A COMMON FIXED POINT THEOREM ON COMPLETE $G-$ METRIC SPACES

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Abstract: In this paper, we obtain common fixed point theorem for two self maps of complete $G-$metric spaces.

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1. Introduction

Metric fixed point theory is an important Mathematical discipline. The generalization of metric spaces was proposed by Gähler [3] (called 2-metric spaces) and Dhage [1, 2] (called $D$-metric spaces). Hsiao [5] showed that every contractive definition, with $x_n = T^n x_0$, every orbit is linearly dependent, thus giving fixed point theorem in such spaces. However HA et. al. [4] have pointed out that the results obtained by Gähler for his 2-metric spaces are independent, rather than the generalizations of corresponding results in metric spaces. While Mustafa and Sims [6] have proved that the Dhage’s notion of $D$-metric space is fundamentally incorrect and most of the results claimed by Dhage and others are invalid.
Mustafa and Sims [6] in 2003 have introduced a more appropriate and robust notion of generalized metric spaces as follows and obtained several properties in [7,8].

**Definition 1.1.** (see [6]) Let $X$ be a non empty set and let $G: X \times X \times X \rightarrow [0, \infty)$ be a function satisfying the following axioms:

(G1) $G(x, y, z) = 0$ if $x = y = z$;

(G2) $0 < G(x, y, z)$ for all $x, y \in X$ with $x \neq y$;

(G3) $G(x, x, y) < G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$;

(G4) $G(x, y, z) = G(\sigma(x, y, z))$ for all $x, y, z \in X$, where $\sigma$ is a permutation of the set $\{x, y, z\}$ (Symmetry in all three variables);

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (Rectangular inequality).

Then the function $G$ is called a generalized metric or more specifically a $G$-metric on $X$. The pair $(X, G)$ is called a $G$-metric space.

**Definition 1.2.** Let $(X, G)$ be a $G$-metric space and let $\{x_n\}$ be a sequence of points of $X$, we say that $\{x_n\}$ is $G$-convergent to $x$ if for every given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ (set of all natural numbers) such that $G(x, x_n, x_m) < \varepsilon$ for all $m, n \geq N$. We denote it as $\lim_{n,m \to \infty} G(x, x_n, x_m) = 0$.

**Definition 1.3.** Let $(X, G)$ be a metric space, a sequence $\{x_n\}$ in $X$ is called $G$-cauchy if for every given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that, $G(x_n, x_m, x_l) < \varepsilon$ for all $n, m, l \geq N$, that is if $\lim_{n,m,l \to \infty} G(x_n, x_m, x_l) = 0$.

**Definition 1.4.** A $G$-metric space $(X, G)$ is said to be $G$-complete (or a complete $G$-metric space) if every $G$-Cauchy sequence in $(X, G)$ is $G$-convergent to some point in $(X, G)$.

**Definition 1.5.** Let $(X, G)$ be a $G$-metric space and let $T: X \rightarrow X$ be a mapping. $T$ is called a contraction of if

$$G(Tx, Ty, Tz) \leq kG(x, y, z) \text{ for all } x, y, z, \in X. \quad (1.1)$$

**Definition 1.6.** A function $\phi: [0, \infty) \rightarrow [0, \infty)$ is called an Altering distance function, if the following properties are satisfied.

$$\phi(0) = 0, \quad (1.2)$$

$\phi$ is continuous and monotonically non-decreasing. \quad (1.3)
Definition 1.7. A mapping $T : X \rightarrow X$, where $(X, G)$ is a $G$-metric space, is said to be weakly contractive if

$$G(Tx, Ty, Tz) \leq G(x, y, z) - \varphi(G(x, y, z)),$$

for all $x, y, z, \in X$, where $\phi : [0, \infty) \rightarrow [0, \infty)$ is a continuous and non decreasing function such that $\phi(t) = 0$ if and only if $t = 0$.

2. Main Theorem

In 1996 Rashwan et al [9] established fixed point theorem for self maps of complete metric spaces by using Altering distance function and weakening the inequality in the earlier results which states as follows.

