

ON THE HYPERSURFACE OF A FINSLER SPACE WITH THE SQUARE METRIC

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Abstract: In this paper, a special metric, known as Z. Shen's Square metric has been studied. Further, a criterion has been obtained for a hypersurface of a Finsler space with this special metric to be a hyperplane of first kind, second kind but not the hyperplane of third kind.

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1. Introduction

Let $F^n = (M^n, L)$ be an n -dimensional Finsler space, where M^n is an n -dimensional differential manifold equipped with a fundamental function $L = L(x, y)$, $x = (x^i)$ is a point and $y = (y^i)$ is a supporting element of differentiable manifold M^n . The fundamental function $L = L(x, y)$ is called Finsler

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metric. The idea of (α, β) metric was introduced by M. Matsumoto ([4] [5]) and has been studied in detail. A Finsler metric $L(\alpha, \beta)$ on a differentiable manifold is a positively homogeneous function of degree one in α and β , where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a differential 1-form on differentiable manifold M^n . There are several important (α, β) metrics, namely Z. Shen's square metric $L = \frac{(\alpha+\beta)^2}{\alpha}$, Kropina metric $L = \frac{\alpha^2}{\beta}$, Randers metric $L = \alpha + \beta$, Matsumoto metric $L = \frac{\alpha^2}{\alpha-\beta}$ and Generalized Kropina metric $L = \frac{\alpha^{n+1}}{\beta^n}$. Among these metrics this was the Z. Shen's square metric that draws our attention to work with hypersurface F^{n-1} of Finsler space $F^n = (M^n, L)$. The study of the properties of special hypersurface in Finsler geometry has been embarked on by M. Matsumoto [3]. He discussed the properties of special hypersurface of Randers space [6]. Not only Matsumoto but many authors ([1], [2], [7], [8], [9], [11]) have been constantly studied and derived the conditions under which a hypersurface of a Finsler space becomes a hyperplane of first kind, second kind but not of the third kind. The goal of the present paper is to investigate the hypersurface of Finsler space using Z. Shen's square metric $L = \frac{(\alpha+\beta)^2}{\alpha}$.

2. Preliminaries

We consider the Finsler space (M^n, L) , where L is the Z. Shen's square metric, that is given by

$$L(\alpha, \beta) = \frac{(\alpha + \beta)^2}{\alpha} \quad (2.1)$$

Differentiating equation (2.1) partially with respect to α and β up to second order, we get

$$L_\alpha = \frac{\alpha^2 - \beta^2}{\alpha^2}, L_\beta = \frac{2(\alpha + \beta)}{\alpha}, L_{\alpha\alpha} = \frac{2\beta^2}{\alpha^3}, L_{\beta\beta} = \frac{2}{\alpha}, L_{\alpha\beta} = \frac{-2\beta}{\alpha^2}. \quad (2.2)$$

In Finsler space (M^n, L) , where $L = \frac{(\alpha+\beta)^2}{\alpha}$, the normalized element of support $l_i = \frac{\partial L}{\partial y^i}$ and the angular metric tensor h_{ij} [6] are respectively given by

$$l_i = \frac{L_\alpha y_i}{\alpha} + L_\beta b_i \quad (2.3)$$

$$h_{ij} = pa_{ij} + q_0b_ib_j + q_1(b_iy_j + b_jy_i) + q_2y_iy_j, \quad (2.4)$$

where the coefficients are defined and calculated as follows

$$y_i = a_{ij}y^j, \quad p = \frac{LL_\alpha}{\alpha} = \frac{(\alpha + \beta)^3(\alpha - \beta)}{\alpha^4}, q_0 = LL_{\beta\beta} = \frac{2(\alpha + \beta)^2}{\alpha^2}, \quad (2.5)$$

$$q_1 = \frac{LL_{\alpha\beta}}{\alpha} = \frac{-2\beta(\alpha + \beta)^2}{\alpha^4}, q_2 = \frac{L(L_{\alpha\alpha} - \frac{L_\alpha}{\alpha})}{\alpha^2} = \frac{(\alpha + \beta)^2(3\beta^2 - \alpha^2)}{\alpha^6}. \quad (2.6)$$

