \( F \in C(Y, f(x)) \) there exists \( U \in \tilde{g}O(X, x) \) such that \( f(U) \subseteq F \).

(4) \( f(\text{cl}_{\tilde{g}}(A)) \subseteq \text{ker}(f(A)) \) for every subset A of X.

(5) \( \text{cl}_{\tilde{g}}(f^{-1}(B)) \subseteq f^{-1}(\text{ker}(B)) \) for every subset B of Y.

Proof:

(1) \( \iff \) (2) and (2) \( \implies \) (3) are obvious.

(3) \( \implies \) (2): Let \( F \) be any closed set of Y and \( x \in f^{-1}(F) \). Then \( f(x) \in F \) and there exists \( U_x \in \tilde{g}O(X, x) \), such that \( f(U_x) \subseteq F \). Therefore we obtain \( f^{-1}(F) = \bigcup \{ U_x : x \in f^{-1}(F) \} \), which is \( \tilde{g} \) open in X.

(2) \( \implies \) (4): Let A be any subset of X. Suppose that \( Y \not\in \text{ker}(f(A)) \). Then by lemma (2.3), there exist \( F \in C(Y, f(x)) \) such that \( f(A) \cap F = \phi \). Thus, we have \( A \cap f^{-1}(F) = \phi \) and since \( f^{-1}(F) \) is \( \tilde{g} \) open, then we have \( \tilde{g}cl(A) \cap f^{-1}(f) = \phi \). Therefore we obtain \( f(\tilde{g}cl(A) \cap f^{-1}(F) = \phi) \) and \( y \not\in \tilde{g}cl(A) \). This implies that \( f[\tilde{g}cl(A)] \subseteq \text{ker}(f(A)) \).

(4) \( \implies \) (5): Let B be any subset of Y. By (4) and lemma (2.3), we have \( f[\tilde{g}(\text{cl}f^{-1}(B))] \subseteq \text{ker}(f(f^{-1}B)) \subseteq \text{ker}(B) \). Thus \( \tilde{g}cl(f^{-1}(B)) \subseteq f^{-1}(\text{ker}(B)) \)

(5) \( \implies \) (1): Let V be any open set of Y. Then by lemma (2.3), We have \( \tilde{g}(\text{cl}(f^{-1}(V))) \subseteq f^{-1}(\text{ker}(V)) = f^{-1}(V) \) and \( \tilde{g}cl(f^{-1}V)) = f^{-1}(V) \). This show that \( f^{-1}(V) \) is \( \tilde{g} \) - closed in X.

Theorem: 3.7

If a function \( f : X \to Y \) is contra \( \tilde{g} \) continuous and Y is regular ,Then \( f \) is \( \tilde{g} \) continuous.

Proof:

Let \( x \) be an arbitrary point of X and let V be an open set of Y containing \( f(x) \), since Y is regular, there exists an open set G in Y containing \( f(x) \) such that \( \text{cl}(G) \subseteq V \). Since \( f \) is contra \( \tilde{g} \) continuous, by theorem: 3.6(3) there exists \( U \subseteq \tilde{g} O(X, x) \) such that \( f(U) \subseteq \text{cl}(G) \). Then \( f(U) \subseteq \text{cl}(G) \subseteq V \) hence \( f \) is \( \tilde{g} \) continuous.

Definition: 3.8

A space \((X, \tau)\) is said to be

(a) \( \tilde{g} \)- space if every \( \tilde{g} \)-open set of X is open in X.
(b) locally $\sim$-indiscrete if every $\sim$-open set of $X$ is closed in $X$

The following two results follow immediately.

**Theorem: 3.9**
If a function $f : X \to Y$ is contra $\sim$ continuous and $X$ is $\sim$ space, then $f$ is contra continuous.

**Proof:**
Let $V \in O(Y)$. Then $f^{-1}(V)$ is $\sim$-closed in $X$. Since $X$ is $\sim$ space, $f^{-1}(V)$ is closed in $X$. Thus, $f$ is contra continuous.

**Theorem: 3.10**
Let $X$ be locally $\sim$-indiscrete. If a function $f : X \to Y$ is contra $\sim$ continuous, then it is continuous.

**Proof:**
Let $V \in O(Y)$. Then $f^{-1}(V)$ is $\sim$-closed in $X$. Since $X$ is locally $\sim$-indiscrete space, $f^{-1}(v)$ is open in $X$. Thus $f$ is continuous.

4. CLOSED GRAPHS

Recall that for a function $f : X \to Y$, the subset $\{(x, f(x)) : x \in X\} \subseteq X \times Y$ is called the graph of $f$ and is denoted by $G(f)$.

