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ON REGULAR TERNARY Γ-SEMIGROUPS

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Abstract: In this paper we study some interesting properties of regular ternary Γ-semigroups, completely regular ternary Γ-semigroups, intra-regular ternary Γ-semigroups and characterize them by using various ternary Γ-ideals of ternary Γ-semigroups.

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1. Introduction

In [10], Vasile introduced and studied the notion of regular ternary rings. T.K. Dutta, S. Kar and B.K. Maity. In [2 3], Dutta and Kar introduced and studied the notion of regular ternary semirings. Recently, a number of mathematicians

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have worked on ternary structures, see [1, 6] and references there in. In [11, 12] M. Vasantha, D. Madhusudhana Rao and M. Venkateswara Rao introduced the concept of ternary Γ-semigroup and made a study of some special element in ternary Γ-semigroup and ternary Γ-ideals in ternary Γ-semigroups and characterised ternary Γ-ideals. In [13] M. Vasantha and D. Madhusudhana Rao discussed about prime Γ-radicals in ternary Γ-semigroups.

2. Preliminaries

Definition 2.1. (see [11]) Let T and Γ be two non-empty set. Then T is said to be Ternary Γ-semigroup if there exist a mapping from $T \times \gamma \times T \times \Gamma \times T$ to T which maps $(x_1, \alpha, x_2, \beta, x_3) \to [x_1 \alpha x_2 \beta x_3]$ satisfying the condition

$$[[x_1 \alpha x_2 \beta x_3] \gamma x_4 \delta x_5] = [x_1 \alpha [x_2 \beta x_3 \gamma x_4] \delta x_5] = [x_1 \alpha x_2 \beta [x_3 \gamma x_4 \delta x_5]],$$

$\forall x_i \in T$, $1 \leq i \leq 5$ and $\alpha, \beta, \gamma, \delta \in \Gamma$.

Note 2.2. (see [11]) For the convenience, we write $x_1 \alpha x_2 \beta x_3$ instead of $[x_1 \alpha x_2 \beta x_3]$.

Note 2.3. (see [11]) Let T be a ternary Γ-semigroup. If A, B and C are three subsets of T, we shall denote the set

$$A\Gamma B \Gamma C = \{a \alpha b \beta c : a \in A, b \in B, c \in C, \alpha, \beta \in \Gamma\}.$$

Definition 2.4. (see [11]) An element a of a ternary Γ-semigroup T is said to be an identity provided $a \alpha a \beta t = t \alpha a \beta a = a \alpha t \beta a = t$ $\forall t \in T$, $\alpha, \beta \in \gamma$.

Definition 2.5. (see [11]) A ternary Γ-semigroup T is said to be commutative provided $a \Gamma b \Gamma c = b \Gamma c \Gamma a = c \Gamma a \Gamma b = b \Gamma a \Gamma a \Gamma c = c \Gamma b \Gamma a = a \Gamma c \Gamma b$ for all $a, b, c \in T$.

Note 2.6. (see [11]) Let T be a ternary Γ-semigroup and $a, b \in T$ Then $a \alpha a \alpha b$ is denoted by $(a\alpha)^2 b$ and consequently $a \alpha a \alpha a \alpha .. (nterms) b$ is denoted by $(a\alpha)^n b$.

Definition 2.7. A ternary Γ-semigroup T is said to be:

(i) strongly left Γ-cancellative (SLC) if $a \Gamma b \Gamma x = a \Gamma b \Gamma y \Rightarrow x = y$ for all $a, b, x, y \in T$.

(ii) strongly right Γ-cancellative (SRC) if $x \Gamma a \Gamma b = y \Gamma a \Gamma b \Rightarrow x = y$ for all $a, b, x, y \in T$.

(iii) strongly lateral Γ-cancellative (SMC) if $a\Gamma x\Gamma b = a\Gamma y\Gamma b \Rightarrow x = y$ for all $a, b, x, y \in T$.

(iv) strongly Γ-cancellative (SC) if it is strongly left Γ-Cancellative, strongly right Γ-Cancellative and strongly lateral Γ-cancellative.

Definition 2.8. (see [11]) An element a of a ternary Γ-semigroup T is said to be an α-idempotent element provided $a\alpha a\alpha a =$ a.

Note 2.9. The set of all idempotent elements in a ternary Γ-semigroup T is denoted by $E_\alpha(T)$.

Example 2.10. Every identity, zero elements are α-idempotent elements.

Definition 2.1. (see [11] An element a of a ternary Γ-semigroup T is said to be an (α, β)-idempotent element provided a α a β a = a for all α , $\beta \in \Gamma$.

Note 2.12. (see [11]) If an element a of a ternary Γ- semigroup T is an idempotent, $a\Gamma a\Gamma a = a$.

In the following we introduce proper idempotent element ternary Γ-semigroup.

Definition 2.13. (see [11]) An element a of a ternary Γ-semigroup T is said to be a proper idempotent element provided a is an idempotent which is not the identity of T if identity exists.

Definition 2.14. (see [11]) A ternary Γ-semigroup T is said to be an idempotent ternary Γ-semigroup provided every element of S is αidempotent for some $\alpha \in \Gamma$.