The fundamental metric tensor $g_{ij} = \frac{1}{2} \frac{\partial^2 L^2}{\partial y^i \partial y^j}$ of Finsler space (M^n, L) is defined by [6]

$$g_{ij} = pa_{ij} + p_0b_ib_j + p_1(b_iy_j + b_jy_i) + p_2y_iy_j, \quad (2.7)$$

where the coefficients p, p_0, p_1 and p_2 are defined and calculated as follows:

$$p = \frac{LL_\alpha}{\alpha} = \frac{(\alpha + \beta)^3(\alpha - \beta)}{\alpha^4}, \quad (2.8)$$

$$p_0 = q_0 + L_\beta^2 = \frac{6(\alpha + \beta)^2}{\alpha^2}, \quad (2.9)$$

$$p_1 = q_1 + \frac{pL_\beta}{L} = \frac{(\alpha + \beta)^2(2\alpha - 4\beta)}{\alpha^4}, \quad (2.10)$$

$$p_2 = q_2 + \frac{p^2}{L^2} = \frac{2\beta(2\beta - \alpha)(\alpha + \beta)^2}{\alpha^6} \quad (2.11)$$

In addition, reciprocal tensor g^{ij} of fundamental metric tensor $g_{ij} = \frac{1}{2} \frac{\partial^2 L^2}{\partial y^i \partial y^j}$ of Finsler space (M^n, L) is defined by [6]

$$g^{ij} = \frac{a^{ij}}{p} - S_0b^ib^j - S_1(b^iy^j + b^jy^i) - S_2y^iy^j, \quad (2.12)$$

where the coefficients b^i, S_0, S_1 and S_2 are defined as follows:

$$b^i = a^{ij}b_j \quad S_0 = \frac{pp_0 + (p_0p_2 - p_1^2)\alpha^2}{p\zeta} \quad (2.13)$$

$$S_1 = \frac{pp_1 + (p_0p_2 - p_1^2)\beta}{p\zeta} \quad (2.14)$$

$$S_2 = \frac{pp_2 + (p_0p_2 - p_1^2)b^2}{p\zeta} \quad (2.15)$$

$$\zeta = p(p + p_0b^2 + p_1\beta) + (p_0p_2 - p_1^2)(\alpha^2b^2 - \beta^2), \quad (2.16)$$

where $b^2 = a_{ij}b^ib^j$. Now the hv -torsion tensor $C_{ijk} = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k}$ is defined by [10]

$$C_{ijk} = \frac{p_1(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j) + \gamma_1 m_i m_j m_k}{2p}, \quad (2.17)$$

where the coefficient γ_1 and m_i are defined as

$$\gamma_1 = p \frac{\partial p_0}{\partial \beta} - 3p_1q_0, m_i = b_i - \frac{y_i\beta}{\alpha^2} \quad (2.18)$$

Here m_i is known as non-zero covariant vector orthogonal to element of support y^i . Let Γ_{jk}^i be the components of Christoffel symbol of the associated Riemannian space R^n and ∇_k be the covariant differentiation with respect to x^k relative to Christoffel symbol. Now we put

$$2E_{ij} = b_{ij} + b_{ji} \quad (2.19)$$

$$2F_{ij} = b_{ij} - b_{ji}, \quad (2.20)$$

where $b_{ij} = \nabla_j b_i$. Let $CT = (\Gamma_{jk}^{*i}, \Gamma_{0k}^{*i}, C_{jk}^i)$ be Cartan connection of F^n . The difference tensor $D_{jk}^i = \Gamma_{jk}^{*i} - \Gamma_{jk}^i$ of the special Finsler space F^n is given by [10]

