SANDWICH-TYPE OUTCOME BASED ON
A DUAL LINEAR OPERATOR

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Abstract: By using a dual linear operator that we define it on meromorphic multivalent functions (MMF) in the punctured unit disk \( U^* \), we derive some third-order differential subordination and superordination outcomes by taking into account the appropriate class of admissible functions. Sufficient conditions are determined to yield the best dominant and the best subordinant, respectively. Moreover, sandwich-type outcome is obtained.

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1. Introduction

Let \( \mathcal{H}(\mathcal{U}) \) be the class of analytic functions defined in the unit disk \( \mathcal{U} = \{ z \in \mathcal{C} : |z| < 1 \} \). For \( k \in \mathcal{N} \) and \( a \in \mathcal{C} \), let \( \mathcal{H}[a,k] \) be the subclass of \( \mathcal{H}(\mathcal{U}) \) consisting of functions of the form \( h(z) = a + a_k z^k + a_{k+1} z^{k+1} + \ldots \) and \( \mathcal{H}_0 = \mathcal{H}[0,1] \) and \( \mathcal{H}_1 = \mathcal{H}[1,1] \).
Let $F_1$ and $F_2$ be two members in the class $\mathcal{H}(U)$, the function $F_1$ is said to be subordinate to $F_2$, or $F_2$ is superordinate to $F_1$, if there exists an analytic function $g(z)$ in the unit open disk $U$ with $g(0) = 0$ and $|g(z)| < 1$ for all $z \in U$, such that $F_1(z) = F_2(g(z))$. In such a case, we write $F_1 \prec F_2$ or $F_1(z) \prec F_2(z)$.

Let $\Sigma_p$ signify the class of all meromorphic multivalent functions (MMF) of the form:

$$h(z) = \frac{1}{z^p} + \sum_{k=1-p}^\infty a_k z^k, \quad (p \in \mathbb{N} = \{1, 2, \ldots\}, \ z \in U^*),$$

which are analytic in the punctured unit disk $U^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$.

For function $h$ defined in (1) and $g$ given by

$$g(z) = \frac{1}{z^p} + \sum_{k=1-p}^\infty b_k z^k,$$

the convolution (or Hadamard product) denoted by $h \ast g$ of the functions $h$ and $g$ is defined by

$$h \ast g(z) = \frac{1}{z^p} + \sum_{k=1-p}^\infty a_k b_k z^k.$$

El-Ashwah et al. [1] imposed the linear operator $L_p(l, \mu) : \Sigma_p \to \Sigma_p$ as follows:

$$L_p(l, \mu)h(z) = \frac{1}{z^p} + \sum_{n=1-p}^\infty \left(\frac{\mu + k + p}{\mu}\right)^l a_k z^k.$$

We consider the following dual linear operator $X_p(l, \mu) : \Sigma_p \to \Sigma_p$:

$$X_p(l, \mu)h(z) = \frac{1}{z^p} + \sum_{n=1-p}^\infty \left(\frac{\mu}{\mu + k + p}\right)^l a_k z^k,$$

equation (5) presents a proof of a recurrence relation as:

$$z (L_p(l + 1, \mu)h(z))' = \mu (L_p(l, \mu)h(z)) - (\mu + p) (L_p(l + 1, \mu)h(z)).$$

The theory of differential subordination was studied and applied by numerous authors to deal with various important problems in the area of geometric function theory (GFT). Research monographs on this subject include those of (see
In 2011, Antonino and Miller [19] investigated and discussed the third-order differential subordination (TDSB) (which is a generalization of second-order differential subordination). In 2014, Tang et al. [20] have contributed to the development of this area. They investigated the third-order differential superordination (TDSP) as the dual problem of the differential subordination (TDSB). Afterwards, several complex analysts were attracted to utilize the methods of the third-order differential subordination (TDSB) and superordination (TDSP). For instance, Tang and Deniz [21], Tang et al. [22], Farzana et al. [23], Ibrahim et al. [24], El-Ashwah and Hassan [25] and Al-janaby et al. [26].

2. Preliminaries

In 2011, Antonino and Miller [19] introduced the following essential concepts in the theory of the third-order differential subordination. In this section, we will mention the essential definitions and fundamental theorems in the theory of third-order differential subordination and superordination which are needed in our results.