Definition 2.15. (see [11]) A ternary Γ-semigroup T is said to be a strongly idempotent ternary Γ-semigroup or ternary Γ-band provided every element in T is an α idempotent for some $\alpha \in \Gamma$.

Definition 2.16. A pair (a, b) of elements in a ternary Γ-semigroup T is said to be an idempotent pair if $a\Gamma b\Gamma(a\Gamma b\Gamma x) = a\Gamma b\Gamma x$ and $(x\Gamma a\Gamma b)\Gamma a\Gamma b = x\Gamma a\Gamma b$ for all $x \in T$.

Definition 2.17. Two idempotent pairs (a, b) and (c, d) of a ternary Γ-semigroup T are said to be equivalent, in notation we write $(a, b) \sim (c, d)$, if $a\alpha b\beta x = c\alpha d\beta x$ and $x\alpha a\beta b = x\alpha c\beta d$ for all $x \in T$ and for all $\alpha, \beta \in \Gamma$.

Note 2.18. Two idempotent pairs (a, b) and (c, d) of a ternary Γ-semigroup T are equivalent, in notation we write $(a, b) \sim (c, d)$, iff $a\Gamma b\Gamma x = c\Gamma d\Gamma x$ and $x\Gamma a\Gamma b = x\Gamma c\Gamma d$ for all $x \in T$.

Definition 2.19. (see [11]) A left ternary Γ-ideal A of a ternary Γ-semigroup T is said to be the principal left (lateral, right) ternary Γ-ideal generated by a if A is a left(lateral, right) ternary Γ-ideal generated by $\{a\}$ for some $a \in T$. It is denoted by $L(a)\,[M\,(a)\,, R\,(a)]$ or $< a >_l$ $(< a >_m, < a >_r)$.

3. Regular Ternary Γ-Semigroups

In the following we are introducing regular element and regular ternary Γ-semigroup.

Definition 3.1. (see [11]) An element a of a ternary Γ-semigroup T is said to be regular if there exist $x, y \in T$ and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $a\alpha x\beta a\gamma y\delta a = a$.

Subsequently, modified the definition of regular ternary Γ-semigroup as follows:

Definition 3.2. An element a in a ternary Γ-semigroup T is called regular if there exists an element $x \in T$ and $\alpha, \beta \in I$ such that $a\alpha x\beta a = a$.

Definition 3.3. (see [11]) A ternary Γ-semigroup T is said to be regular ternary Γ-semigroup provided every element is regular.

Clearly, we see that the above definition of regular ternary Γ-semigroup is equivalent to the definition of regular ternary Γ-semigroup.

Example 3.4. (see [11]) Let $T = \{0, a, b\}$ and Γ be any nonempty set. If we define a binary operation on T as the following Cayley table, then T is a ternary Γ-semigroup.

*	0	a	b
0	0	0	0
a	0	a	a
b	0	b	b

Define a mapping from $T \times \Gamma \times T \times \Gamma \times T$ to T as $a\alpha b\beta c = abc$ for all $a, b, c \in T$ and $\alpha, \beta \in \Gamma$. Then T is regular ternary Γ-semigroup.

Theorem 3.5. (see [11]) Every α-idempotent element in a ternary Γ-semigroup is regular.

Theorem 3.6. If a ternary Γ-semigroup T is a regular ternary Γ-semigroup then every principal ternary Γ-ideal is generated by a α-idempotent for some $\alpha \in \Gamma$.

In the following we are introducing the notion of left regular, lateral regular right regular, intra regular and completely regular elements of a ternary Γ-semigroup and completely regular ternary Γ-semigroup.

Definition 3.7. An element a of a ternary Γ-semigroup T is said to be left regular if there exist $x \in T$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha a\beta x$. i.e, $a \in a\Gamma a\Gamma T$.

Definition 3.8. An element a of a ternary Γ-semigroup T is said to be right regular if there exist $x \in T$ and $\alpha, \beta \in \Gamma$ such that $a = x\alpha a\beta a$. i.e., $a \in T\Gamma a\Gamma a$.

Definition 3.9. An element a of a ternary Γ-semigroup T is said to be completely regular if there exist $x \in T$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$ and the idempotent pair $(a, x), (x, a)$ are equivalent.

Definition 3.10. (see [11]) An element a of a ternary Γ-semigroup T is said to be intra regular if there exist $x, y \in T$ and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $a = x\alpha a\beta a\gamma a\delta y$.

Theorem 3.11. (see [11]) Let T be a ternary Γ-semigroup and $a \in T$. If a is a completely regular element, then a is regular, left regular and right regular.

In the following we are introducing the notion of mid unit of a ternary Γ-semigroup.

Definition 3.12. An element a of a ternary Γ-semigroup T is said to be a mid-unit provided $x\Gamma a\Gamma y\Gamma a\Gamma z = x\Gamma y\Gamma z$ for all $x, y, z \in T$.

