

ON THE ADJACENCY AND INCIDENCE MATRICES OF
THE ZERO DIVISOR GRAPHS OF A CLASS OF
THE SQUARE RADICAL ZERO FINITE
COMMUTATIVE RINGS

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Abstract: The characterization of the zero divisor graphs of commutative finite rings has attracted a lot of research for quite sometime, however not so much has been done concerning their Adjacency and Incidence matrices. In computer modelling, matrices are better understood than graphs and therefore the representation of graphs by matrices is worth studying. Given an arbitrary square matrix M_n , it is not known in general the classes of finite rings for which it represents the zero divisors. In spite of that, there exist some expositions on the adjacency and incidence matrices of the zero divisor graphs of commutative finite rings (reference can be made to [3, 7, 9] among others). Let R be a square radical zero finite commutative ring. This paper characterizes the adjacency and incidence matrices of the zero divisor graphs $\Gamma(R)$ of such rings of characteristic p and p^2 . We have drawn a zero divisor graph of the classes of rings studied using TikZ software and studied its properties, then generalized the properties of such graphs in the same category. By using the standard algebraic concepts, we have formulated the Adjacency and Incidence matrices of the graphs. A cursory study of these matrices has been undertaken on some of their algebraic properties. We also extend our findings on the adjacency matrices $[A_{ij}]$ as transformations. The results provide an extension

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on the classification problem of rings in which the product of any two zero divisors is zero.

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1. Introduction

The definitions of terms and the notations are standard and can be obtained from the references.

Much work has been done on the zero divisor graphs for commutative rings with identity. A few expositions have been given by Anderson and Livingston [1], Beck [2], Mulay [4] and Redmond [8]. For instance, Beck in [2] constructed a graph of zero divisors in which every element is a vertex and focused on the determination of the coloring of the graphs. Anderson and Livingston in [1] re-defined the concept of the zero divisor graph and constructed graphs in which every vertex is a nonzero-zero divisor. Their motivation was to give a better illustration of the zero divisor structure of the ring. Mulay in [4] introduced another zero divisor graph which is constructed from the classes of zero divisors determined by annihilator ideals rather than the individual zero divisors themselves. A further study was done by Redmond in [8] who introduced another zero divisor graph in which an element $x \in R \setminus I$ is a vertex when I is an ideal of R and $x, y \in R \setminus I$ are adjacent when $x \cdot y = 0$. He discussed the relationship between $\Gamma_I(R)$ and $\Gamma(R/I)$. Their results did not however extend to the adjacency and the incidence matrices of the said graphs.

Godsil, Sin and Xiang in [3] gave some results on the invariants of incidence matrices of some arbitrary graphs. They posed a general problem concerning the computation of the invariants of matrices representing linear maps over fields by computing the eigenvalues and p -ranks.

Sharma et al in [9] analyzed the adjacency matrices and the neighbourhood associated with the zero divisor graphs of finite commutative rings. Some of their findings were that, for $p \neq 2$, the adjacency matrix is always singular for $\mathbb{Z}_p \times \mathbb{Z}_p$. Moreover, the number of the zero divisors in $\mathbb{Z}_p \times \mathbb{Z}_p$ are $2(p-1)$, the Eigenvalues of the matrix with respect to the zero divisor graph are $(p-1)$ and 0, the rank of the matrix is 2 and the matrix is singular, among others. A similar study was done by Patra and Baruah in [7] on the Adjacency matrix and the neighbourhood associated with the zero divisor graphs of $\mathbb{Z}_p \times \mathbb{Z}_{p^2-2}$ and $\mathbb{Z}_p \times \mathbb{Z}_{2p}$ together with $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$ where p is a prime integer. They established that the determinant of the adjacency matrices corresponding to the zero

divisor graphs are 0, the matrices have a rank of 2 and both symmetric and singular.

This paper presents an investigation on the adjacency and incidence matrices of the zero divisor graphs of finite commutative rings of the square radical zero, constructed through the idealization of R_0 -modules.

2. Preliminaries

Definition 1. For any vertices x_i, x_j and for all $i, j \in \mathbb{N}$, the adjacency matrix of the zero divisor graph $\Gamma(R)$ of R is defined as $\mathbf{A}=[A_{ij}]$, where

$$A_{ij} = \begin{cases} 1, & x_i x_j = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2. If G is a graph with n vertices, m edges and without self-loop, for any vertices $x_i, x_j \forall i, j \in \mathbb{N}$, the incidence matrix $\mathbf{D}=[D_{ij}]$ of G is an $n \times m$ matrix $\mathbf{D}=[D_{ij}]$ whose n rows correspond to the n vertices and m columns correspond to m edges such that

$$D_{ij} = \begin{cases} 1, & \text{when there is an edge between } i \text{ and } j; \\ 0, & \text{otherwise.} \end{cases}$$

Definition 3. A function $Q(x_1, x_2, \dots, x_n)$ from $\mathbb{R}^n$ to $\mathbb{R}$ is called a quadratic form if it is a linear combination of functions of the form $x_i x_j$. A quadratic form can be written as $Q(\vec{x}) = \vec{x} \cdot A \vec{x} = \vec{x}^T A \vec{x}$ for any symmetric $n \times n$ matrix such that a_{ij} = coefficient of x_i^2 while $a_{ij} = \frac{1}{2}$ (coefficient of $x_i x_j$, for $i \neq j$).