$$\begin{aligned} D_{jk}^i &= B^i E_{jk} + F_k^i B_j + F_j^i B_k + B_j^i b_{0k} + B_k^i b_{0j} - b_{0m} g^{im} B_{jk} \\ &\quad - C_{jm}^i A_k^m - C_{km}^i A_j^m + C_{jkm} A_s^m g^{is} \\ &\quad + \lambda^s (C_{jm}^i C_s k^m + C_{km}^i C_{sj}^m - C_{jk}^m C_{ms}^i), \end{aligned} \quad (2.21)$$

where

$$B_k = p_0 b_k + p_1 y_k \quad (2.22)$$

$$B^i = g^{ij} B_j \quad (2.23)$$

$$B_{ij} = \frac{p_1(a_{ij} - \frac{y_i y_j}{\alpha^2}) + \frac{\partial p_0}{\partial \beta} m_i m_j}{2} \quad (2.24)$$

$$B_i^k = g^{kj} B_{ji} \quad (2.25)$$

$$A_k^m = B_k^m E_{00} + B^m E_{k0} + B_k F_0^m + B_0 F_k^m \quad (2.26)$$

$$\lambda^m = B^m E_{00} + 2B_0 F_0^m \quad (2.27)$$

$$F_i^k = g^{kj} F_{ji} \quad (2.28)$$

$$B_0 = B_i Y^i \quad (2.29)$$

Here as well as henceforward '0' denotes tensorial contraction with y^i besides p_0, q_0 and S_0 .

3. Induced Cartan Connection

Let $F^n = (M^n, L)$, where $L = \frac{(\alpha+\beta)^2}{\alpha}$, be a Finsler space and let F^{n-1} be it's hypersurface having equation $x^i = x^i(u^\alpha)$, $i = 1, 2, 3, \dots, (n-1)$. Let the matrix of projection factor be $B_\alpha^i = \frac{\partial x^i}{\partial u^\alpha}$ and it's rank is $(n-1)$. The tangential component of element of support y^i of Finsler space F^n along it's hypersurface F^{n-1} is given by

$$y^i = B_\alpha^i(u) v^\alpha. \quad (3.1)$$

Therefore v^α is the element of support of hypersurface F^{n-1} at the point u^α . The metric tensor $g_{\alpha\beta}$ and hv-torsion tensor $C_{\alpha\beta\gamma}$ of hypersurface F^{n-1} are defined by

$$g_{\alpha\beta} = g_{ij} B_\alpha^i B_\beta^j, C_{\alpha\beta\gamma} = C_{ijk} B_\alpha^i B_\beta^j B_\gamma^k \quad (3.2)$$

Now the unit normal vector $N^i(u, v)$ at an arbitrary point u^α of the hypersurface F^{n-1} is defined by the following property

$$g_{ij}(x(u, v), y(u, v)) B_\alpha^i N^j = 0, g_{ij}(x(u, v), y(u, v)) N^i N^j = 1 \quad (3.3)$$

As for the angular metric tensor h_{ij} , we have

$$h_{\alpha\beta} = h_{ij} B_\alpha^i B_\beta^j, h_{ij} B_\alpha^i N^j = 0, h_{ij} N^i N^j = 1 \quad (3.4)$$

Let (B_i^α, N_i) be the inverse of (B_α^i, N^i) , then we have

$$\begin{aligned} B_i^\alpha &= g^{\alpha\beta} g_{ij} B_\beta^j, B_\alpha^i B_i^\beta = \delta_\alpha^\beta, B_i^\alpha N^i = 0, \\ B_\alpha^i N_i &= 0, N_i = g_{ij} N^j, B_i^k = g^{kj} B_{ji}, \\ B_\alpha^i B_j^\alpha + N^i N_j &= \delta_j^i \end{aligned}$$