Definition 3.13. Let T be a ternary Γ-semigroup, $a \in T$ and $\alpha, \beta \in \Gamma$. An element $b \in T$ is said to be an (α, β)-inverse of a if $a = a\alpha b\beta a$ and $b = b\alpha a\beta b$.

Theorem 3.14. Let T be a ternary Γ-semigroup and $a \in T$. Then a is a regular element if and only if a has an (α, β)-inverse.

Theorem 3.15. Let T be a ternary Γ -semigroup and $a \in T$. Then a is a regular element if and only if a has an (α, β)-inverse.

Proof. Suppose that a is a regular element. Then $a = a\alpha b\beta a$ for some $b \in S$ and $\alpha, \beta \in \Gamma$. Let $x = b\alpha a\beta b \in S$. Now

$$a\alpha x\beta a = a\alpha\,(b\beta a\alpha b)\,\beta a = (a\alpha b\beta a)\,\alpha b\beta a = a\alpha b\beta a = a$$

and

$$\begin{aligned} x\beta a\alpha x = (b\beta a\alpha b)\beta a\alpha(b\beta a\alpha b) &= b\beta(a\alpha b\beta a)\alpha(b\beta a\alpha b) \\ &= \beta a\alpha(b\beta a\alpha b) = b\beta(a\alpha b\beta a)\alpha b = b\beta a\alpha b = x. \end{aligned}$$

Therefore, $x = b\beta a\alpha b$ is the (α, β)-inverse of a.

Conversely suppose that b is an (α, β)-inverse of a. Then $a = a\alpha b\beta a$ and $b = b\beta a\alpha b$. Therefore, $a = a\alpha b\beta a$ and hence a is regular.

Note 3.16. A nonempty subset A of a ternary Γ-semigroup T is said to be left ternary Γ-idealof T if $b, c \in T, a \in A$ and $\alpha, \beta \in \Gamma$ implies $b\alpha c\beta a \in A$.

Note 3.17. A nonempty subset A of a ternary Γ-semigroup T is said to be lateral ternary Γ-idealof T if $b, c \in T, a \in A$ and $\alpha, \beta \in \Gamma$ implies $b\alpha a\beta c \in A$.

Note 3.18. A nonempty subset A of a ternary Γ-semigroup T is said to be right ternary Γ-idealof T if $b, c \in T, a \in A$ and $\alpha, \beta \in \Gamma$ implies $a\alpha b\beta c \in A$.

Note 3.19. A nonempty subset A of a ternary Γ-semigroup T is said to be two sided ternary Γ-idealof T if $b, c \in T, a \in A$ and $\alpha, \beta \in \Gamma$ implies $b\alpha c\beta a \in A$ and $a\alpha b\beta c \in A$.

Note 3.20. A nonempty subset A of a ternary Γ-semigroup T is said to be a ternary Γ-idealof T if $b, c \in T, a \in A$ and $\alpha, \beta \in \Gamma$ implies $b\alpha c\beta a \in A$, $b\alpha a\beta c \in A$ and $a\alpha b\beta c \in A$.

Note 3.21. We note that every left and right ternary Γ-ideal of a regular ternary Γ-semigroup maynot be a regular ternary Γ-semigroup; however, for a lateral ternary Γ-ideal of a regular ternary Γ-semigroup, we have the following result:

Lemma 3.22. Every lateral ternary Γ-ideal of a regular ternary Γ-semigroup T is a regular ternary Γ-semigroup.

Proof. Let M be a lateral ternary Γ-ideal of a regular ternary Γ-semigroup T. Then for each $a \in M$, there exists $x \in T, \alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$. Now $a = a\alpha x\beta a = a\alpha x\beta a\alpha x\beta a = a\alpha\,(x\beta a\alpha x)\,\beta a = a\alpha b\beta a$, where $b = x\beta a\alpha x \in M$. This implies that M is a regular ternary Γ-semigroup.

Note 3.23. Every ternary Γ-ideal of a regular ternary Γ-semigroup T is a regular ternary Γ-semigroup.

Now we have the following characterization theorem for regular ternary Γ-semigroup:

Theorem 3.24. The following conditions in a ternary Γ-semigroup T are equivalent:

(i) T is regular;

(ii) For any right ternary Γ-ideal R, lateral ternary Γ-ideal M and left ternary Γ-ideal L of T, $R\Gamma M\Gamma L = R \cap M \cap L$;

(iii) For $a, b, c \in T$; $< a >_r \Gamma < b >_m \Gamma < c >_l = < a >_r \cap < b >_m \cap < c >_l$;

(iv) For $a \in T$; $< a >_r \Gamma < a >_m \Gamma < a >_l = < a >_r \cap < a >_m \cap < a >_l$.

Proof. (i) Suppose that T is a regular ternary Γ -semigroup. Let R, M and L be a right ternary Γ-ideal, a lateral ternary Γ-ideal and a left ternary

Γ-ideal of T respectively. Then clearly, $R\Gamma M\Gamma L \subseteq R \cap M \cap L$. Now for $a \in R \cap M \cap L$, we have $a \in a\Gamma x\Gamma a$ for some $x \in T$. This implies that $a \in a\Gamma x\Gamma a = (a\Gamma x\Gamma a)\,\Gamma\,(x\Gamma a\Gamma x)\,\Gamma\,(a\Gamma x\Gamma a) \in R\Gamma M\Gamma L$. Thus we have $R \cap M \cap L \subseteq R\Gamma M\Gamma L$. So we find that $R\Gamma M\Gamma L = R \cap M \cap L$.