3. Constructions I and II of R in which $(Z(R))^2 = (0)$

3.1. Construction I: Finite Rings of Characteristic p

For any prime integer p and a positive integer r , let $R_0 = GR(p^r, p)$ be a Galois ring of order p^r and characteristic p . For each $i = 1, 2, \dots, h$, let $u_i \in Z(R)$ and U be an h -dimensional R_0 -module generated by $\{u_1, \dots, u_h\}$ such that $R = R_0 \oplus U$ is an additive Abelian group. On R , define multiplication by; $(r_0, r_1, \dots, r_h)(s_0, s_1, \dots, s_h) = (r_0 s_0, r_0 s_1 + r_1 s_0^{\sigma_1}, \dots, r_0 s_h + r_h s_0^{\sigma_h})$, for $(r_0, r_1, \dots, r_h), (s_0, s_1, \dots, s_h) \in R$ and $\sigma_1, \dots, \sigma_h \in \text{Aut}(R_0)$. It is verifiable that R is a finite ring with identity $(1, 0, \dots, 0)$ and it is commutative iff $\sigma_i = \text{id}_{R_0} \forall i = 1, \dots, h$. (see [6]).

3.2. Construction II: Finite Rings of Characteristic p^2

For any prime integer p and a positive integer r , let $R_o = GR(p^{2r}, p^2)$ be a Galois ring of order p^{2r} and characteristic p^2 . Suppose U is finitely generated R_o -module such that U is generated by h elements so that $R = R_o \oplus U$ where $U = R_o/pR_o \oplus R_o/pR_o \oplus \cdots \oplus R_o/pR_o$ is an additive Abelian group. On R , define multiplication as follows;

$$(r_o, \bar{r}_1, \dots, \bar{r}_h)(s_o, \bar{s}_1, \dots, \bar{s}_h) = (r_o s_o, r_o \bar{s}_1 + \bar{r}_1 s_o^{\sigma_1}, \dots, r_o \bar{s}_h + \bar{r}_h s_o^{\sigma_h}),$$

where $\sigma_i \in \text{Aut}(R_o)$, $i = 1, \dots, h$. The additive abelian group R defined above is a commutative finite ring with identity $(1, \bar{0}, \dots, \bar{0})$. The ring R is completely primary of characteristic p^2 . The ring is commutative if and only if $\sigma_i = \text{Id}_{R_o}$ (*see*[6]).

4. The Zero Divisor Graphs $\Gamma(R)$

Consider the ring R of construction *I*. Let $p = 2, h = 3, k = 1$ and $r = 1$ so that $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

The set of non-zero, zero divisors is

$$(Z(R))^* = \{(0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), (0, 1, 1, 0), (0, 1, 0, 1), (0, 1, 1, 1)\}.$$

The zero divisor graph $\Gamma(R)$ is given by:

From the graph, $|V(\Gamma(R))| = 7$ which corresponds to the number of edges, $\text{diam}(\Gamma(R)) = 1$, $\Delta(\Gamma(R)) = 6$, $\text{girth}(\Gamma(R)) = 3$ and $\chi(\Gamma(R)) \leq 7$.

Remark 1. From constructions *I* and *II*, the invariants p, k, r and h can be suitably chosen so as to have appropriate structures of the ring R , the zero divisors $Z(R)$, the non-zero zero divisors $Z(R)^*$ and consequently the graph $\Gamma(R)$.

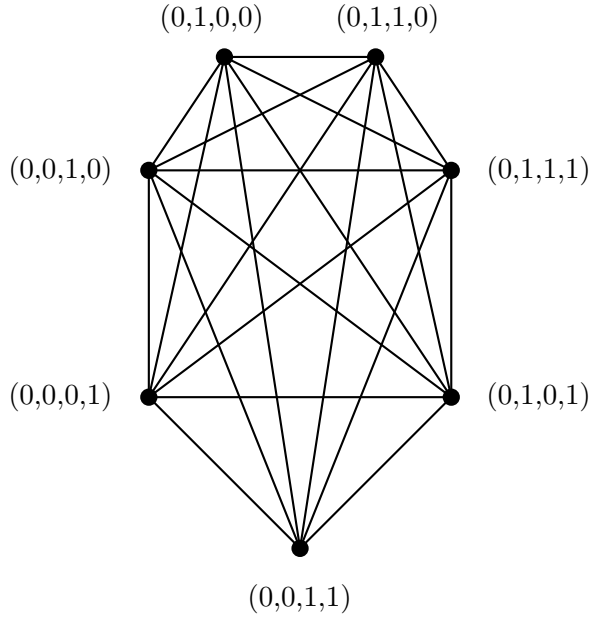
The following results in the sequel hold:

Lemma 1. *Let R be the ring described by construction *I* and $\Gamma(R)$ be its zero divisor graph. Then,*

$$(i) |V(\Gamma(R))| = p^{rh} - 1.$$

(ii) $\Gamma(R)$ is complete.