Theorem 2.1. Let $S$ and $T$ be self map of a complete metric space $(X, d)$ and $\phi : [0, \infty) \rightarrow [0, \infty)$ satisfying:

1. $\phi$ is continuous and strictly increasing;
2. $\phi(t) = 0$ if and only if $t = 0$;
3. The following inequality hold true

$$\phi(d(Sx, Ty)) \leq a(d(x, y)) \phi(d(x, y)) + b(d(x, y)) [\phi(d(x, Tx)) + \phi(d(y, Ty))]$$
$$+ c(d(x, y)) \min\{\phi(d(x, Ty)), \phi(d(y, Tx))\},$$

where $a(t) + 3b(t) + c(t) < 1$ for all $t \geq 0$. Then $S$ and $T$ have unique common fixed point.

In this paper we prove a fixed point theorem analogous to the above theorem.

Theorem 2.2. Let $S$ and $T$ be self map of a complete $G$-metric space $(X, G)$, $a, b, c$ are three decreasing functions from $R^+$ to $[0,1)$ and let $\phi : [0, \infty) \rightarrow [0, \infty)$ be a mapping satisfying:

1. $\phi$ is continuous and strictly increasing on $R^+$;
2. $\phi(t) = 0$ if and only if $t = 0$;
3. The following inequality is valid

$$\phi(G(Sx, Ty, Tz)) \leq a(G(x, y, z)) \phi(G(x, y, z)) + b(G(x, y, z)) [\phi(G(x, Sx, Sx))$$
$$+ \phi(G(y, Ty, Ty)) + \phi(G(z, Tz, Tz))] + c(G(x, y, z))$$
$$\min\{\phi(G(x, Ty, Ty)), \phi(G(x, Tz, Tz)), \phi(G(y, Sx, Sx)), \phi(G(y, Tz, Tz))\},$$

where $a(t) + 3b(t) + c(t) < 1$ for all $t \geq 0$. Then $S$ and $T$ have unique common fixed point.
Proof. Let \( x_0 \epsilon X \). Define the \( \{x_n\} \) in \( X \) as follows:

\[
x_{2n} = Sx_{2n-1}, \quad n = 1, 2, 3, \ldots
\]
\[
x_{2n+1} = Tx_{2n}, \quad n = 1, 2, 3, \ldots
\]

Let \( \tau_n = G(x_n, x_{n+1}, x_{n+1}) \). Assume that \( \tau_n > 0 \) for every \( n \in N \). Now for an even integer

\[
\phi(\tau_n) = \phi(G(x_n, x_{n+1}, x_{n+1})) = \phi(G(Sx_{n-1}, Tx_n, Tx_n)) \leq A1 + A2 + A3, \quad (2.1)
\]

where:

\[
A1 = a(G(x_{n-1}, x_n, x_n)) \phi(G(x_{n-1}, x_n, x_n)),
\]
\[
A2 = b(G(x_{n-1}, x_n, x_n)) \phi(G(x_{n-1}, x_n, x_n)) + \phi(G(x_n, x_{n+1}, x_{n+1})) + \phi(G(x_n, x_{n+1}, x_{n+1}))
\]
\[
A3 = c(G(x_{n-1}, x_n, x_n)) \min\{\phi(G(x_{n-1}, x_{n+1}, x_{n+1})), \phi(G(x_n, x_n, x_n)), \phi(G(x_n, x_{n+1}, x_{n+1}))\}.
\]

Now (2.1) becomes

\[
\phi(\tau_n) \leq \frac{a(\tau_{n-1}) + b(\tau_{n-1})}{1 - 2b(\tau_{n-1})} \phi(\tau_{n-1}) < \phi(\tau_{n-1}), \quad (2.2)
\]

since \( a(t) + 3b(t) + c(t) < 1 \).

As \( \phi \) is increasing, \( \{\tau_n\} \) is a decreasing sequence and bounded below by 0 and hence \( \{\tau_n\} \) convergent to some \( \tau \). That is,

\[
\lim_{n \to \infty} \tau_n = \tau.
\]

Assume that \( \tau \neq 0 \) then, clearly \( \tau_n \geq \tau \) and using this in (2.2), we receive

\[
\phi(\tau_n) \leq \frac{a(\tau) + b(\tau)}{1 - 2b(\tau)} \phi(\tau) < \phi(\tau).
\]

Let \( n \to \infty \), the above inequality reduces to the form

\[
\phi(\tau) \leq \frac{a(\tau) + b(\tau)}{1 - 2b(\tau)} \phi(\tau) < \phi(\tau).
\]

This is a contradiction. Hence \( \tau = 0 \).
Similarly for an odd integer \( n \) we can show that \( \{x_n\} \) is a \( G \)-Cauchy sequence. Since \( \tau = 0 \), it is suffices to prove that \( \{x_{2n}\} \) is a \( G \)-Cauchy sequence.