Clearly, $(ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (iv)$.

$(iv) \Rightarrow (i)$ Let $a \in T$. Clearly, $a \in <a>_r \cap <a>_m \cap <a>_l = <a>_r \Gamma <a>_m \Gamma <a>_l$. Then we have,

$$a \in (a\Gamma T\Gamma T \cup a)\Gamma(T\Gamma a\Gamma T \cup T\Gamma T\Gamma a\Gamma T\Gamma T \cup a)\Gamma(T\Gamma T\Gamma a \cup a) \subseteq a\Gamma T\Gamma a.$$

So we find that $a \in a\Gamma T\Gamma a$ and hence there exists an element $x \in T, \alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$. This implies that a is regular and hence T is regular.

Corollary 3.25. The following conditions on a ternary Γ-semigroup T are equivalent:

(i) T is regular;

(ii) $R \cap L = R\Gamma T\Gamma L$ for every right ternary Γ-ideal R and every left ternary Γ-ideal L of T;

(iii) For $a, b \in T$; $<a>_r \cap <b>_l = <a>_r \Gamma T\Gamma <b>_l$;

(iv) For $a \in T$; $<a>_r \cap <a>_l = <a>_r \Gamma T\Gamma <a>_l$.

Definition 3.26. A ternary Γ-semigroup T is said to be a ternary Γ-group if for $a, b, c \in T, \alpha, \beta \in \Gamma$, the equations $a\alpha b\beta x = c$, $a\alpha x\beta b = c$ and $x\alpha a\beta b = c$ have solutions in T.

Theorem 3.27. The following conditions in a ternary Γ-semigroup T are equivalent:

(i) T is regular and strongly Γ-cancellative;

(ii) T is regular and the idempotent pairs in T are all equivalent;

(iii) Every element of T is invertible in T;

(iv) T is a ternary Γ-group;

(v) T contains no proper one-sided ternary Γ-ideals.

Theorem 3.28. A ternary Γ-semigroup T is regular if and only if every ternary Γ-ideal of T is idempotent.

Proof. Let T be a regular ternary Γ-semigroup and A be any ternary Γ-ideal of T. Then $(A\Gamma)^2 A = A\Gamma A\Gamma A \subseteq T\Gamma T\Gamma A \subseteq A$. Let $a \in A$. Then there exists $x \in T, \alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a = a\alpha x\beta a\alpha x\beta a$. Since A is a ternary Γ-ideal and $a \in A, x\beta a\alpha x \in A$. Thus $a = a\alpha x\beta a = a\alpha x\beta a\alpha x\beta a \in (A\Gamma)^2 A$. Consequently, $A \subseteq (A\Gamma)^2 A$ and hence $(A\Gamma)^2 A = A\Gamma A\Gamma A = A$ i.e.,A is idempotent.

Conversely, suppose that every ternary Γ-ideal of T is idempotent. Let A, B and C be three ternary Γ-ideals of T. Then

$$A\Gamma B\Gamma C \subseteq A\Gamma T\Gamma T \subseteq A, A\Gamma B\Gamma C \subseteq T\Gamma B\Gamma T \subseteq B$$

and

$$A\Gamma B\Gamma C \subseteq T\Gamma T\Gamma C \subseteq C.$$

This implies that $A\Gamma B\Gamma C \subseteq A \cap B \cap C$. Also,

$$(A \cap B \cap C)\,\Gamma\,(A \cap B \cap C)\,\Gamma\,(A \cap B \cap C) \subseteq A\Gamma B\Gamma C.$$

Again, since $A \cap B \cap C$ is a ternary Γ-ideal of T,

$$(A \cap B \cap C)\,\Gamma\,(A \cap B \cap C)\,\Gamma\,(A \cap B \cap C) = A \cap B \cap C.$$

Thus $A \cap B \cap C \subseteq A\Gamma B\Gamma C$ and hence $A \cap B \cap C = A\Gamma B\Gamma C$. Therefore, by Theorem 3.24, T is a regular ternary Γ-semigroup.

We now introduce a semisimple element of a ternary Γ-semigroup and a semisimple ternary Γ-semigroup.

Definition 3.29. An element a of ternary Γ- semigroup T is said to be semisimple provided $a \in < a > \Gamma < a > \Gamma < a >$, that is, $< a > \Gamma < a > \Gamma < a >=< a >$.

Definition 3.30. A ternary Γ- semigroup T is said to be semisimple ternary Γ- semigroup provided every element of T is a semisimple element.

Definition 3.31. (see [12]) A ternary Γ-ideal A of a ternary Γ-semigroup T is said to be a prime Γ-ideal of T provided X, Y, Z are Ternary Γ-ideals of T and $X\Gamma Y\Gamma Z \subseteq A \Rightarrow X \subseteq A$ or $Y \subseteq A$ or $Z \subseteq A$.