$$(iii) \Gamma(R) = K_{p^{rh}-1}.$$



$$(iv) \text{diam}(\Gamma(R)) = 1.$$

$$(v) \text{girth}(\Gamma(R)) = \begin{cases} \infty, & \text{if } r=1 \text{ and } p=2,3 \\ 3, & \text{elsewhere} \end{cases}$$

$$(vi) b(\Gamma(R)) = \infty.$$

$$(vii) \Delta(\Gamma(R)) = p^{rh} - 2.$$

$$(viii) \chi(\Gamma(R)) \leq p^{rh} - 1.$$

Proof. (i) From the construction I , we have that $R_o = GF(p^r, p)$ and $F \simeq R_o/pR_o$. So $U = F^h$ is an R_o -module generated by $u_1, \dots, u_h$ over F . It is clear that $Z(R) = R_o u_1 \oplus R_o u_2 \oplus \dots \oplus R_o u_h$ and every non-zero element in $Z(R)$ is of the form $(0, r_1, r_2, \dots, r_h)$. We show that any element not in $Z(R)$ is invertible. Suppose $(r_o, r_1, r_2, \dots, r_h)$ is not contained in $Z(R)$. Choose an element say $(s_o, s_1, s_2, \dots, s_h)$ not in $Z(R)$ such that $(r_o, r_1, r_2, \dots, r_h)(s_o, s_1, s_2, \dots, s_h) = (1, 0, \dots, 0)$. This implies that $r_o s_o = 1 \Rightarrow s_o = r_o^{-1}$ and $r_o s_i + r_i s_o = 0 \Rightarrow s_i = -r_i r_o^{-2}$ for $1 < i \leq h$. Since this holds in the reverse order as well, we have established that $(r_o, r_1, r_2, \dots, r_h)^{-1} = (r_o^{-1}, -r_1 r_o^{-2}, \dots, -r_h r_o^{-2})$. Since $|R| = |R_o| \cdot |U| = p^{(h+1)r} \Rightarrow |Z(R)| = p^{rh}$ and $Z(R)^* = Z(R) - \{0\}$, then $|V(\Gamma(R))| = p^{rh} - 1$. which establishes (i).

- (ii) To establish (ii), note that the product of every pair of zero divisors of the form $(0, u_1, \dots, u_h)$ and $(0, v_1, \dots, v_h)$ is $(0, 0, \dots, 0)$ so that every pair of vertices in $\Gamma(R)$ are adjacent. Hence $\Gamma(R)$ is complete.
- (iii) This is clear since by (ii), $\Gamma(R)$ is a complete graph having $p^{rh} - 1$ vertices.
- (iv) $\text{Diam}(\Gamma(R)) = \sup\{d(x, y) | x, y \in Z(R)^*\}$. Since $Z(R)^* = Z(R) - \{0\}$, and for all distinct $x, y \in Z(R) - \{0\}$, $xy = 0$, we have that $d(x, y) = 1$. So, $\sup\{d(x, y)\} = 1 \forall x, y \in \Gamma(R)$. This implies that $\text{diam}(\Gamma(R)) = 1$.
- (v) A complete graph $\Gamma(R)$ with n vertices is denoted as K_n . When $r = 1, h = 1$ and $p = 2, 3$, $n = (p^{rh} - 1) \leq 2$. So $\text{girth}(\Gamma(R)) = \infty$. Otherwise, $\forall n = (p^{rh} - 1) > 2$, it is well known from [1] that $\text{girth}(\Gamma(R)) \leq 2\text{diam}(\Gamma(R)) + 1$. Since $\text{diam}(\Gamma(R)) = 1$, the result readily follows.
- (vi) Let S be the set of vertices of minimal degree. Clearly $S = \Phi$, thus $b(\Gamma(R)) = \infty$.
- (vii) $|V(\Gamma(R))| = p^{rh} - 1$. So let $u_i \in \Gamma(R)$ be a vertex such that $u_i \cdot u_j = 0; \forall i, j$, thus each vertex is adjacent to every other vertex except itself for avoidance of loops. Therefore, the number of vertices adjacent to u_i is $(p^{rh} - 1) - 1$. So, $\Delta(\Gamma(R)) = p^{rh} - 2$.
- (viii) The minimum number of colors that can be assigned to each vertex relates to the maximum degree of each vertex in $\Gamma(R)$. Therefore it can be established that $\chi(\Gamma(R)) \leq p^{rh} - 1$.

□

Lemma 2. *Let R be the ring described by construction II and $\Gamma(R)$ be its zero divisors graph. Then,*

- (i) $|V(\Gamma(R))| = p^{(h+1)r} - 1$.
- (ii) $\Gamma(R)$ is complete.
- (iii) $\Gamma(R) = K_{p^{(h+1)r} - 1}$.
- (iv) $\text{diam}(\Gamma(R)) = 1$.
- (v) $\text{girth}(\Gamma(R)) = \begin{cases} \infty, & \text{if } r=1 \text{ and } p=2,3 \\ 3, & \text{elsewhere} \end{cases}$
- (vi) $b(\Gamma(R)) = \infty$.

$$(vii) \Delta(\Gamma(R)) = p^{(h+1)r} - 2.$$

Proof. (i) By the construction II, $x \in Z(R)^*$ if and only if x is in the form $(0, \bar{r}_1, \bar{r}_2, \dots, \bar{r}_h)$. Now let $x = (0, \bar{r}_1, \bar{r}_2, \dots, \bar{r}_h)$ not in $Z(R)$ be an element in R , then x is invertible and indeed x^{-1} is $(s_o, \bar{s}_1, \dots, \bar{s}_h)$ such that $s_o = r_o^{-1}$ and $\bar{s}_i = -\bar{r}_i r_o^{-2}$ for $1 \leq i \leq h$. Since $|R| = |R_o| + |U| = p^{(h+2)r}$, $|Z(R)| = p^{(h+1)r}$ and $Z(R)^* = Z(R) - \{0\}$, then $|V(\Gamma(R))| = p^{(h+1)r} - 1$.