The induced Cartan connection $\mathcal{ICT} = (\Gamma_{\beta\gamma}^{*\alpha}, G_{\beta}^{\alpha}, C_{\beta\gamma}^{\alpha})$ on hypersurface F^{n-1} induced from the Cartan's connection $CT = (\Gamma_{jk}^{*i}, \Gamma_{0k}^{*i}, C_{jk}^i)$ is given by [3]

$$\Gamma_{\beta\gamma}^{*\alpha} = B_i^{\alpha}(B_{\beta\gamma}^i + \Gamma_{jk}^{*i} B_{\beta}^j B_{\gamma}^k) + M_{\beta}^{\alpha} H_{\gamma} \quad (3.5)$$

$$G_{\beta}^{\alpha} = B_i^{\alpha}(B_{0\beta}^i + \Gamma_{0j}^{*i} B_{\beta}^j) \quad (3.6)$$

$$C_{\beta\gamma}^{\alpha} = B_i^{\alpha} C_{jk}^i B_{\beta}^j B_{\gamma}^k, \quad (3.7)$$

where

$$M_{\beta\gamma} = N_i C_{jk}^i B_{\beta}^j B_{\gamma}^k \quad (3.8)$$

$$M_{\beta}^{\alpha} = g^{\alpha\gamma} M_{\beta\gamma} \quad (3.9)$$

$$H_{\beta} = N_i (B_{0\beta}^i + \Gamma_{0j}^{*i} B_{\beta}^j) \quad (3.10)$$

$$B_{\beta\gamma}^i = \frac{\partial B_{\beta}^i}{\partial U^{\gamma}} \quad (3.11)$$

$$B_{0\beta}^i = B_{\alpha\beta}^i v^{\alpha}. \quad (3.12)$$

The quantities $M_{\beta\gamma}$ and H_{β} appeared in above equations are called the second fundamental v-tensor and normal curvature respectively [3]. The second fundamental h-tensor $H_{\beta\gamma}$ is defined as [3]

$$H_{\beta\gamma} = N_i (B_{\beta\gamma}^i + \Gamma_{jk}^{*i} B_{\beta}^j B_{\gamma}^k) + M_{\beta} H_{\gamma}, \quad (3.13)$$

where

$$M_{\beta} = N_i C_{jk}^i B_{\beta}^j N^k. \quad (3.14)$$

The relative h-covariant derivative and v-covariant derivative of projection factor B_{α}^i with respect to induced Cartan connection $\mathcal{ICT}$ are respectively given by

$$B_{\alpha|\beta}^i = H_{\alpha\beta} N^i \quad (3.15)$$

$$B_{\alpha|\beta}^i = M_{\alpha\beta} N^i \quad (3.16)$$

The equation (3.10) shows that $H_{\beta\gamma}$ is not always symmetric and

$$H_{\beta\gamma} - H_{\gamma\beta} = M_{\beta} H_{\gamma} - M_{\gamma} H_{\beta} \quad (3.17)$$

Thus the above equation simplifies to

$$H_{0\gamma} = H_{\gamma}, H_{\gamma 0} = H_{\gamma} + M_{\gamma} H_0 \quad (3.18)$$

We utilize the following lemma in order to prove our hypothesis. The following lemma has already been derived by M. Matsumoto [3]:

Lemma 1 *The normal curvature $H_0 = H_\beta v^\beta$ vanishes if and only if normal curvature vector H_β vanishes.*

Lemma 2 *A hypersurface F^{n-1} is a hyperplane of first kind if and only if $H_\alpha = 0$.*

Lemma 3 *A hypersurface F^{n-1} is a hyperplane of second kind with respect to Cartan connection CT if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = 0$.*

Lemma 4 *A hypersurface F^{n-1} is a hyperplane of third kind with respect to Cartan connection CT if and only if $H_\alpha = 0$, $H_{\alpha\beta} = 0$ and $M_{\alpha\beta} = 0$.*