Definition 3.32. (see [12]) A ternaty Γ- ideal A of a ternary Γ-semigroup T is said to be a completely semiprime Γ-ideal provided $x \in T$, $(x\Gamma)^n x \subseteq A$ for some odd natural number $n > 1$ implies $x \in A$.

Definition 3.33. (see [12]) A ternary Γ-ideal A of a ternary Γ-semigroup T is said to be semiprime ternary Γ-ideal provided X is a ternary Γ-ideal of T and $(X\Gamma)^{n-1} X \subseteq A$ for some odd natural number n implies $X \subseteq A$.

Theorem 3.34. A commutative ternary Γ-semigroup T is regular if and only if every ternary Γ-ideal of T is semiprime.

Proof. Let T be a commutative regular ternary Γ-semigroup and Q be any ternary Γ-ideal of T such that $A\Gamma A\Gamma A \subseteq Q$ for any ternary Γ-ideal A of

T. From Theorem 3.24, it follows that $A\Gamma A\Gamma A = A$. Consequently, $A \subseteq Q$ and hence Q is a semiprime ternary Γ-ideal of T. Conversely, suppose every ternary Γ-ideal of a commutative ternary Γ-semigroup T is semiprime. Let $a \in T$. Then $a\Gamma T\Gamma a$ is aternary Γ-ideal of T. Now by hypothesis, $a\Gamma T\Gamma a$ is a semiprime ternary Γ-ideal of T. If $a\Gamma T\Gamma a = T$, then we are done. Now suppose that $a\Gamma T\Gamma a \neq T$. Then

$$\begin{aligned} < a > \Gamma < a > \Gamma < a >= & (T\Gamma T\Gamma a \cup a\Gamma T\Gamma T \cup T\Gamma a\Gamma T \cup T\Gamma T\Gamma a\Gamma T\Gamma T \cup \{a\}) \\ & \Gamma(T\Gamma T\Gamma a \cup a\Gamma T\Gamma T \cup T\Gamma a\Gamma T \cup T\Gamma T\Gamma a\Gamma T\Gamma T \cup \{a\}) \\ & \Gamma(T\Gamma T\Gamma a \cup a\Gamma T\Gamma T \cup T\Gamma a\Gamma T \cup T\Gamma T\Gamma a\Gamma T\Gamma T \cup a) \\ \subseteq & a\Gamma T\Gamma a \end{aligned}$$

i.e.,

$$< a > \Gamma < a > \Gamma < a > \subseteq a\Gamma T\Gamma a.$$

This implies that $< a > \subseteq a\Gamma T\Gamma a$, since $a\Gamma T\Gamma a$ is a semiprime ternary Γ-ideal of T. Consequently, $a = a\alpha x\beta a$ for some $x \in T, \alpha, \beta \in \Gamma$ and hence T is a regular ternary Γ-semigroup.

Theorem 3.35. A ternary Γ-semigroup T is right (resp. left) regular if and only if every right(resp. left) ternary Γ-ideal of T is completely semiprime.

Proof. Let T be a right regular ternary Γ-semigroup and R be any right ternary Γ-ideal of T. Suppose $(a\Gamma)^2 a = a\Gamma a\Gamma a \subseteq R$ for $a \in T$. Since T is right regular, there exists an element $x \in T$, $\alpha, \beta \in \Gamma$ such that $a = x\alpha a\beta a = x\alpha\left(x\alpha a\beta a\right)\beta a = x\alpha x\left(\alpha a\beta a\beta a\right) \in T\Gamma T\Gamma R \subseteq R$. Thus R is completely semiprime.

Conversely, suppose that every right Γ-ideal of T is completely semiprime. Now for any $a \in T$, $T\Gamma a\Gamma a$ is a right ternary Γ-ideal of T. Then by hypothesis, $T\Gamma a\Gamma a$ is a completely semiprime ternary Γ-ideal of T. Now $(a\Gamma)^2 a = a\Gamma a\Gamma a \subseteq T\Gamma a\Gamma a$. Since $T\Gamma a\Gamma a$ is completely semiprime, it follows that $a \in T\Gamma a\Gamma a$. So there exists an element $x \in T$ such that $a = x\alpha a\beta a$. Consequently, a is right regular. Since a is arbitrary, it follows that T is right regular. Similarly, we can prove the theorem for left regularity.

Theorem 3.36. The following conditions in a ternary Γ-semigroup T are equivalent:

(i) T is completely regular;

(ii) T is left and right regular i.e., $a \in a\Gamma a\Gamma T \cap T\Gamma a\Gamma a$ for all $a \in T$;

(iii) $a \in a\Gamma a\Gamma T\Gamma a\Gamma a$ for all $a \in T$.