**Definition 1.** ([19], Definition 1, p. 440) Let $\tau : C^4 \times U \rightarrow C$ and the function $\eta(z)$ be univalent in $U$. If $p(z)$ is analytic in $U$ and satisfies the following third-order differential subordination:

$$\tau(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \prec \eta(z),$$

then $p(z)$ is called a solution of the differential subordination. A univalent function $q(z)$ is called a dominant of the solutions of the differential subordination, or, more simply, a dominant if $p(z) \prec q(z)$ for all $p(z)$ satisfying (7). A dominant $\tilde{q}(z)$ that satisfies $\tilde{q}(z) \prec q(z)$ for all dominants $q(z)$ of (7) is said to be the best dominant.

**Definition 2.** ([19], Definition 2, p. 441) Let $Q$ denote the set of functions $q$ that are analytic and univalent on the set $\overline{U \setminus E(q)}$, where

$$E(q) = \{ \zeta : \zeta \in \partial U : \lim_{z \to \zeta} q(z) = \infty \}$$

is such that $\min |q'(\zeta)| = \rho > 0$ for $\zeta \in \partial U \setminus E(q)$. Further, let the subclass of $Q$ for which $q(0) = a$ be denoted by $Q(a)$, $Q(0) \equiv Q_0$ and $Q(1) \equiv Q_1$.

The next definition of the class of admissible functions (ADF) is due to Antonino and Miller [19].
Definition 3. ([19], Definition 3, p. 449) Let $\Lambda$ be a set in $\mathcal{C}$ and $q \in \mathcal{Q}$ and $n \in N \setminus \{1\}$. The class of admissible functions $\Psi_n[\Lambda, q]$ consists of those functions $\tau : C^4 \times U \to \mathcal{C}$ that satisfy the following admissibility condition:

$$\tau(r, s, t, u; z) \notin \Lambda$$

whenever

$$r = q(\zeta), \quad s = k\zeta q'(\zeta), \quad \Re \left( \frac{t}{s} + 1 \right) \geq k\Re \left( \frac{\zeta q''(\zeta)}{q'(\zeta)} + 1 \right),$$

and

$$\Re \left( \frac{u}{s} \right) \geq k^2\Re \left( \frac{\zeta^2 q''(\zeta)}{q'(\zeta)} \right),$$

where $z \in U$, $\zeta \in \partial U \setminus E(q)$, and $k \geq n$.

Antonino and Miller [19] have given the foundation result in the theory of third-order differential subordination.

Lemma 4. ([19], Theorem 1, p. 449) Let $p \in \mathcal{H}[a, n]$ be with $n \geq 2$. Also, let $q \in \mathcal{Q}(a)$ and satisfy the following conditions:

$$\Re \left( \frac{\zeta q''(\zeta)}{q'(\zeta)} \right) \geq 0, \text{ and } \left| \frac{zp'(z)}{q'(\zeta)} \right| \leq k,$$

where $z \in U$, $\zeta \in \partial U \setminus E(q)$, and $k \geq n$. If $\Lambda$ is a set in $\mathcal{C}$, $\tau \in \Psi_n[\Lambda, q]$ and

$$\tau(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \in \Lambda,$$

then $p(z) \prec q(z) z \in U$.

We now describe the basic concepts and the main results in the theory of the third-order differential superordination that are considered by Tang et al. [20].

Definition 5. [20] Let $\tau : C^4 \times U \to \mathcal{C}$ and the function $j(z)$ be analytic in $U$. If the functions $p(z)$ and $\tau(p(z), zp'(z), z^2p''(z), z^3p'''(z); z)$ are univalent in $U$ and satisfy the following third-order differential superordination:

$$j(z) \prec \tau(p(z), zp'(z), z^2p''(z), z^3p'''(z); z),$$

then $p(z)$ is called a solution of the differential superordination. An analytic function $q(z)$ is called a subordinant of the solutions of the differential superordination, or, more simply, a subordinant if $q(z) \prec p(z)$ for all $p(z)$ satisfying (9). A univalent subordinant $\tilde{q}(z)$ that satisfies the condition $q(z) \prec \tilde{q}(z)$ for all subordinants $q(z)$ of (9) is said to be the best subordinant.
Definition 6. [20] Let $\Lambda$ be a set in $C$, $q \in H[a, n]$ with $q'(z) \neq 0$ and $n \in N\{1\}$. The class of admissible functions $\Psi'_n[\Lambda, q]$ consists of those functions $\tau : C^4 \times \mathcal{U} \rightarrow C$ that satisfy the following admissibility condition:

$$\tau(r, s, t, u; \zeta) \in \Lambda$$

whenever

$$r = q(z), \quad s = \frac{zq'(z)}{m}, \quad \Re\left(\frac{t}{s} + 1\right) \leq \frac{1}{m} \Re\left(\frac{zq''(z)}{q'(z)} + 1\right),$$

and

$$\Re\left(\frac{u}{s}\right) \leq \frac{1}{m^2} \Re\left(\frac{z^2q'''(z)}{q'(z)}\right),$$

where $z \in \mathcal{U}$, $\zeta \in \partial\mathcal{U}$, and $m \geq n \geq 2$.