(ii) For all $x, y \in Z(R)^*$, $xy = 0$ so, $\Gamma(R)$ is complete.

(iii) Clearly follows from (ii).

(iv) It can be seen that $x, y \in Z(R)^*$, $d(x, y) = 1$ so, $\text{Sup}\{d(x, y)\} = 1$ for all $x, y \in \Gamma(R)$. Therefore the result follows immediately.

(v) When $R = 1$, $h = 0$ and $p = 2$ or 3 , $\Gamma(R) = K_3$ then $n = (p^{(h+1)r} - 1) \leq 2$ and hence $\Gamma(R)$ has no cycles so $\text{girth}(\Gamma(R)) = \infty$. Otherwise, $\forall r, h \geq 1$, $n = (p^{(h+1)r} - 1) > 2$ so the completeness of $\Gamma(R)$ implies that $\text{girth}(\Gamma(R)) = 2\text{diam}(\Gamma(R)) + 1 = 3$, since $\text{diam}(\Gamma(R)) = 1$.

(vi) Since the set of vertices of minimal degree in $\Gamma(R)$ is empty, i.e, the minimum degree of each vertex is zero. Therefore, it follows that $b(\Gamma(R)) = \infty$.

(vii) Let $u_i \in Z(R)$ be a vertex such that $u_i \cdot u_j = 0 \forall i, j \in \mathbb{N}$. Each vertex being adjacent to any other vertex except its self for avoidance of a loop, the vertices adjacent to u_i are $p^{(h+1)r} - 1$ in number. Therefore, $\Delta(\Gamma(R)) = p^{(h+1)r} - 2$.

□

Proposition 1. *If R is the ring of construction I or II then the graph $\Gamma(R)$ is triangular if :*

$$(i) R = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

$$(ii) R = \mathbb{F}_4 \oplus \mathbb{F}_4.$$

$$(iii) R = \mathbb{Z}_4 \oplus \mathbb{Z}_2.$$

Proof. A zero divisor graph $\Gamma(R)$ is triangular if $\Gamma(R) = K_3$. Let R be a ring given by the constructions I and II, then $\Gamma(R) = K_{p^{rh}-1}$ or $\Gamma(R) = K_{p^{(h+1)r}-1}$ respectively for the rings of characteristic p and p^2 . It suffices to find the values of p , r and h for which $p^{hr} - 1$ or $p^{(h+1)r} - 1$ equals to 3. Now let R be of

characteristic $p = 2$, then $\Gamma(R) = K_3$ when $r = 1$ and $h = 2$ or vice versa. When R is of characteristic p^2 , the graph is triangular when $p = 2$, $r = 1$ and $h = 1$. $\square$

5. The Adjacency and Incidence Matrices of $\Gamma(R)$

Example 1. Let R be the ring of construction I such that $p = 2, k = 1, r = 1$ and $h = 1$. So, $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$, $(Z(R))^* = \{(0, 1)\}$.

The matrix $[A_{11}]$ of $\Gamma(R)$ is given by $[A_{11}] = \begin{bmatrix} 0 \end{bmatrix}$. Clearly, (i) $[A_{11}]$ is of order 1×1 , (ii) $\det | [A_{11}] | = 0$, (iii) $[A_{11}] = [A_{11}]^T$ thus symmetric, (iv) $Q([A_{11}]) = 0$, a zero form. Moreover, the incidence matrix $[D_{11}]$ corresponding to $\Gamma(R)$ is same as $[A_{11}]$

Example 2. When $p = 2, h = 2$ and $r = 1$,

$$R = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

$$= \{(0, 0, 0), (0, 1, 0), (0, 0, 1), (0, 1, 1), (1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1)\},$$

$$(Z(R))^* = \{(0, 1, 0), (0, 0, 1), (0, 1, 1)\}. [A_{33}] = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Evidently, (i) $\text{tr}([A_{33}]) = 0$, (ii) $[A_{33}] = [A_{33}]^T$ so $[A_{33}]$ is symmetric, (iii) $\text{rank}([A_{33}]) = 3$, (iv) $\det | [A_{33}] | = 2$, (v) the eigenvalues of $[A_{33}]$ are 2 and -1 of multiplicity 2. (vi) $Q([A_{33}]) = Q(x_1, x_2, x_3) = 0x_1^2 + 0x_2^2 + 0x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3 = 2x_1x_2 + 2x_1x_3 + 2x_2x_3$.

$$\text{The incidence matrix } [D_{33}] \text{ corresponding to } \Gamma(R) \text{ is } [D_{33}] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(i) $\det | [D_{33}] | = 0$, (ii) $\text{tr}([D_{33}]) = 3$, (iii) $\text{rank}([D_{33}]) = 1$, (iv) the eigenvalues of $[D_{33}]$ are 3 and 0 of multiplicity 3, (v) the quadratic form $Q([D_{33}]) = Q(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3$.