4. Hypersurface $F^{n-1}(c)$ of the special Finsler space

We consider here a special Finsler metric $L = \frac{(\alpha+\beta)^2}{\alpha}$, known as Z. Shen's square metric. Now consider a hypersurface $F^{n-1}(c)$, whose equation is given by $b(x) = c$, where c is a fixed constant. Thus the gradient of the function representing hypersurface $F^{n-1}(c)$ is given by in tensor notation $b_i(x) = \partial_i b$. Again consider the parametric equation $x^i = x^i(u^\alpha)$ of F^{n-1} . Differentiating the equation of hypersurface $b(x(u)) = c$ with respect to parameter u^α , we get $\partial_\alpha b(x(u)) = 0 = b_i B_\alpha^i$. It is obvious that $b_i(x)$ are the covariant component of normal vector field of hypersurface F^{n-1} .

Therefore along the hypersurface F^{n-1} we have

$$b_i B_\alpha^i = 0 \quad (4.1)$$

$$b_i y^i = 0 \quad (4.2)$$

The metric induced $L(u, v)$ from the special Finsler space (M^n, L) , where $L = \frac{(\alpha+\beta)^2}{\alpha}$, on the hypersurface F^{n-1} is given by

$$L(u, v) = a_{\alpha\beta} v^\alpha v^\beta, \quad (4.3)$$

where $a_{\alpha\beta} = a_{ij} B_\alpha^i B_\beta^j$. The induced metric in equation (4.3) do not have β component, (i.e. $\beta = b_i y^i = 0$), of the Finsler metric of the original space (M^n, L) , therefore induced metric in equation (4.3) is a Riemannian metric. Therefore at any point on the hypersurface $F^{n-1}(c)$ equations (2.5), (2.6), (2.8),

(2.9), (2.10), (2.11), (2.13),
(2.14), (2.15) and (2.16) reduce to

$$p = 1, q_0 = 2, q_1 = 0, q_2 = -\frac{1}{\alpha^2}, p_0 = 6, p_1 = \frac{2}{\alpha}, p_2 = 0 \quad (4.4)$$

$$S_1 = \frac{2}{1+2b^2}, S_1 = \frac{2}{\alpha(1+2b^2)}, S_2 = -\frac{4b^2}{\alpha^2(1+2b^2)}, \zeta = 1+2b^2 \quad (4.5)$$

Using the values of p, S_0, S_1, S_2 from above equation into equation (2.12), we have

$$g^{ij} = a^{ij} - \frac{2}{1+2b^2} b^i b^j - \frac{2}{\alpha(1+2b^2)} (b^i y^j + b^j y^i) + \frac{4b^2}{\alpha^2(1+2b^2)} y^i y^j \quad (4.6)$$

Multiplying equation (4.6) by $b_i b_j$ and using the fact that $\beta = b_i y^i = 0$, it becomes

$$g^{ij} b_i b_j = \frac{b^2}{1+2b^2}.$$

Thus we get

$$b_i(x(u)) = \sqrt{\frac{b^2}{1+2b^2}} N_i, b^2 = a^{ij} b_i b_j, \quad (4.7)$$

where b is the length of the vector b^i . Now from (4.6) and (4.7) we get

$$b^i = a^{ij} b_j = \sqrt{b^2(1+2b^2)} N^i + \frac{b^2}{\alpha} y^i \quad (4.8)$$

Theorem 1. *In the special Finsler hypersurface F^{n-1} the induced metric is a Riemannian metric given by (4.3) and scalar function $b(x)$ is given by (4.7) and (4.8).*

Using the values of p, p_0, p_1 and p_2 from equation (4.4) into equation (2.7), Finsler metric tensor of F^n reduces to

$$g_{ij} = a_{ij} + 6b_i b_j + \frac{2}{\alpha} (b_i y_j + b_j y_i) \quad (4.9)$$

and using the value of p, q_0, q_1 and q_2 in equation (2.4), angular metric tensor of F^n reduces to

$$h_{ij} = a_{ij} + 2b_i b_j - \frac{1}{\alpha^2} y_i y_j \quad (4.10)$$

From equations (4.10), (4.2) and (3.4) it follows that if $h_{\alpha\beta}^{(a)}$ denotes the angular metric tensor of Riemannian $a_{ij}(x)$, then along hypersurface $F^{n-1}(c)$, $h_{\alpha\beta} = h_{\alpha\beta}^{(a)}$.