Lemma 7. [20] Let $\tau \in \Psi'_n[\Lambda, q]$. If $\tau \left(p(z), zp'(z), z^2p''(z), z^3p'''(z) ; z\right)$ is univalent in $\mathcal{U}$ and $p \in Q(a)$ and $q \in H[a, n]$ satisfy the following conditions:

$$\Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right) \geq 0, \quad \text{and} \quad \left|\frac{zp'(z)}{q'(\zeta)}\right| \leq m,$$

where $z \in \mathcal{U}$, $\zeta \in \partial\mathcal{U}$, and $m \geq n \geq 2$, then

$$\Lambda \subset \{\tau \left(p(z), zp'(z), z^2p''(z), z^3p'''(z) ; z\right) : z \in \mathcal{U}\}$$

implies that

$$q(z) \prec p(z) \quad (z \in \mathcal{U}).$$

In this study, we introduce certain suitable classes of admissible functions and investigate several applications of the third-order differential subordination and superordination of meromorphic multivalent functions associated with a dual linear operator $X_p(l, \mu)h(z)$ defined in (5).
3. Subordination of the Dual Linear Operator $X_p(l, \mu)$

In this section, by using of the excellent outcomes for the third order differential subordination (TDSB) due to authors Antonino and Miller [19] in the open unit disk $\mathcal{U}$, we define a suitable class of (ADF) which is a first step required to prove the differential subordination outcomes for meromorphically multivalent functions associated with the operator $X_p(l, \mu)h(z)$ given in (5).

**Definition 8.** Let $\Lambda$ be a set in $\mathbb{C}$ and $q \in \mathcal{Q}_1 \cap \mathcal{H}_1$. The class of admissible functions $\Gamma_{X}[\Lambda, q]$ consists of those functions $\omega : \mathbb{C}^4 \times \mathcal{U} \rightarrow \mathbb{C}$ that satisfy the following admissibility condition:

$$\omega(a, b, c, d; z) \notin \Lambda$$

whenever

$$a = q(\zeta), \quad b = \frac{k\zeta q'(\zeta) + \mu q(\zeta)}{\mu},$$

$$\Re\left(\frac{\mu(c - a)}{b - a} - 2\mu\right) \geq k \Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right),$$

and

$$\Re\left(\frac{\mu^2(d - a)}{b - a} - 3[\mu + 1]\left[\frac{\mu(c - a)}{b - a} - [2\mu + 1]\right] - [3\mu^3 + 3\mu + 1]\right) \geq k^2 \Re\left(\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\right),$$

where $z \in \mathcal{U}$, $\zeta \in \partial\mathcal{U} \setminus E(q)$, and $k \geq n$.

We now state and prove our main third-order differential subordinations outcomes.

**Theorem 9.** Let $\omega \in \Gamma_{\mathcal{H}}[\Lambda, q]$. If the function $h \in \Sigma_p$ and $q \in \mathcal{Q}_1$ satisfy the following conditions

$$\Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right) \geq 0, \quad \left|\frac{z^p(X_p(l + 2, \mu)h(z))}{q'(\zeta)}\right| \leq k,$$  \hspace{1cm} (11)

and

$$\left\{\omega\left(z^p(X_p(l + 3, \mu)h(z))\right), z^p(X_p(l + 2, \mu)h(z)), z^p(X_p(l + 1, \mu)h(z)), z^p(X_p(l, \mu)h(z)) : z \in \mathcal{U}\right\} \subset \Lambda,$$  \hspace{1cm} (12)

then

$$z^p(X_p(l + 3, \mu)h(z)) \prec q(z).$$
Proof. We define the analytic function \( p(z) \) in the unit disk \( U \) by

\[
p(z) = z^p(X_p(l + 3, \mu)h(z)).
\]  

(13)

Because of the relation in equation (6), we have

\[
z(L_p(l + 3, \mu)h(z))' = \mu(L_p(l + 2, \mu)h(z)) - (\mu + p)(L_p(l + 3, \mu)h(z)).
\]  

(14)

Using equations (13) and (14), it follows that

\[
z^p(X_p(l + 2, \mu)h(z)) = \frac{zp'(z) + \mu p(z)}{\mu}
\]  

(15)

From the relation in equation (6), we get

\[
z(L_p(l + 2, \mu)h(z))' = \mu(L_p(l + 1, \mu)h(z)) - (\mu + p)(L_p(l + 2, \mu)h(z)).
\]  