Proposition 2. Let R be the ring given by construction I and $[A_{ij}], [D_{ij}]$ be the adjacency and incidence matrices respectively for its $\Gamma(R)$, then;

$$(i) \text{tr}([A_{ij}]) = 0 \text{ and } \text{tr}([D_{ij}]) = p^{rh} - 1 \forall i, j \in \mathbb{N}.$$

$$(ii) [D_{ij}] \text{ and } [A_{ij}] \text{ are symmetric.}$$

- (iii) The order of $[A_{ij}]$ = order of $[D_{ij}] = (p^{rh} - 1) \times (p^{rh} - 1)$.
- (iv) $\text{rank}([A_{ij}]) = p^{rh} - 1$ and $\text{rank}([D_{ij}]) = 1$.
- (v) The $\det | [D_{ij}] | = 0$ and the $\det | [A_{ij}] | = \begin{cases} p^{rh} - 2, & \text{for an odd } p^{rh} - 1; \\ 2 - p^{rh}, & \text{for an even } p^{rh} - 1. \end{cases}$
where $p^{rh} - 1$ is the number of vertices of $\Gamma(R)$.
- (vi) The eigenvalues of $[A_{ij}] = \begin{cases} p^{rh} - 2, & \text{or} \\ -1, & \text{of multiplicity } p^{rh} - 2. \end{cases}$
The eigenvalues of $[D_{ij}] = \begin{cases} p^{rh} - 1, & \text{or} \\ 0, & \text{of multiplicity } p^{rh} - 2. \end{cases}$

Proof. (i) Let $A = [A_{ij}]$ of order $p^{rh} - 1$ be the adjacency matrix of $R = \mathbb{F}_p \oplus \cdots \oplus \mathbb{F}_p$.

It is well known that, $\text{tr}(A) = \sum_{i,i}^{p^{rh}-1} A_{ij}$. Since $\Gamma(R)$ is simple with no self loop allowed for any vertex v_i , $i = 1, 2, \dots, h$, the diagonal entries are all zeros thus, $\sum_{i,i}^{p^{rh}-1} A_{ij} = 0 \Rightarrow \text{tr}([A_{ij}]) = 0$. Similarly, $\text{tr}([D_{ij}]) = \sum_{i,i}^{p^{rh}-1} D_{ij}$; where D_{ij} 's are the diagonal entries of $[D_{ij}]$. By the connectedness of $\Gamma(R)$, $D_{ij} = 1$ for there exist an edge between i^{th} and j^{th} vertices $\forall i, j = 1, 2, \dots, h$.

Therefore, $\sum_{i=j}^{p^{rh}-1} D_{ij} = p^{rh} - 1$.

- (ii) It can easily be seen that $[A_{ij}] = [A_{ij}]^T$ and $[D_{ij}] = [D_{ij}]^T$ which establishes symmetry of $[A_{ij}]$ and $[D_{ij}]$.
- (iii) From Lemma 1, the order $| (Z(R))^* |$ is $p^{rh} - 1$ which corresponds to the number of vertices in $\Gamma(R)$. It follows that for any prime integer p , the order of $[A_{ij}]$ and $[D_{ij}]$ is $(p^{rh} - 1) \times (p^{rh} - 1)$.
- (iv) Upon carrying out an elementary row reductions for the adjacency ma-

$$\text{trix } [A_{ij}] = \begin{bmatrix} 0 & 1 & \cdot & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ \vdots & 1 & \ddots & \cdot & \vdots \\ \cdot & \vdots & \cdots & \ddots & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix} \text{ and for the incidence matrix } [D_{ij}] = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} \text{ we end up with reduced row echelon form of matrices}$$

$$\begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \vdots \\ \vdots & 0 & \ddots & 0 & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \text{ for } [A_{ij}] \text{ and } \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \text{ for } [D_{ij}].$$

For the rank of $[A_{ij}]$, let $V = \{v_1, v_2, \dots, v_{p^{rh}-1}\}$ be the set of linearly

$$\text{independent vectors such that } v_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, v_{p^{rh}-1} =$$

$$\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \text{ Clearly, the set } V \text{ spans the whole matrix space. Hence the rank}$$

of $[A_{ij}]$ is $p^{rh} - 1$. For $[D_{ij}]$, the only row which is non-zero is the first row. Hence, the rank of $[D_{ij}]$ is 1.

- (v) For the incidence matrix, it is clear that the elements in the rows are the same elements in the column. All the matrix minors have a determinant of 0, therefore $|[D_{ij}]| = 0$. For the adjacency matrix $[A_{ij}] =$

$$\begin{bmatrix} 0 & 1 & \cdot & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ \vdots & 1 & \ddots & \cdots & \vdots \\ \cdot & \vdots & \cdots & \ddots & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix}, \text{ on conducting row operations, we obtain an up-}$$

per triangular matrix of the form

$$\begin{bmatrix} 1 & 1 & \cdot & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 1 \\ \vdots & 0 & \ddots & \cdot & 2 \\ \cdot & \vdots & \cdots & \ddots & \cdot \\ 0 & \cdots & \cdot & 0 & p^{rh}-2 \end{bmatrix}. \text{ Expanding the determinant along the first}$$

row, we obtain $p^{rh} - 2$ for an odd $p^{rh} - 1$ and $2 - p^{rh}$ when $p^{rh} - 1$ is even.