Differentiating equation (2.9) with respect to β , we have

$$\frac{\partial p_0}{\partial \beta} = \frac{12(\alpha + \beta)}{\alpha^2}.$$

Thus along the hypersurface $F^{n-1}(c)$, above equation reduces to $\frac{\partial p_0}{\partial \beta} = \frac{12}{\alpha}$ and (2.18) also reduces to $\gamma_1 = 0$, $m_i = b_i$. Using the values of p, p_1, γ_1 and m_i in equation (2.17), hv-torsion tensor in hypersurface F^{n-1} , becomes

$$C_{ijk} = \frac{h_{ij}b_k + h_{jk}b_i + h_{ki}b_j}{\alpha}. \quad (4.11)$$

Using equations (4.1) and (4.11) in equation (3.5), we get

$$M_{\alpha\beta} = \frac{1}{\alpha} \sqrt{\frac{b^2}{1+2b^2}} h_{\alpha\beta} \quad (4.12)$$

Again, using equations (4.1) and (4.11) in (3.11), we get

$$M_\alpha = 0 \quad (4.13)$$

Using the equation (4.13) in equation (3.13), it shows that $H_{\alpha\beta} = H_{\beta\alpha}$, i.e. $H_{\alpha\beta}$ is symmetric. Thus we have the following theorem:

Theorem 2. *The second fundamental v-tensor of Finsler hypersurface F^{n-1} is given by (4.12) and (4.13) and the second fundamental h-tensor $H_{\alpha\beta}$ is symmetric.*

Now differentiating equation (4.1) with respect to β , we get

$$b_{i|j}B_\alpha^i + b_iB_{\alpha|j}^i = 0 \quad (4.14)$$

Using equation (3.12) and $b_{i|j} = B_\beta^j + b_{i|j}N^jH_\beta$, equation (4.14) reduces to

$$b_{i|j}B_\beta^jB_\alpha^i + b_{i|j}N^jH_\beta B_\alpha^i + b_iH_{\alpha\beta}N^i = 0 \quad (4.15)$$

Since $b_{i|j} = -b_h C_{ij}^h$, from (3.11), (4.7) and (4.13) we get

$$b_{i|j}B_\alpha^iN^j = \sqrt{\frac{b^2}{1+2b^2}}M_\alpha = 0$$

Using above expression in equation (4.15), we get

$$b_{i|j}B_{\beta}^jB_{\alpha}^i + \sqrt{\frac{b^2}{1+2b^2}}H_{\alpha\beta} = 0 \quad (4.16)$$

It is obvious that $b_{i|j}$ is symmetric. Now contracting (4.16) with v^{β} first and then with v^{α} respectively and using the equations (3.1), (3.15) and (4.13) we get

$$b_{i|j}B_{\alpha}^jy^j + \sqrt{\frac{b^2}{1+2b^2}}H_{\alpha} = 0 \quad (4.17)$$