**Definition: 4.1**
The graph $G(f)$ of a function $f : (x, \tau) \to (y, \sigma)$ is said to be contra $\sim$-closed graph in $X \times Y$ if for each $(x, y) \in (X, Y) - G(f)$ there exist $U \in \sim O(X, x)$ and $V \in C(Y, y)$ such that $(U \times V) \cap G(f) = \emptyset$.

**Lemma: 4.2**
The graph $G(f)$ of a function $f : (x, \tau) \to (y, \sigma)$ is contra $\sim$ closed in $X \times Y$ if and only if for each $(x, y) \in (X, Y) - G(f)$ there exist $U \in \sim O(X, x)$ and $V \in C(Y, y)$ such that $f(U) \cap V = \emptyset$.

**Proof:**
We shall prove that $f(U) \cap V = \emptyset \iff (U \times V) \cap G(f) = \emptyset$. Let $(U \times V) \cap G(f) \neq \emptyset$. Then there exist $(x, y) \in (X, Y)$ and $(x, y) \in G(f)$. This implies that $x \in U$, $y \in V$ and $y = f(x) \in V$. Therefore, $f(U) \cap V \neq \emptyset$. Hence the result follows.

**Theorem: 4.3**
Let $f : X \to Y$ be a function and let $g : X \to X \times Y$ be the graph function
of $f$ defined by $g(x) = (x, f(x))$ for every $x \in X$. If $g$ is contra $\check{g}$-continuous, then $f$ is contra $\check{g}$ continuous.

Proof:
Let $U$ be an open set in $Y$, then $X \times U$ is an open set in $X \times Y$. Since $g$ is contra $\check{g}$ continuous, it follows that $f^{-1}(U) = g^{-1}(X \times U)$ is an $\check{g}$-closed in $X$. Thus $f$ is contra $\check{g}$ continuous.

Theorem: 4.4
If $f : X \to Y$ is contra $\check{g}$ continuous and $Y$ is Urysohn, then $G(f)$ is contra $\check{g}$-closed in $X \times Y$

Proof:
Let $(x, y) \in (X, Y)$- $G(f)$. Then $y \neq f(x)$ and there exist open sets $V, W$ such that $f(x) \in V, y \in W$ and $\text{cl}(V) \cap \text{cl}(W) = \emptyset$. Since $f$ is contra $\check{g}$ continuous, there exists $U \in \check{g} \text{O}(X, x)$ such that $f(U) \subseteq \text{cl}(V)$. Therefore we obtain $f(U) \cap \text{cl}(W) = \emptyset$. This shows that $G(f)$ is contra $\check{g}$-closed.

Theorem: 4.5
If $f : X \to Y$ is contra $\check{g}$ continuous, $g : X \to Y$ contra continuous and $Y$ is Urysohn, then $E = \{x \in X : f(x) = g(x)\}$ is $\check{g}$-closed in $X$.

Proof:
Let $x \in X - E$. Then $f(x) \neq g(x)$. Since $Y$ is Urysohn, there exists open sets $V$ and $W$ such that $f(x) \in V, g(x) \in W$ and $\text{cl}(V) \cap \text{cl}(W) = \emptyset$. Since $f$ is contra $\check{g}$ continuous, then $f^{-1}(\text{cl}(V))$ is $\check{g}$-open in $X$ and $g$ is contra continuous, then $g^{-1}(\text{cl}(W))$ is open in $X$. Let $U = f^{-1}(\text{cl}(V))$ and $G = g^{-1}(\text{cl}(W))$. Then $U$ and $G$ contain $x$. Set $A = U \cap G$ is $\check{g}$-open in $X$. And $f(A) \cap g(A) \subseteq f(U) \cap g(G) \subseteq \text{cl}(V) \cap \text{cl}(W) = \emptyset$. Hence $f(A) \cap g(A) = \emptyset$ and $A \cap E = \emptyset$ where $A$ is $\check{g}$-open therefore $x \notin \check{g} \text{cl}(E)$. Thus $E$ is $\check{g}$-closed in $X$.

Theorem: 4.6
If $f : X \to Y$ is $\check{g}$-continuous and $Y$ is $T_1$, then $G(f)$ is contra $\check{g}$-closed in $X \times Y$.

Proof:
Let $(x, y) \in (X \times Y) - G(f)$. Then $Y \neq f(x)$ and there exist open set $V$ of $Y$, such that $f(x) \in V$ and $y \notin V$. Since $f$ is $\check{g}$ continuous, there exists $U \in \check{g} \text{O}(X, x)$ such that $f(U) \subseteq V$. Therefore, we obtain $f(U) \cap (Y - V) = \emptyset$ and $(Y - V) \in \text{C}(Y, y)$. This shows that $G(f)$ is contra $\check{g}$-closed in $X \times Y$. 

References