Proof. $(i) \Rightarrow (ii)$: Suppose T is a completely regular ternary Γ-semigroup. Let $a \in T$. Then there exists an element $x \in T, \alpha, \beta \in \Gamma$ such that $a\alpha x\beta a = a$ and the idempotent pairs (a, x) and (x, a) are equivalent i.e., $a\Gamma x\Gamma y = x\Gamma a\Gamma y$ and $y\Gamma a\Gamma x = y\Gamma x\Gamma a$ for all $y \in T$. Now in particular, putting $y = a$ we find that $a\Gamma x\Gamma a = x\Gamma a\Gamma a$ and $a\Gamma a\Gamma x = a\Gamma x\Gamma a$. This implies that $a \in a\Gamma a\Gamma T$ and $a \in T\Gamma a\Gamma a$ i.e., $a \in a\Gamma a\Gamma T \cap T\Gamma a\Gamma a$.

$(ii) \Rightarrow (iii)$: Suppose that T is both left and right regular. Let $a \in T$. Then there exist $x, y \in T$, $\alpha, \beta \in \Gamma$ such that $a = a\alpha a\beta x$ and $a = y\alpha a\beta a$. This implies that $a\gamma x\delta z = y\alpha a\beta a\gamma x\delta z = y\alpha a\delta z$ for all $z \in T, \gamma, \delta \in \Gamma$. Now

$$\begin{aligned} a =& a\alpha a\beta x = a\alpha\,(a\alpha a\beta x)\,\beta x \\ =& (a\alpha a)\,\alpha a\beta x\beta x = (a\alpha a)\,\alpha y\beta a\beta x \\ =& a\alpha a\alpha y\beta\,(y\alpha a\beta a)\,\beta x = a\alpha a\alpha y\beta y\alpha\,(a\beta a\beta x) \\ =& a\alpha a\alpha y\beta y\alpha a = a\alpha a\alpha y\beta y\alpha\,(y\alpha a\beta a) \\ =& (a\alpha)^2\,(y\beta y\alpha y)\,\alpha\,(a\beta a) \in a\Gamma a\Gamma T\Gamma a\Gamma a. \end{aligned}$$

$(iii) \Rightarrow (i)$: Suppose $a \in a\Gamma a\Gamma Ta\Gamma a$ for all $a \in T$. Then there exists $x \in T$ such that $a \in a\Gamma a\Gamma x\Gamma a\Gamma a$. Now $a \in a\Gamma a\Gamma x\Gamma a\Gamma a = a\Gamma(a\Gamma x\Gamma a)\Gamma a = a\Gamma y\Gamma a$, where $y \in a\Gamma x\Gamma a \in T$. This implies that T is regular. Also

$$a\Gamma y\Gamma z = a\Gamma(a\Gamma x\Gamma a)\Gamma z = (a\Gamma)^2 x(a\Gamma)^2 x(a\Gamma)^2 z$$

and

$$y\Gamma a\Gamma z = (a\Gamma x\Gamma a)\Gamma a\Gamma z = (a\Gamma)^2 x(a\Gamma)^2 x(a\Gamma)^2 z$$

for all $z \in T$. This shows that the idempotent pairs (a, y) and (y, a) are equivalent. Consequently, T is a completely regular ternary Γ-semigroup.

Defnition 3.37. A ternary Γ-subsemigroup B of a ternary Γ-semigroup T is called a bi-ternary Γ-ideal of T if $B\Gamma T\Gamma B\Gamma TB \subseteq B$.

Theorem 3.38. A ternary Γ-semigroup T is completely regular if and only if every bi-ternary Γ-ideal of T is completely semiprime.

Proof. First suppose that T is a completely regular ternary Γ-semigroup. Let B be any bi-ternary Γ-ideal of T. Let $(b\Gamma)^2 b \subseteq B$ for $b \in T$. Since T is completely regular,from Theorem 3.35, it follows that $b \in (b\Gamma)^2 T(\Gamma b)^2$. This implies that there exists $x \in T$ such that

$$\begin{aligned} b \in& (b\Gamma)^2 x(\Gamma b)^2 = b\Gamma[(b\Gamma)^2 x(\Gamma b)^2]\Gamma x\Gamma[(b\Gamma)^2 x(\Gamma b)^2]\Gamma b \\ =& (b\Gamma)^3(x\Gamma(b\Gamma)^2 x)\Gamma b\Gamma[(b\Gamma)^2 x\Gamma(b\Gamma)^2]\Gamma x\Gamma(b\Gamma)^2 b \end{aligned}$$

$$=(b\Gamma)^3(x\Gamma(b\Gamma)^2x)\Gamma(b\Gamma)^3(x\Gamma(b\Gamma)^2\Gamma x)\Gamma(b\Gamma)^2 b$$
$$\subseteq B\Gamma T\Gamma B\Gamma T\Gamma B \subseteq B.$$

This shows that B is completely semiprime.

Conversely, suppose that every bi-ternary Γ-ideal of T is completely semiprime. Since every left and right ternary Γ-ideal of a ternary Γ-semigroup T is a bi-ternary Γ-ideal of T, it follows that every left and right ternary Γ-ideal of T is completely semiprime. Consequently, we have from Theorem 3.34, that T is both left and right regular. Now by using Theorem 3.35, we find that T is a completely regular ternary Γ-semigroup.

Theorem 3.39. If T is a completely regular ternary Γ-semigroup, then every bi-ternary Γ-ideal of T is idempotent.