(16)

From equations (15) and (16), we have

\[
z^p(X_p(l + 1, \mu)h(z)) = \frac{z^2p''(z) + (1 + 2\mu)zp'(z) + \mu^2p(z)}{\mu^2}
\]  

(17)

From equations (17) and (6), further computations show that

\[
z^3p'''(z) + 3(1 + \mu)z^2p''(z) + (1 + 3\mu + 3\mu^2)zp'(z) + \mu^3p(z)
\]  

\[
= \frac{z^3p'''(z) + 3(1 + \mu)z^2p''(z) + (1 + 3\mu + 3\mu^2)zp'(z) + \mu^3p(z)}{\mu^2}
\]  

(18)

We now define the transformation from \( C^4 \) to \( C \) by

\[
a(r, s, t, u) = r, \quad b(r, s, t, u) = \frac{s + \mu r}{\mu},
\]

\[
c(r, s, t, u) = t + (1 + 2\mu)s + \mu^2r,
\]

\[
d(r, s, t, u) = u + 3(1 + \mu)t + (1 + 3\mu + 3\mu^2)s + \mu^3r
\]  

(19)

Let

\[
\tau(r, s, t, u; z) = \omega(a, b, c, d; z) = \omega\left(\frac{r}{\mu}, \frac{t + (1 + 2\mu)s + \mu^2r}{\mu^2}, \frac{u + 3(1 + \mu)t + (1 + 3\mu + 3\mu^2)s + \mu^3r}{\mu^3}\right)
\]  

(20)
The proof will make use of Lemma 4. Using equations (13), (15), (17) and (18), we deduce from the equation (20) that

\[ \tau(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) = \omega \left( z^p(X_p(l + 3, \mu)h(z)), z^p(X_p(l + 2, \mu)h(z)), z^p(X_p(l, \mu)h(z)); z \right). \]

Hence equation (12) becomes

\[ \tau(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \in \Lambda. \]

Utilizing (19), then we have

\[ \frac{t}{s} + 1 = \frac{\mu(c - a)}{b - a} - 2\mu, \]

and

\[ \frac{u}{s} = \frac{\mu^2(d - a)}{b - a} - 3[\mu + 1] \left[ \frac{\mu(c - a)}{b - a} - [2\mu + 1] \right] - [3\mu^3 + 3\mu + 1]. \]

Thus, the admissibility condition for \( \omega \in \Gamma_X[\Lambda, q] \) in Definition 8 is equivalent to the admissibility condition for \( \tau \in \Psi_n[\Lambda, q] \) as given in Definition 3. Therefore, by using (11) and Lemma 4, we have \( p(z) \prec q(z) \) or equivalently,

\[ z^p(X_p(l + 3, \mu)h(z)) \prec q(z). \]

The proof of Theorem 9 is completed. \( \square \)

The following result is an extension of Theorem 9 to the case where the behaviour of \( q(z) \) on \( \partial U \) is not known.

**Corollary 10.** Let \( \Lambda \subset C \) and let the function \( q \) be univalent in \( U \) with \( q(0) = 1 \). Suppose also that \( \omega \in \Gamma_X[\Lambda, q_\rho] \) for some \( \rho \in (0,1) \), where \( q_\rho(z) = q(\rho z) \). If the function \( h \in \Sigma_p \) and \( q_\rho \) satisfy the following conditions:

\[ \Re \left( \frac{\zeta q_\rho''(\zeta)}{q_\rho'(\zeta)} \right) \geq 0, \quad \left| \frac{z^p(X_p(l + 2, \mu)h(z))}{q_\rho'(\zeta)} \right| \leq k \quad (z \in U, \zeta \in \partial U \setminus E(q_\rho)), \]

and

\[ \omega \left( z^p(X_p(l + 3, \mu)h(z)), z^p(X_p(l + 2, \mu)h(z)), z^p(X_p(l + 1, \mu)h(z)), z^p(X_p(l, \mu)h(z)); z \right) \in \Lambda. \]
then

\[ z^p(X_p(l + 3, \mu)h(z)) \prec q(z). \]

**Proof.** As a consequence of Theorem 9, we have \( z^p(X_p(l + 3, \mu)h(z)) \prec q(z). \) The result asserted by Corollary 10 is now deduced from the subordination property \( q_\rho(z) \prec q(z). \)

If \( \Lambda \neq C \) is a simply connected domain, then \( \Lambda = \gamma(U) \), for some conformal mapping \( \gamma(z) \) of \( U \) onto \( \Lambda \). In this case, the class \( \Gamma_X[j(U), q] \) is written as \( \Gamma_X[X, q] \). The following result follows immediate consequence from Theorem 9.