If it is possible to assume that \( \{x_{2n}\} \) is not a \( G \)-Cauchy sequence then there is an \( \varepsilon > 0 \) and a even integer \( 2k \) and two sequences \( \{2p_k\}, \{2q_k\} \) with \( 2p_k > 2q_k \geq 2k \) for which

\[
G(x_{2p_k}, x_{2q_k}, x_{2q_k}) > \varepsilon. \tag{2.3}
\]

For each even integer \( 2k \), let \( 2p_k \) be the least even integer exceeding \( 2q_k \) and satisfying (2.3). Then

\[
G(x_{2p_k-2}, x_{2q_k}, x_{2q_k}) \leq \varepsilon \quad \text{and} \quad G(x_{2p_k}, x_{2q_k}, x_{2q_k}) > \varepsilon.
\]

Putting \( d_{2k} = G(x_{2p_k}, x_{2q_k}, x_{2q_k}) \), we have

\[
\varepsilon < d_{2k} = G(x_{2p_k}, x_{2q_k}, x_{2q_k}) \\
\leq G(x_{2p_k}, x_{2q_k-2}, x_{2q_k-2}) + G(x_{2q_k-2}, x_{2q_k-1}, x_{2q_k-1}) \\
+ G(x_{2q_k-1}, x_{2q_k}, x_{2q_k}) \\
\leq \varepsilon + (\tau_{2p_k-2}) + (\tau_{2p_k-1}).
\]

That is

\[
d_{2k} \leq \varepsilon + (\tau_{2p_k-2}) + (\tau_{2p_k-1}). \tag{2.4}
\]

Letting \( k \to \infty \) in (2.4), we get

\[
d_{2k} \to \varepsilon \text{ as } k \to \infty.
\]

Further more, by rectangular inequality it follows that

\[
G(x_{2p_k+1}, x_{2q_k+1}, x_{2q_k+1}) \leq G(x_{2p_k+1}, x_{2p_k}, x_{2p_k}) + G(x_{2p_k}, x_{2q_k}, x_{2q_k}) \\
+ G(x_{2q_k}, x_{2q_k+1}, x_{2q_k+1}).
\]

This up on using the fact that, \( G(x, x, y) \leq 2G(y, y, x) \) for all, we get

\[
G(x_{2p_k+1}, x_{2q_k+1}, x_{2q_k+1}) \leq 2\tau_{2p_k} + d_{2k} + \tau_{2q_k}.
\]

Letting \( k \to \infty \) on both the sides, we receive

\[
\lim_{n \to \infty} G(x_{2p_k+1}, x_{2q_k+1}, x_{2q_k+1}) = \varepsilon.
\]
Moreover,
\[ G(x_{2p_k+1}, x_{2q_k}, x_{2q_k}) \leq G(x_{2p_k+1}, x_{2p_k}, x_{2p_k}) + G(x_{2p_k}, x_{2q_k}, x_{2q_k}) \leq 2\tau_{2p_k} + d_{2k} \rightarrow \varepsilon \text{ as } k \rightarrow \infty. \]

Therefore
\[ G(x_{2p_k+1}, x_{2q_k}, x_{2q_k}) \rightarrow \varepsilon \text{ as } k \rightarrow \infty \]

and
\[ G(x_{2p_k+2}, x_{2q_k+1}, x_{2q_k+1}) \leq G(x_{2p_k+2}, x_{2p_k+1}, x_{2p_k+1}) + G(x_{2p_k+1}, x_{2q_k+1}, x_{2q_k+1}) \leq 2\tau_{2p_k+1} + \varepsilon \rightarrow \varepsilon \text{ as } k \rightarrow \infty. \]

That is
\[ G(x_{2p_k+2}, x_{2q_k+1}, x_{2q_k+1}) \rightarrow \varepsilon \text{ as } k \rightarrow \infty. \]