- (vi) Clearly, the equations $|[A_{ij}] - \lambda I| = 0$ and $|[D_{ij}] - \lambda I| = 0$ yields the char-

acteristic polynomial equations for $[A_{ij}]$ and $[D_{ij}]$ respectively. The solutions for these equations gives the eigenvalues for the respective matrices. Now, for any $i, j \in \mathbb{N}$ the characteristic polynomial for the adjacency $[A_{ij}]$ is given by $p(\lambda) = \lambda^{p^{rh}-1} - (p^{rh}-1)\lambda^{p^{rh}-3} - (p^{rh}-2)\lambda^{p^{rh}-5} - \dots - 1 = 0$. Let the eigenvalues of $[A_{ij}]$ be $\lambda_1, \dots, \lambda_{p^{rh}-1}$. Denote the characteristic polynomial of $[A_{ij}]$ by $p(\lambda)$. Since the eigenvalues of $[A_{ij}]$ are the zeros of $p(\lambda)$, this implies that $p(\lambda)$ can be factorised as $p(\lambda) = (\lambda - \lambda_1) \cdots (\lambda - \lambda_{p^{rh}-1})$. Consider the constant term of $p(\lambda)$, c_o . The constant term of $p(\lambda)$ is given by $p(0)$, which can be calculated in two ways; firstly, $p(0) = (0 - \lambda_1) \cdots (0 - \lambda_{p^{rh}-1}) = (-1)^{p^{rh}-1} \lambda_1 \cdots \lambda_{p^{rh}-1}$. Secondly, $p(0) = |0I - [A_{ij}]| = |-A| = (-1)^{p^{rh}-1} |[A_{ij}]|$. Therefore, $c_o = (-1)^{p^{rh}-1} \lambda_1 \cdots \lambda_{p^{rh}-1} = (-1)^{p^{rh}-1} |[A_{ij}]|$, and so $\lambda_1 \cdots \lambda_{p^{rh}-1} = |[A_{ij}]|$. Consider the coefficient of $\lambda^{p^{rh}-1}$. This coefficient can be calculated by expanding $|\lambda I - [A_{ij}]|$. This is achieved by multiplying the elements in positions $1_{j_1}, 2_{j_2}, \dots, p^{rh} - 1_{j_{p^{rh}-1}}$ for each permutation $j_1, \dots, j_{p^{rh}-1}$ of $1, \dots, p^{rh} - 1$. If the permutation is odd, the product is also multiplied by -1 then all these $p^{rh} - 1$ are added to produce the determinant. One of these products is $(\lambda - a_{11}) \cdots (\lambda - a_{(p^{rh}-1)(p^{rh}-1)})$. Every other possible product can contain at most $p^{rh} - 2$ elements on the diagonal of the matrix and so will contain at most $p^{rh} - 2$ λ 's. For $[D_{ij}]$, the characteristic polynomial equation

$$p(\lambda) = \lambda^{p^{rh}-1} - (p^{rh} - 1)\lambda^{p^{rh}-2} = 0.$$

$$\Rightarrow \lambda^{p^{rh}-2}(\lambda - (p^{rh} - 1)) = 0 \Rightarrow \lambda = p^{rh} - 1 \text{ or } 0 \text{ of multiplicity } p^{rh} - 2.$$

□

Proposition 3. *Let R be a ring of the construction II and $Z(R)$ be the set of zero divisors of R . If $[A_{ij}]$ and $[D_{ij}]$ are the adjacency and incidence matrices respectively for $\Gamma(R)$ then,*

$$(i) [A_{ij}] = [A_{ij}]^T \text{ and } [D_{ij}] = [D_{ij}]^T.$$

$$(ii) \text{tr}([A_{ij}]) = 0 \text{ and } \text{tr}([D_{ij}]) = p^{(h+1)r} - 1.$$

$$(iii) |[D_{ij}]| = 0 \text{ and } |[A_{ij}]| = \begin{cases} p^{(h+1)r} - 2, & \text{when } p^{(h+1)r} - 1 \text{ is odd;} \\ 2 - p^{(h+1)r}, & \text{when } p^{(h+1)r} - 1 \text{ is even.} \end{cases}$$

$$(iv) \text{rank}([A_{ij}]) = p^{(h+1)r} - 1 \text{ and } \text{rank}([D_{ij}]) = 1.$$

- (v) The eigenvalues of $[A_{ij}] = \begin{cases} p^{(h+1)r} - 2, & \text{or} \\ -1, & \text{of multiplicity } p^{(h+1)r} - 2. \end{cases}$
 and the eigenvalues of $[D_{ij}] = \begin{cases} p^{(h+1)r} - 1, & \text{or} \\ 0, & \text{of multiplicity } p^{(h+1)r} - 2. \end{cases}$
- (vi) Order of $[A_{ij}] = (p^{(h+1)r} - 1) \times (p^{(h+1)r} - 1) = \text{order of } [D_{ij}]$.

Proof. (i),(ii),(iii), (iv) and (v) are similar to proofs of proposition 2 with some little modifications.