$$b_{i|j}y^iy^j + \sqrt{\frac{b^2}{1+2b^2}}H_0 = 0 \quad (4.18)$$

According to the lemma (1) and lemma (2), a hypersurface $F^{n-1}(c)$ is a hyperplane of first kind if and only if normal curvature vanishes i.e $H_0 = 0$. Using the value $H_0 = 0$ in equation (4.18) we find that hypersurface $F^{n-1}(c)$ is again a hyperplane of first kind if and only if $b_{i|j}y^iy^j = 0$. This $b_{i|j}$ is the covariant derivative of with respect to Cartan connection $C\Gamma$ of Finsler space F^n , it may depend on y^i . Moreover $\nabla_j b_i = b_{ij}$ is the covariant derivative with respect to Rieamannian connection Γ_{jk}^i constructed from $a_{ij}(x)$, therefore b_{ij} dose not depend on y^i . We shall consider the difference $b_{i|j} - b_{ij}$ of above covariant derivatives in further discussion. The difference tensor $D_{jk}^i = \Gamma_{jk}^{*i} - \Gamma_{jk}^i$ is given by (2.21). Since b_i is a gradient vector, from equation (2.19) and (2.20), we have

$$E_{ij} = b_{ij}, F_{ij} = 0, F_j^i = 0 \quad (4.19)$$

Using equation (4.19) into equation (2.21), we get

$$\begin{aligned} D_{jk}^i &= b_{jk}B^i + b_{0k}B_j^i + b_{0j}B_k^i - b_{0m}g^{im}B_{jk} \\ &\quad - A_k^m C_{jm}^i - A_j^m C_{km}^i + A_s^m C_{jkm}g^{is} \\ &\quad + \lambda^s (C_{sk}^m C_{jm}^i + C_{sj}^m C_{km}^i - C_{ms}^i C_{jk}^m) \end{aligned} \quad (4.20)$$

Using the equations (4.2), (4.4), (4.5) and (4.6) into equations (2.22) to (2.27), we get

$$B_k = 6b_k + \frac{2}{\alpha}y_k, B^i = \frac{2}{1+2b^2}b^i + \frac{2}{\alpha(1+2b^2)}y^i \quad (4.21)$$

$$B_{ij} = \frac{1}{\alpha}a_{ij} - \frac{1}{\alpha^3}y_i y_j + \frac{6}{\alpha}b_i b_j \quad (4.22)$$

$$B_j^i = \frac{1}{\alpha}(\delta_j^i - \frac{1}{\alpha^2}y^i y_j) + \frac{4}{\alpha(1+2b^2)}b^i b_j - \frac{(2+12b^2)}{\alpha^2(1+2b^2)}b_j y^i \quad (4.23)$$

$$A_k^m = B_k^m b_{00} + B^m b_{k0}, \lambda^m = B^m b_{00}. \quad (4.24)$$

Using tensor contraction operation with equations (4.24) and (4.25) by y^j , we get $B_{i0} = 0$, $B_0^i = 0$. Further contracting equation (4.26) by y^k and using the fact that $B_0^i = 0$, we get $A_0^m = B^m b_{00}$. Contracting equation (4.20) by y^k and using the facts $B_{i0} = 0$, $B_0^i = 0$, $A_0^m = B^m b_{00}$ and $C_{s0}^m = 0$, $C_{0m}^i = 0$, $C_{j0}^m = 0$ obtained by contracting (3.2) and (3.7), we get

$$D_{j0}^i = B^i b_{j0} + B_j^i b_{00} - b_{00} B^m C_{jmi}^i \quad (4.25)$$

$$D_{00}^i = \frac{2}{1+2b^2}b^i b_{00} + \frac{2}{\alpha(1+2b^2)}y^i b_{00} \quad (4.26)$$

Multiplying equation (4.25) by b_i and then using equations (4.2), (4.21), (4.22) and (4.23) and we get

$$b_i D_{j0}^i = \frac{2b^2}{1+2b^2}b_{j0} + \frac{1+6b^2}{\alpha(1+2b^2)}b_j b_{00} - \frac{2}{1+2b^2}b_i b^m C_{jmi}^i b_{00} \quad (4.27)$$