Consider,
\[ \phi(G(x_{2p_k+2}, x_{2q_k+1}, x_{2q_k+1})) = \phi(G(Sx_{2p_k+1}, Tx_{2p_k}, Tx_{2p_k})) \leq L1 + L2 + L3 + L4 + L5 + L6, \]

where:
\[
\begin{align*}
L1 & = a(G(x_{2p_k+1}, x_{2q_k}, x_{2q_k})). \phi(G(x_{2p_k+1}, x_{2q_k}, x_{2q_k})), \\
L2 & = b(G(x_{2p_k+1}, x_{2q_k}, x_{2q_k})). \left[ \phi(G(x_{2p_k+1}, x_{2p_k+2}, x_{2p_k+2})) \\
& + \phi(G(x_{2q_k}, x_{2q_k+1}, x_{2q_k+1})) + \phi(G(x_{2q_k}, x_{2q_k+1}, x_{2q_k+1})) \right], \\
L3 & = c(G(x_{2p_k+1}, x_{2q_k}, x_{2q_k})). \min \left\{ \phi(G(x_{2p_k+1}, x_{2q_k+1}, x_{2q_k+1})), \\
& \phi(G(x_{2p_k+1}, x_{2q_k+1}, x_{2q_k+1})), \phi(G(x_{2q_k}, x_{2p_k+2}, x_{2p_k+2})), \\
& \phi(G(x_{2q_k}, x_{2q_k+1}, x_{2q_k+1})) \right\}, \\
L4 & = a(2\tau_{2p_k} + d_{2k}). \phi(2\tau_{2p_k} + d_{2k}), \\
L5 & = b(2\tau_{2p_k} + d_{2k}). \left[ \phi(\tau_{2p_k+1}) + \phi(\tau_{2q_k}) + \phi(\tau_{2q_k}) \right], \\
L6 & = c(2\tau_{2p_k} + d_{2k}). \min \left\{ \phi(2\tau_{2p_k} + G(x_{2p_k}, x_{2q_k+1}, x_{2q_k+1})), \\
& \phi(d_{2q_k} + (G(x_{2p_k}, x_{2p_k+2}, x_{2p_k+2}))) \right\}.
\end{align*}
\]

Letting \( k \rightarrow \infty \), we have
\[ \phi(\varepsilon) \leq a(\varepsilon) \phi(\varepsilon) < \phi(\varepsilon). \]

This is a contradiction, hence \( \{x_n\} \) is a \( G \)-Cauchy sequence in complete \( G \)-metric space \( (X, G) \). Therefore there exist \( u \in X \) such that \( \{x_n\} \) is \( G \)-convergent to \( u \).
Now we shall show that $u$ is the common fixed point of $S$ and $T$. Taking $x = x_{2p+2}, y = u, z = u$ in (2.1),and letting $k \to \infty$ on both the sides, we get

$$\phi(G(u, Tu, Tu)) \leq a(0)\phi(0) + b(0)[2\phi(G(u, Tu, Tu))] + c(0)0.$$ 

Therefore

$$\phi(G(u, Tu, Tu)) \leq 2b(0)\phi(G(u, Tu, Tu)) < \phi(G(u, Tu, Tu)).$$

This is a contradiction if $(G(u, Tu, Tu) \neq 0$).

Hence $G(u, Tu, Tu) = 0$ this implies $Tu = u$, showing '$u'$ is a fixed point of $T$.

Similarly it can be shown that $Su = u$.

Now we claim that $u$ is the unique common fixed point of $S$ and $T$. For this, suppose that $v \ (v \neq u)$ is another common fixed point of $S$ and $T$. Then:

$$\phi(G(u, v, v)) = \phi(G(Su, Tv, Tv)) \leq a(G(u, v, v))\phi(G(u, v, v)) + b(G(u, v, v))\phi(G(u, Su, Su)) + \phi(G(v, Tv, Tv)) + \phi(G(v, Tv, Tv)) + c(G(u, v, v))\min\{\phi(G(u, Tu, Tu)), \phi(G(v, Tv, Tv)), \phi(G(v, Su, Su)), \phi(G(v, Tv, Tv))\}.$$ 

That is

$$\phi(G(u, v, v)) \leq a(G(u, v, v))\phi(G(u, v, v)) < \phi(G(u, v, v)),$$

a contradiction. Hence $u = v$.

This completes proof of the theorem.

References