- (vi) Since $Z(R)^* = Z(R) - \{0\}$ and $|\Gamma(R)| = p^{(h+1)r} - 1$, it is already seen that the number of vertices of $\Gamma(R)$ corresponds to the number of rows and columns for the $[A_{ij}]$ and $[D_{ij}]$ matrices. Therefore the order of $[A_{ij}]$ and $[D_{ij}]$ is $(p^{(h+1)r} - 1) \times (p^{(h+1)r} - 1)$. □

Corollary 4. *If $[A_{ij}]$ be the adjacency matrix corresponding to $\Gamma(R)$ then the quadratic form of $[A_{ij}]$ is linear.*

Proof. Since the quadratic form of $[A_{ij}]$, $Q(x_1, x_2, \dots, x_h) = Q(\bar{x})$ and A_{ij} are the entries of the $n \times n$ matrix. Now for $i = j$, a_{ij} are the coefficients of x_i^2 . We know that, $\text{tr}([A_{ij}]) = 0$ thus x_i^2 are all 0. Thus we obtain an equation of the form $Q([A_{ij}]) (x_1, x_2, \dots, x_{p^h-1}) = 2x_i(x_{i+1} + x_{i+2} + \dots + x_{p^h-1}) + 2x_{i+1}(x_{i+2} + x_{i+3} + \dots + x_{p^h-1}) + 2x_{i+2}(x_{i+3} + x_{i+4} + \dots + x_{p^h-1}) + \dots + 2x_{p^h-2}x_{p^h-1}$, which is a polynomial of maximum degree 1 hence linear. □

Theorem 5. *If $[A_{ij}]$ and $[D_{ij}]$ are the adjacency and incidence matrices respectively for $\Gamma(R)$ of classes of rings constructed, then the following statements are true;*

- (i) *The quadratic form $Q([A_{ij}])$ of the adjacency matrix is indefinite.*
- (ii) *The quadratic form $Q([D_{ij}])$ of the incidence matrix is positive semi-definite.*

Proof. (i) Intuitively from the constructed quadratic forms, $Q([A_{ij}]) > 0$ for some x and $Q([A_{ij}]) < 0$ for other values of x . So, by definition of indefiniteness, $Q([A_{ij}])$ is indefinite.

- (ii) For positive semi-definiteness, consider $D = S'S$ where S is a square symmetric matrix of order $p^{rh} - 1$ or $p^{(h+1)r} - 1$. Let $v \neq 0$ be a vector in D, S then $v'Dv = vS'Sv$. But $vS'Sv$ is a sum of squares and therefore

$vS'Sv = 0$ if and only if $Sv = 0$. Thus there exist at least one non-null v such that $v'Dv = 0$. This shows that $[D_{ij}]$ is positive semi definite. $\square$

6. The Adjacency Matrices $[A_{ij}]$'s as Transformations

Definition 6. Let $[A_{ij}]$ be an $(p^{rh} - 1) \times p^{(rh-1)}$ matrix. The null space of $[A_{ij}]$ denoted by $N([A_{ij}])$ is the set of vectors in $\mathbb{R}^{p^{rh-1}}$ defined by $N(A) = \{x : Ax = 0, x \in \mathbb{R}^{p^{rh-1}}\}$.

Proposition 4. Let $A = [A_{ij}]$ be the $(p^{rh} - 1) \times (p^{rh} - 1)$ adjacency matrix of the ring of construction I , the null space of A , $N(A)$ is the set of all $(p^{rh} - 1)$ -dimensional column vector x such that $Ax = 0$ and the Nullity of A is 0.

Proof. For any p, h and $r = 1$, the matrix $[A_{ij}] = \begin{bmatrix} 0 & 1 & \cdot & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ \vdots & 1 & \ddots & \cdots & \vdots \\ \cdot & \vdots & \cdots & \ddots & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix}$

can be reduced by an elementary row reduction to echelon form. The augmented

form of matrix $[A_{ij}]$, $[A \mid 0]$ is equivalent to $\begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \vdots \\ \vdots & 0 & \ddots & 0 & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$. Now, the

solutions for the systems are;

$x_1 = x_2 = x_3 = \cdots = x_{p^{rh-1}} = 0$. From the system, it can be seen that the

solution space is the set $\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{p^{rh-1}} \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\}$. Clearly, $N([A_{ij}]) =$

$\{0\}$. Also, from the dimension Theorem, the nullity of $[A_{ij}]$ is $(p^{rh} - 1) - (p^{rh} - 1) = 0$. $\square$

Definition 7. If $[A_{ij}]$ is a $(p^{rh} - 1) \times (p^{rh} - 1)$ adjacency matrix, the row space of $[A_{ij}]$ is the $\text{span}\{v_1, v_2, \dots, v_{p^{rh}-1}\}$ and the rows of $[A_{ij}]$ can be regarded as vectors $v_1, v_2, \dots, v_{p^{rh}-1} \in \mathbb{R}^{p^{rh}-1}$.

Proposition 5. The row space of $(p^{rh} - 1) \times (p^{rh} - 1)$ adjacency matrix $[A_{ij}]$ of the ring R of construction I is the subspace of $\mathbb{R}^{p^{rh}-1}$ spanned by rows of $[A_{ij}]$. The dimension for the row space of the adjacency matrix is $p^{rh} - 1$.

Proof. From the elementary row operation on $[A_{ij}]$, we obtain

$$\begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \vdots \\ \vdots & 0 & \ddots & 0 & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

The row vectors with the leading 1's forms the basis for the row space of $[A_{ij}]$. The vectors in the row space of $[A_{ij}]$ are

$$\left(\begin{matrix} 0 & 1 & 1 & \cdots & 1 \end{matrix} \right)_1, \left(\begin{matrix} 1 & 0 & 1 & \cdots & 1 \end{matrix} \right)_2, \dots, \left(\begin{matrix} 1 & 1 & 1 & \cdots & 0 \end{matrix} \right)_{p^{rh}-1}.$$

Therefore,

$$\begin{aligned} &\text{Row space of } [A_{ij}] \\ &= \left\{ \left(\begin{matrix} 0 & 1 & 1 & \cdots & 1 \end{matrix} \right), \left(\begin{matrix} 1 & 0 & 1 & \cdots & 1 \end{matrix} \right), \dots, \left(\begin{matrix} 1 & 1 & 1 & \cdots & 0 \end{matrix} \right) \right\}. \end{aligned}$$

From this, it is clear that the dimension for the row space of $[A_{ij}]$ is $p^{rh} - 1$. $\square$

Definition 8. The column space of a $(p^{rh} - 1) \times (p^{rh} - 1)$ adjacency matrix $[A_{ij}]$ of construction I is the subspace of $\mathbb{R}^{p^{rh}-1}$ spanned by columns of $[A_{ij}]$.