Proof. Let T be a completely regular ternary Γ-semigroup and B be a bi-ternary Γ-ideal of T. Since T is a completely regular ternary Γ-semigroup, it is also a regular ternary Γ-semigroup. Let $b \in B$. Then there exists $x \in T$ suchthat $b \in b\Gamma x\Gamma b$. This implies that $b \in B\Gamma T\Gamma B and hence B \subseteq B\Gamma T\Gamma B$. Also $B\Gamma T\Gamma B \subseteq B\Gamma T\Gamma B\Gamma T\Gamma B \subseteq B$. Thus we find that $B = B\Gamma T\Gamma B$. Again, we have from Theorem 3.35, that $b \in (b\Gamma)^2T(\Gamma b)^2 \subseteq B\Gamma B\Gamma T\Gamma B\Gamma B$. This implies that

$$B \subseteq B\Gamma B\Gamma T\Gamma B\Gamma B = B\Gamma(B\Gamma T\Gamma B)\Gamma B = B\Gamma B\Gamma B \subseteq B.$$

Consequently, $B\Gamma B\Gamma B = B$.

Theorem 3.40. If T is an intra-regular ternary Γ-semigroup then for any left ternary Γ-ideal L, lateral ternary Γ-ideal M and right ternary Γ-ideal R of T, $L \cap M \cap R \subseteq L\Gamma M\Gamma R$.

Proof. Suppose that T is an intra-regular ternary Γ-semigroup. Let L, M and R be a left ternary Γ-ideal, a lateral ternary Γ-ideal and a right ternary Γ-ideal of T respectively. Now for $a \in L \cap M \cap R$, we have $a \in x\Gamma a\Gamma a\Gamma a\Gamma y$ for some $x, y \in T$. This implies that

$$a \in x\Gamma a\Gamma a\Gamma a\Gamma y = (x\Gamma x\Gamma a\Gamma a\Gamma a)G(y\Gamma x\Gamma a\Gamma a\Gamma a\Gamma y\Gamma x)\Gamma(a\Gamma a\Gamma a\Gamma y\Gamma y) \in L\Gamma M\Gamma R.$$

Thus we have $L\Gamma M\Gamma R \subseteq L\Gamma M\Gamma R$.

Theorem 3.41. Let T be an intra-regular ternary Γ-semigroup. Then a non-empty subset I of T is a ternary Γ-ideal of T if and only if I is a lateral ternary Γ-ideal of T.

Proof. Clearly, if I is a ternary Γ-ideal of T, then I is a lateral ternary Γ-ideal of T. Conversely, let I be a lateral ternary Γ-ideal of an intra-regular ternary Γ-semigroup. Let $a \in I$ and $s, t \in T$. Then $a \in T$ and hence there exist elements $x, y \in T$ such that $a \in x\Gamma a\Gamma a\Gamma a\Gamma y$. Now $s\Gamma t\Gamma a = s\Gamma t\Gamma x\Gamma a\Gamma a\Gamma a\Gamma y \subseteq T\Gamma I\Gamma T \subseteq I$ and $a\Gamma s\Gamma t = x\Gamma a\Gamma a\Gamma a\Gamma y\Gamma s\Gamma t \subseteq T\Gamma I\Gamma T \subseteq I$. This implies that I is both a left ternary Γ-ideal and a right ternary Γ-ideal of T. Consequently, I is a ternary Γ-ideal of T.

Lemma 3.42. Every lateral ternary Γ-ideal of an intra-regular ternary Γ -semigroup T is an intra-regular ternary Γ-semigroup.

Proof. Let L be a lateral ternary Γ-ideal of an intra-regular ternary Γ-semigroup T. Then for each $a \in L$, there exists $x, y \in T$ such that $a \in x\Gamma a\Gamma a\Gamma a\Gamma y$. Now

$$\begin{aligned} a \in & x\Gamma a\Gamma a\Gamma a\Gamma y \\ =& x\Gamma(x\Gamma a\Gamma a\Gamma a\Gamma y)\Gamma(x\Gamma a\Gamma a\Gamma a\Gamma y)\Gamma(x\Gamma a\Gamma a\Gamma a\Gamma y)\Gamma y \\ =& (x\Gamma x\Gamma a\Gamma a\Gamma a\Gamma y\Gamma x)\Gamma a\Gamma a\Gamma a\Gamma(y\Gamma x\Gamma a\Gamma a\Gamma a\Gamma y\Gamma y) \\ \in & L\Gamma a\Gamma a\Gamma a L. \end{aligned}$$

This implies that there exist $u, v \in L$ such that $a = u\Gamma a\Gamma a\Gamma a\Gamma v$. Consequently, L is an intra-regular ternary Γ-semigroup. From Theorem 3.40, we have the following result:

Corollary 3.43. Every ternary Γ-ideal of an intra-regular ternary Γ -semigroup T is an intra-regular ternary Γ-semigroup.