**Theorem 11.** Let \( \omega \in \Gamma_X[j, q] \). If the function \( h \in \Sigma_p \) and \( q \in Q_1 \) satisfy the following conditions

\[
\Re \left( \frac{\zeta q''(\zeta)}{q'(\zeta)} \right) \geq 0, \quad \left| z^p(X_p(l + 2, \mu)h(z)) \right| \leq k,
\]

and

\[
\omega \left( z^p(X_p(l + 3, \mu)h(z)), z^p(X_p(l + 2, \mu)h(z)), z^p(X_p(l + 1, \mu)h(z)), z^p(X_p(l, \mu)h(z)); z \right) \prec \gamma(z),
\]

then

\[ z^p(X_p(l + 3, \mu)h(z)) \prec q(z). \]

The next outcome is an immediate consequence of Corollary 10.

**Corollary 12.** Let \( \Lambda \subset C \) and let the function \( q \) be univalent in \( U \) with \( q(0) = 1 \). Suppose also that \( \omega \in \Gamma_X[\Lambda, q_\rho] \) for some \( \rho \in (0, 1) \), where \( q_\rho(z) = q(\rho z) \). If the function \( h \in \Sigma_p \) and \( q_\rho \) satisfy the following conditions:

\[
\Re \left( \frac{\zeta q''_\rho(\zeta)}{q'_\rho(\zeta)} \right) \geq 0, \quad \left| z^p(X_p(l + 2, \mu)h(z)) \right| \leq k \quad (z \in U, \zeta \in \partial U \setminus E(q_\rho)),
\]

and

\[
\omega \left( z^p(X_p(l + 3, \mu)h(z)), z^p(X_p(l + 2, \mu)h(z)), z^p(X_p(l + 1, \mu)h(z)), z^p(X_p(l, \mu)h(z)); z \right) \prec \gamma(z).
\]

then

\[ z^p(X_p(l + 3, \mu)h(z)) \prec q(z). \]
The following outcome yields the best dominant of the differential subordination (12) or (22).

**Theorem 13.** Let the function \( j \) be univalent in \( U \). Also, let the function \( \omega : \mathbb{C}^4 \times U \to \mathbb{C} \) and \( \tau \) be given by (20). Suppose that the differential equation

\[
\tau(q(z), zq'(z), z^2q''(z), z^3q'''(z); z) = j(z),
\]

has a solution \( q(z) \) with \( q(0) = 0 \), which satisfies condition (11). If the function \( h \in \Sigma_p \) satisfies condition (22) and functions

\[
\omega \left( z^p(X_p(l + 3, \mu)h(z)), z^p(X_p(l + 2, \mu)h(z)), z^p(X_p(l + 1, \mu)h(z)), z^p(X_p(l, \mu)h(z)); z \right)
\]

are analytic in \( U \), then

\[
z^p(X_p(l + 3, \mu)h(z)) \prec q(z).
\]

and \( q(z) \) is the best dominant.

**Proof.** In view of Theorem 9, we deduce that \( q \) is a dominant of (22). Since \( q \) satisfies (23), it is also a solution of (22) and therefore \( q \) will be dominated by all dominants. Hence, \( q \) is the best dominant. \( \square \)

In view of Definition 8, and in the special case \( q(z) = 1 + Mz, M > 0 \), the class of admissible functions \( \Gamma_X[\Lambda, q] \), denoted by \( \Gamma_X[\Lambda, M] \), is expressed as follows.

**Definition 14.** Let \( \Lambda \) be a set in \( \mathbb{C} \) and \( M > 0 \). The class \( \Gamma_X[\Lambda, M] \) of admissible functions consists of those functions \( \omega : \mathbb{C}^4 \times U \to \mathbb{C} \) such that

\[
\omega \left( 1 + Me^{i\theta}, 1 + \frac{(\mu + k)Me^{i\theta}}{\mu}, 1 + \frac{L + [\mu^2 + (1 + 2\mu)k]Me^{i\theta}}{\mu^2}; \frac{N + 3(1 + \mu)L + [\mu^3 + (1 + 3\mu + 3\mu^2)k]Me^{i\theta}}{\mu^3}; z \right) \notin \Lambda,
\]

whenever \( z \in U, \Re(Le^{-i\theta}) \geq (k - 1)kM, \) and \( \Re(Ne^{-i\theta}) \geq 0 \) for all \( \theta \in \mathbb{R} \) and \( k \geq 2 \).