Remark 2. The column space of $[A_{ij}]$ coincides with the row space of $[A_{ij}]^T$. Since $[A_{ij}]^T = [A_{ij}]$, the dimension for the row space of the adjacency matrix equals the dimension of the column space.

Proposition 6. The kernel of the linear transformation $T : \mathbb{R}^{p^{rh}-1} \rightarrow \mathbb{R}^{p^{rh}-1}$ is the set of all $x \in \mathbb{R}^{p^{rh}-1}$ such that $Tx = 0 \in \mathbb{R}^{p^{rh}-1}$. The range of T is the set of all $y \in \mathbb{R}^{p^{rh}-1}$ such that $y = T(x)$ for all $x \in \mathbb{R}^{p^{rh}-1}$.

Proof. Finding the kernel of T amounts to finding the null space of $[A_{ij}]$. So, we solve the system $Ax = 0$. The rest follows from proposition 4. The range

of T can be obtained from the column space of $[A_{ij}]$. It is a subspace of $\mathbb{R}^{p^{rh}-1}$. We therefore choose the original column of the matrix to form the basis for the

$$\text{range} = \left\{ \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ \vdots \end{pmatrix}, \dots, \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \right\}.$$

□

Corollary 9. Let $[A_{ij}]$ be the adjacency matrix of $\Gamma(R)$ of the ring in construction I. Given the transformation T such that $Tx = [A_{ij}]x$ then;

- (i) the Dimension for the row space of $[A_{ij}] = \text{dimension column space of } [A_{ij}] = p^{rh} - 1$
- (ii) the Nullity of $[A_{ij}] = 0$.
- (iii) the Range of T is $p^{rh} - 1$.
- (iv) the Range($[A_{ij}]$) + Nullity ($[A_{ij}]$) = $p^{rh} - 1$.

Remark 3. The adjacency matrix $[A_{ij}]$ as transformations in construction II is similar to construction I.

To this end, we present the following general results:

Theorem 10. Let R be the ring of constructions I and II and $[A_{ij}]$ and $[D_{ij}]$ be the adjacency and incidence matrices respectively for $\Gamma(R)$ then:

- (i) $[A_{ij}]$ and $[D_{ij}]$ are both symmetric.
- (ii) order of

$$[A_{ij}] = \text{order } [D_{ij}] = \begin{cases} (p^{rh} - 1) \times (p^{rh} - 1), & \text{for char}(R)=p \\ (p^{(h+1)r} - 1) \times (p^{(h+1)r} - 1), & \text{for char}(R)=p^2. \end{cases}$$

$$(iii) \text{rank}([A_{ij}]) = \text{trace}([A_{ij}]) = \begin{cases} p^{rh} - 1, & \text{for char}(R)=p \\ p^{(h+1)r} - 1, & \text{for char}(R)=p^2. \end{cases}$$

$$(iv) \text{rank}([D_{ij}]) = 1 \text{ when char}(R) \text{ are both } p \text{ and } p^2.$$

(v) The eigenvalues of

$$[A_{ij}] = \begin{cases} p^{rh} - 2 \text{ or } -1 \text{ of multiplicity } p^{rh} - 2, & \text{for char}(R)=p \\ p^{(h+1)r} - 2 \text{ or } -1 \text{ of multiplicity } p^{(h+1)r} - 2, & \text{for char}(R)=p^2. \end{cases}$$

(vi) The eigenvalues of

$$[D_{ij}] = \begin{cases} p^{rh} - 1 \text{ or } 0 \text{ of multiplicity } p^{rh} - 2, & \text{for } \text{char}(R)=p \\ p^{(h+1)r} - 1 \text{ or } 0 \text{ of multiplicity } p^{(h+1)r} - 2, & \text{for } \text{char}(R)=p^2. \end{cases}$$

$$(vii) | [A_{ij}] | = \begin{cases} p^{rh} - 2 \text{ and } 2 - p^{rh} \text{ for odd respective even orders,} & \text{when } \text{char}(R)=p \\ p^{(h+1)r} - 2 \text{ and } 2 - p^{(h+1)r} \text{ for odd respective even orders,} & \text{when } \text{char}(R)=p^2. \end{cases}$$

$$(viii) | [D_{ij}] | = 0 \text{ for } \text{char}(R) = p \text{ and } p^2.$$

$$(ix) \text{trace}([A_{ij}]) = 0 \text{ when } \text{char}(R) = p \text{ and } p^2.$$

(x) The dimension of the row space and the column spaces for the adjacency matrices are $p^{rh} - 1$ and $p^{(h+1)r} - 1$ respectively for constructions I and II.

(xi) The $N([A_{ij}])$ of the adjacency matrices is $\{0\}$.

Proof. Follows from the results of constructions I and II as captured in the previous propositions. $\square$

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