Note 3.44. In ring theory, we note that if A is an ideal of a ring R and B is an ideal of A, then B need not be an ideal of the entire ring R. But it is well known that the result is true for regular ring. Like ring theory, we note that if A is a ternaryΓ-ideal of a ternary Γ-semigroup T and B is a ternary Γ-ideal of A, then B need not be a ternary Γ-ideal of the entire ternary Γ-semigroup T. But in particular, for an intra-regular ternary Γ-semigroup T, we have the following result:

Theorem 3.45. Let A be a ternary Γ-ideal of an intra-regular ternary Γ-semigroup T and B be a ternary Γ-ideal of A. Then B is a ternary Γ-ideal of the entire ternaryΓ-semigroup T.

Proof. It is sufficient to show that B is a lateral ternary Γ-ideal of T. Let $a \in B \subseteq A$ and $s, t \in T$, $\alpha, \beta \in \Gamma$. Then $s\alpha a\beta t \in A$. We have to show that $s\alpha a\beta t \in B$. From Corollary 3.42, it follows that A is an intra-regular ternary Γ-semigroup. Thus there exist $u, v \in A$ such that $s\alpha a\beta t = u\gamma(s\alpha a\beta t\delta)^3 v =$

$u\gamma(s\alpha a\beta t)\delta(s\alpha a\beta t)\delta(s\alpha a\beta t)\delta v = (u\gamma s\alpha a\beta t\delta s)\alpha a\beta(t\delta s\alpha a\beta t\delta v) \in A\Gamma B\Gamma A \subseteq B$. Consequently, B is a lateral ternary Γ-ideal of T.

Theorem 3.46. A ternary Γ-semigroup T is intra-regular if and only if every ternary Γ-ideal of T is completely semiprime.

Proof. Let T be an intra-regular ternary Γ-semigroup and A be a ternary G -ideal of T. Let $a\Gamma a\Gamma a \subseteq A$ for $a \in T$. Since T is intra-regular, there exist $x, y \in T$ such that $a \in x\Gamma a\Gamma a\Gamma a\Gamma y \subseteq A$. Consequently, A is completely semiprime. Conversely, suppose that every ternary Γ-ideal of T is completely semiprime. Let $a \in T$. Then $a\Gamma a\Gamma a \subseteq< a\Gamma a\Gamma a >$. This implies that $a \in< a\Gamma a\Gamma a >$, since $< a\Gamma a\Gamma a >$ is completely semiprime. Now $< a\Gamma a\Gamma a >= T\Gamma T\Gamma a\Gamma a\Gamma a \cup a\Gamma a\Gamma a\Gamma T\Gamma T \cup T\Gamma a\Gamma a\Gamma a\Gamma T \cup T\Gamma T a\Gamma a\Gamma a\Gamma T\Gamma T \cup a\Gamma a\Gamma a$. So we have the following cases:

If $a \in T\Gamma T\Gamma a\Gamma a\Gamma a$, then $a\Gamma a\Gamma a \subseteq T\Gamma T\Gamma a\Gamma a\Gamma a\Gamma a$. Hence

$$a \in T\Gamma T\Gamma T\Gamma T\Gamma a\Gamma a\Gamma a\Gamma a\Gamma a \subseteq T\Gamma T\Gamma T\Gamma a\Gamma a\Gamma a\Gamma T \subseteq T\Gamma a\Gamma a\Gamma a\Gamma T.$$

If $a \in a\Gamma a\Gamma a\Gamma T\Gamma T$, then $a\Gamma a\Gamma a \subseteq a\Gamma a\Gamma a\Gamma a\Gamma a\Gamma T\Gamma T$. Hence

$$a \in a\Gamma a\Gamma a\Gamma a\Gamma a\Gamma T\Gamma T\Gamma T\Gamma T \subseteq T\Gamma a\Gamma a\Gamma a\Gamma T\Gamma T\Gamma T \subseteq T\Gamma a\Gamma a\Gamma a\Gamma T.$$

If $a \in T\Gamma a\Gamma a\Gamma a\Gamma T$, then we are done.
If $a \in T\Gamma T\Gamma a\Gamma a\Gamma a\Gamma T\Gamma T$, then

$$a\Gamma a\Gamma a \subseteq a\Gamma T\Gamma T\Gamma a\Gamma a\Gamma a\Gamma T\Gamma T\Gamma a.$$

Hence

$$a \in T\Gamma T\Gamma a\Gamma T\Gamma T\Gamma a\Gamma a\Gamma a\Gamma T\Gamma T\Gamma a\Gamma T \subseteq T\Gamma T\Gamma T\Gamma a\Gamma a\Gamma a\Gamma T\Gamma T\Gamma T \subseteq T\Gamma a\Gamma a\Gamma a\Gamma T.$$

If $a \in a\Gamma a\Gamma a$, then

$$a \in a\Gamma a\Gamma a = (a\Gamma a\Gamma a)\Gamma(a\Gamma a\Gamma a)\Gamma(a\Gamma a\Gamma a) \subseteq T\Gamma a\Gamma a\Gamma a\Gamma T.$$

So we find that in any case, T is intra-regular.

4. Conclusion

In this paper we mainly introduced the theory of ternary Γ-ideals in ternary Γ-semigroups and characterised these ternary Γ-ideals.

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