**Corollary 15.** Let \( \omega \in \Gamma_X[\Lambda, M] \). If the function \( h \in \Sigma_p \) satisfies

\[
|z^p(X_p(l + 2, \mu)h(z))| \leq kM \quad (k \geq 2; M > 0),
\]
and
\[
\begin{align*}
\omega(z^p(X_p(l+3,\mu)h(z)), z^p(X_p(l+2,\mu)h(z)), \\
z^p(X_p(l+1,\mu)h(z)), z^p(X_p(l,\mu)h(z)); z) &\in \Lambda.
\end{align*}
\]

then
\[
|z^p(X_p(l+3,\mu)h(z)) - 1| < M.
\]

Proof. The outcome follows from Theorem 9 by taking \( q(z) = 1 + Mz \).

In the special case \( \Lambda = q(U) = \{\nu : |\nu - 1| < M\} \), the class of admissible functions \( \Gamma_X[\Lambda, M] \) is simply denoted by \( \Gamma_X[M] \).

**Corollary 16.** Let \( \omega \in \Gamma_X[M] \). If the function \( h \in \Sigma_p \) satisfies the following conditions:
\[
|z^p(X_p(l+2,\mu)h(z))| \leq kM \quad (k \geq 2; M > 0),
\]
and
\[
\begin{align*}
\omega(z^p(X_p(l+3,\mu)h(z)), z^p(X_p(l+2,\mu)h(z)), \\
z^p(X_p(l+1,\mu)h(z)), z^p(X_p(l,\mu)h(z)); z) - 1 &< M
\end{align*}
\]
then
\[
|z^p(X_p(l+3,\mu)h(z)) - 1| < M.
\]

Proof. By setting \( \Lambda = q(U) \) in Corollary 15. This outcome is obtained.

**Corollary 17.** Let \( k \in N \setminus \{1\} \) and \( M > 0 \). If the function \( h \in \Sigma_p \) satisfies the following condition:
\[
|z^p(X_p(l+2,\mu)h(z))| \leq kM,
\]
and satisfies one of the following:

1. \[
|z^p(X_p(l+2,\mu)h(z)) + (z^p(X_p(l+3,\mu)h(z)) - 2| < 2 \left(1 + \frac{1}{\mu}\right)M.
\]

2. \[
|z^p(X_p(l+1,\mu)h(z)) - 2z^p(X_p(l+2,\mu)h(z)) + (z^p(X_p(l+3,\mu)h(z)) - 1| < \frac{4M}{\mu^2}.
\]

Then
\[
|z^p(X_p(l+3,\mu)h(z)) - 1| < M.
\]
**Proof.** It is enough to show that $\omega \in \Gamma_X[\mathcal{M}]$, that is, the admissibility condition (24) is satisfied so that the required result follows from Corollary 15.

1. Take $\omega(a, b, c, d; z) = b + a - 2$, and $\omega = \gamma(\mathcal{U})$, where $\gamma(z) = 2 \left(1 + \frac{1}{\mu}\right)\mathcal{M}$, $\mathcal{M} > 0$. Hence

\[
\left| \omega \left(1 + \mathcal{M}e^{i\theta}, 1 + \frac{(\mu + k)\mathcal{M}e^{i\theta}}{\mu}, 1 + \frac{\mathcal{L} + [\mu^2 + (1 + 2\mu)k]\mathcal{M}e^{i\theta}}{\mu^2}, \frac{1 + \mathcal{N} + 3(1 + \mu)\mathcal{L} + [\mu^3 + (1 + 3\mu + 3\mu^2)k]\mathcal{M}e^{i\theta}}{\mu^3}; z \right) \right| > 2 \left(1 + \frac{1}{\mu}\right)\mathcal{M},
\]

whenever $z \in \mathcal{U}$, $\Re(\mathcal{L}e^{-i\theta}) \geq (k - 1)k\mathcal{M}$, and $\Re(\mathcal{N}e^{-i\theta}) \geq 0$ for all $\theta \in \mathbb{R}$ and $k \geq 2$.

2. Take $\omega(a, b, c, d; z) = c - 2b + a - 1$, and $\omega = \gamma(\mathcal{U})$, where $\gamma(z) = \frac{4\mathcal{M}}{\mu^2}$, $\mathcal{M} > 0$. Hence