Now multiplying (4.26) by b_i and then using equation (4.2) we get

$$b_i D_{00}^i = \frac{2b^2}{1+2b^2}b_{00} \quad (4.28)$$

From equation (4.11) it is clear that

$$b^m b_i C_{jmi}^i B_\alpha^j = b^2 M_\alpha = 0 \quad (4.29)$$

Contracting the expression $b_{i|j} = b_{ij} - b_r D_{ij}^r$ by y^i and y^j respectively and then using equation (4.28) we get

$$b_{i|j} y^i y^j = b_{00} - b_r D_{00}^r = \frac{1}{1+2b^2}b_{00}$$

Using equations (4.27) and (4.29), equations (4.17) and (4.18) can be written as

$$\frac{1}{\sqrt{1+2b^2}}b_{i0} B_\alpha^i + \sqrt{b^2} H_\alpha = 0 \quad (4.30)$$

$$\frac{1}{\sqrt{1+2b^2}}b_{00} + \sqrt{b^2} H_0 = 0 \quad (4.31)$$

From the equation (4.31) it is clear that the condition $H_0 = 0$ is equivalent to $b_{00} = 0$, where b_{ij} is independent of y^i . Since y^i satisfy equation (4.2), the condition can be written as $b_{ij}y^iy^j = (b_iy^i)(c_jy^j)$ for some $c_j(x)$, so that we have

$$2b_{ij} = b_ic_j + b_jc_i \quad (4.32)$$

Contracting (4.32) and using the fact that $b_iy^j = 0$, we get $b_{00} = 0$. Multiplying equation (4.3) by B_α^i and then B_β^j and using equation (4.1) gives $b_{ij}B_\alpha^iB_\beta^j = 0$. Similarly we get $b_{ij}B_\alpha^iy^j = 0$. This further gives $b_{i0}B_\alpha^iy^j = 0$. Using this in equation (4.30) gives $H_\alpha = 0$. Again contracting (4.32) and then using equation (4.2) gives $b_{i0} = \frac{b^2c_0}{2}$. Now using (4.23) and (4.24) and using $b_{00} = 0$ and $b_{ij}B_\alpha^iB_\beta^j = 0$ we get $\lambda^m = 0$, $A_j^iB_\beta^j = 0$ and $B_{ij}B_\alpha^iB_\beta^j = \frac{1}{2\alpha}h_{\alpha\beta}$. Thus using the equations (4.6), (4.7), (4.8), (4.12) and (4.20) we get

$$b_rD_{ij}^rB_\alpha^iB_\beta^j = -\frac{c_0b^2}{4\alpha(1+2b^2)^2}h_{\alpha\beta} \quad (4.33)$$

Thus using the relation $b_{i|j} = b_{ij} - b_rD_{ij}^r$ and equation (4.33), equation (4.16) reduces to

$$\frac{c_0b^2}{4\alpha(1+2b^2)^2}h_{\alpha\beta} + \sqrt{\frac{b^2}{1+2b^2}}H_{\alpha\beta} = 0. \quad (4.34)$$

Hence the hypersurface $F^{n-1}(c)$ is umbilic.

Theorem 3. *The necessary and sufficient condition for hypersurface $F^{n-1}(c)$ to be a hyperplane of first kind is equation (4.32) and in this case second fundamental tensor of $F^{n-1}(c)$ is proportional to it's angular metric tensor.*

From the Lemma 3, $F^{n-1}(c)$ is a hyperplane of second kind if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = 0$. Therefore from equation (4.32), we get $c_0 = c_i(x)y^i = 0$. Thus there exist a function $e(x)$ such that $c_i(x) = e(x)b_i(x)$. Therefore from the equation (4.32) we have

$$b_{ij} = eb_ib_j. \quad (4.35)$$

Theorem 4. *The necessary and sufficient condition for hypersurface $F^{n-1}(c)$ to be a hyperplane of second kind is that the equation (4.35) is satisfied.*

Finally from equation (4.12) and lemma (4), we deduce that hypersurface $F^{n-1}(c)$ is not a hyperplane of third kind.

Theorem 5. *The hypersurface $F^{n-1}(c)$ is not a hyperplane of the third kind.*

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