\[
\left| \omega \left(1 + \mathcal{M}e^{i\theta}, 1 + \frac{(\mu + k)\mathcal{M}e^{i\theta}}{\mu}, 1 + \frac{\mathcal{L} + [\mu^2 + (1 + 2\mu)k]\mathcal{M}e^{i\theta}}{\mu^2}, \frac{1 + \mathcal{N} + 3(1 + \mu)\mathcal{L} + [\mu^3 + (1 + 3\mu + 3\mu^2)k]\mathcal{M}e^{i\theta}}{\mu^3}; z \right) \right| = 1 + \frac{\mathcal{L} + [\mu^2 + (1 + 2\mu)k]\mathcal{M}e^{i\theta}}{\mu^2} - 2 \left(1 + \frac{(\mu + k)\mathcal{M}e^{i\theta}}{\mu}\right) + \left(1 + \mathcal{M}e^{i\theta}\right) - 1 = \frac{\Re(\mathcal{L}e^{-i\theta}) + k\mathcal{M}}{\mu^2} > \frac{4\mathcal{M}}{\mu^2}.
\]

whenever $z \in \mathcal{U}$, $\Re(\mathcal{L}e^{-i\theta}) \geq (k - 1)k\mathcal{M}$, and $\Re(\mathcal{N}e^{-i\theta}) \geq 0$ for all $\theta \in \mathbb{R}$ and $k \geq 2$.

\[\square\]

4. **Superordination of the Dual Linear Operator $X_p(l, \mu)$**

**Definition 18.** Let $\Lambda$ be a set in $\mathcal{C}$ and $q \in \mathcal{H}_1$ with $q'(z) \neq 0$. The class of admissible functions $\Gamma_X[\Lambda, q]$ consists of those functions $\omega : \mathcal{C}^4 \times \overline{\mathcal{U}} \to \mathcal{C}$ that satisfy the admissibility condition:

$\omega(a, b, c, d; \zeta) \in \Lambda$
whenever
\[ a = q(z), \quad b = \frac{zq'(z) + m\mu q(z)}{m\mu}, \]
\[ \Re \left( \frac{\mu(c-a)}{b-a} - 2\mu \right) \leq \frac{1}{m} \Re \left( \frac{zq''(z)}{q'(z)} + 1 \right), \]

and
\[ \Re \left( \frac{\mu^2(d-a)}{b-a} - 3[\mu + 1] \left[ \frac{\mu(c-a)}{b-a} - [2\mu + 1] \right] - [3\mu^3 + 3\mu + 1] \right) \]
\[ \leq \frac{1}{m^2} \Re \left( \frac{z^2q''(z)}{q'(z)} \right), \]

where \( z \in U, \zeta \in \partial U, \) and \( m \in N \setminus \{1\}. \)

**Theorem 19.** Let \( \omega \in \Gamma_X[\Lambda, q]. \) If the function \( h \in \Sigma_p, \) \( z^p(X_p(l + 2, \mu)h(z)) \in Q_1 \) satisfy the following conditions
\[ \Re \left( \frac{zq''(z)}{q'(z)} \right) \geq 0, \quad \left| \frac{z^p(X_p(l + 2, \mu)h(z))}{q'(z)} \right| \leq m, \tag{25} \]

and
\[ \omega \left( z^p(X_p(l + 3, \mu)h(z)), z^p(X_p(l + 2, \mu)h(z)), \right. \]
\[ \left. z^p(X_p(l + 1, \mu)h(z)), z^p(X_p(l, \mu)h(z)); z \right), \]
is univalent in \( U, \) then
\[ \Lambda \subset \left\{ \omega \left( z^p(X_p(l + 3, \mu)h(z)), z^p(X_p(l + 2, \mu)h(z)), \right. \right. \]
\[ \left. \left. z^p(X_p(l + 1, \mu)h(z)), z^p(X_p(l, \mu)h(z)) : z \in U \right\} \tag{26} \]

which implies that
\[ q(z) \prec z^p(X_p(l + 3, \mu)h(z)). \]

**Proof.** Let the function \( p(z) \) be defined by (13) and \( \tau \) by (20). Since \( \omega \in \Gamma_X[\Lambda, q], \) (21) and (26), we have
\[ \Lambda \subset \{ \tau \left( p(z), zp'(z), z^2p''(z), z^3p'''(z); z \right) : z \in U \}. \]
From (20), we deduce that the admissible condition for \( \omega \in \Gamma'_X[\Lambda, q] \) in Definition 18 is equivalent to the admissibility condition for \( \tau \) in Definition 6. Hence, by utilizing (25) and utilizing Lemma 7, we have

\[
q(z) \prec p(z).
\]
or

\[
q(z) \prec z^p(X_p(l + 3, \mu)h(z)).
\]

\[\square\]

If \( \Lambda \neq C \) is a simply connected domain, then \( \Lambda = j(\mathcal{U}) \), for some conformal mapping \( j(z) \) of \( \mathcal{U} \) onto \( \Lambda \) in this case, and the class \( \Gamma'_X[j(\mathcal{U}), q] \) is written as \( \Gamma'_X[j, q] \). The following result follows immediate consequence from Theorem 19.

**Theorem 20.** Let \( \omega \in \Gamma'_X[\Lambda, q] \). Also, let the function \( j \) be analytic in \( \mathcal{U} \). If the function \( h \in \Sigma_p \) and \( z^p(X_p(l + 2, \mu)h(z)) \in Q_1 \) satisfy the condition (25) and

\[
\omega\left(z^p(X_p(l + 3, \mu)h(z)), z^p(X_p(l + 2, \mu)h(z)),
\right.
\]

\[
z^p(X_p(l + 1, \mu)h(z)), z^p(X_p(l, \mu)h(z)); z)
\]
is univalent in \( \mathcal{U} \), then

\[
j(z) \prec \omega\left(z^p(X_p(l + 3, \mu)h(z)), z^p(X_p(l + 2, \mu)h(z)),
\right.
\]

\[
z^p(X_p(l + 1, \mu)h(z)), z^p(X_p(l, \mu)h(z)); z)
\]

which implies that

\[
q(z) \prec z^p(X_p(l + 3, \mu)h(z)).
\]

The next result proves the existence of the best subordinant of (27) for appropriate chosen \( \omega \).

**Theorem 21.** Let the function \( j \) be analytic in \( \mathcal{U} \), and let \( \omega : C^4 \times \overline{\mathcal{U}} \to C \) and \( \tau \) be given by (20). Suppose that the differential equation

\[
\tau \left(q(z), zq'(z), z^2q''(z), z^3q'''(z); z\right) = j(z),
\]

has a solution \( q(z) \in Q_1 \). If the function \( h \in \Sigma_p \) and \( z^p(X_p(l + 2, \mu)h(z)) \in Q_1 \) satisfies condition (25) and

\[
\omega\left(z^p(X_p(l + 3, \mu)h(z)), z^p(X_p(l + 2, \mu)h(z)),
\right.
\]

\[
z^p(X_p(l + 1, \mu)h(z)), z^p(X_p(l, \mu)h(z)); z)
\]
are univalent in $\mathcal{U}$, then
\[ j(z) \prec \omega \left( z^p \left( X_p(l + 3, \mu)h(z) \right), z^p \left( X_p(l + 2, \mu)h(z) \right) \right), \]
\[ z^p \left( X_p(l + 1, \mu)h(z) \right), z^p \left( X_p(l, \mu)h(z) \right); z \]
which implies that
\[ q(z) \prec z^p \left( X_p(l + 3, \mu)h(z) \right). \]
and $q(z)$ is the best subordinant.

Proof. The proof of this result resembles that of Theorem 13 and it has been omitted here. \qed

Combining Theorem 11 and Theorem 20, we obtain the following sandwich-type result.

**Corollary 22.** Let the functions $j_1$ and $q_1$ be analytic functions in $\mathcal{U}$, $j_2$ be univalent function in $\mathcal{U}$, $q_2 \in \mathcal{Q}_1$ with $q_1(0) = q_2(0) = 1$ and $\omega \in \Gamma_X[j_1, q] \cap \Gamma'_X[j_2, q]$. If the function $h \in \Sigma_p$ and $z^p \left( X_p(l + 2, \mu)h(z) \right) \in \mathcal{Q}_1 \cap \mathcal{H}_1$ and
\[ \omega \left( z^p \left( X_p(l + 3, \mu)h(z) \right), z^p \left( X_p(l + 2, \mu)h(z) \right) \right), \]
\[ z^p \left( X_p(l + 1, \mu)h(z) \right), z^p \left( X_p(l, \mu)h(z) \right); z \]
are univalent in $\mathcal{U}$, and the conditions (11) and (25) are satisfied, then
\[ j_1(z) \prec \omega \left( z^p \left( X_p(l + 3, \mu)h(z) \right), z^p \left( X_p(l + 2, \mu)h(z) \right) \right), \]
\[ z^p \left( X_p(l + 1, \mu)h(z) \right), z^p \left( X_p(l, \mu)h(z) \right); z \prec j_2(z) \]
implicate that
\[ q_1(z) \prec z^p \left( X_p(l + 3, \mu)h(z) \right) \prec q_2(z). \]

5. Conclusion

In this paper, we have introduced third-order differential subordination and differential superordination for meromorphically multivalent functions in the punctured unit disk that are associated with a dual linear operator outcomes. These results are acquired by considering appropriate classes of admissible functions. Moreover, we have determined sufficient conditions to obtain the best dominant and the best subordinant, respectively. Furthermore, we have gained sandwich-type outcome.
References


