SOME RESULTS ON PRIME NUMBERS

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\textbf{Abstract:} In this article using the functions $f_1(k) = 10k + 1$, $k \neq 3 + 2$; $f_2(k) = 10k + 3$, $k \neq 3 - \{0\}$; $f_3(k) = 10k + 7$, $k \neq 3 + 2$, $k \neq 7 - \{0\}$; and $f_4(k) = 10k + 9$, $k \neq 3$, where

$k \in \mathbb{N}_0$, we obtain two important results on prime numbers. The first result indicates that if $p$ is a prime number that ends in 7, then $p + 10l$ will be a prime number under certain conditions. The second result states that if $k$ is a number ending in 7, then $k + 10$ will be also a prime number under certain conditions.

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\section{1. Introduction}

There are no general methods in the literature, to our knowledge, on how to generate a prime number starting from a given prime number.

We know of the existence of large prime numbers and that to determine the
primality of these numbers there are special algorithms, and that there are also prizes for those who show the largest prime number; encouraged by these facts, in this work, we obtain two important results on prime numbers.

The first result states that if \( p \) is a large number that ends in 7, then for \( l \in \mathbb{N} \), \( 10l + p \) is a prime number provided that

\[
\frac{p - 27}{90}, \frac{p - 27}{30}, \frac{p + 10l - 27}{90}, \frac{p + 10l - 27}{30}, \frac{p + 10l - 7}{70}, \frac{p - 7}{70},
\]

are not integers and the \( (x, y) \) natural numbers belonging to the intervals

\[
\left(\frac{p - 27}{30}, \frac{p + 10l - 27}{30}\right) \times \left(\frac{p - 27}{90}, \frac{p + 10l - 27}{90}\right),
\]

are not integer solutions of the diophantine equation:

\[
p + 10l = (10x + 9)(10y + 3)
\]

and, in addition, the \( (x, y) \) natural numbers belonging to the intervals

\[
\left(\frac{p - 7}{70}, \frac{p + 10l - 7}{70}\right) \times \left[\frac{p - 7}{10}, \frac{p + 10l - 7}{10}\right],
\]

are not solutions of the diophantine equation:

\[
p + 10l = (10x + 1)(10y + 7).
\]

The second result states that if \( k \) is a large number ending in 7, \( k \neq 3 \), \( \frac{k - 27}{30}, \frac{k - 27}{90} \notin \mathbb{N} \) and there exists an unique \( (a, b) \in \mathbb{N} \times \mathbb{N} \) such that

\[
k = (10a + 9)(10b + 3),
\]

then \( k + 10 \) is a prime number provided that the equation

\[
k + 10 = (10x + 1)(10y + 7), x \neq \frac{\circ}{3} + 2, y \neq \frac{\circ}{3} + 2, y \neq \frac{\circ}{7},
\]

does not have an integer solution.

In this work \( \mathbb{N} \) represents the set of natural numbers, \( \circ \) represents the set of multiples of \( n \), and \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \).

If \( (X, d) \) is a metric space, the distance between subsets of \( X \) is defined as:

\[
d(A, B) = \inf_{x \in A, y \in B} d(x, y).
\]
2. Some results on prime numbers

In this section we state the Lemma (2.1) and the Lemma (2.2), and then we obtain the results mentioned in the introduction.

**Lemma 2.1.** Let \( f_1, f_2, f_3, f_4 : \mathbb{N}_0 \to \mathbb{N} \) be functions defined by:

\[
\begin{align*}
  f_1(k) &= 10k + 1, \quad k \neq 3 + 2 \\
  f_2(k) &= 10k + 3, \quad k \neq 3 - \{0\} \\
  f_3(k) &= 10k + 7, \quad k \neq 3 + 2, k \neq 7 - \{0\} \\
  f_4(k) &= 10k + 9, \quad k \neq 3.
\end{align*}
\]

So, let \( p \) be a natural number ending in seven, and the following diophantine equations

\[
\begin{align*}
  p &= (10x + 1)(10y + 7); \quad x \neq 3 + 2, x \geq 1, y \neq 3 - \{0\}, y \neq 3 + 2 \\
  p &= (10z + 3)(10w + 9); \quad z \neq 3 - \{0\}, w \neq 3,
\end{align*}
\]

do not possess integer solutions, then \( p \) is a prime number.

**Proof.** Is immediate, see \cite{1}.

\[\square\]

**Lemma 2.2.** If \( p \) and \( l \) are fixed numbers,

\[
\begin{align*}
  C_1 &= \{(x, y) \in \mathbb{R}^2 : (10x + 9)(10y + 3) = p, x \geq 0, y \geq 0\}, \\
  C_2 &= \{(x, y) \in \mathbb{R}^2 : (10x + 9)(10y + 3) = p + 10l, x \geq 0, y \geq 0\}, \\
  C_3 &= \{(x, y) \in \mathbb{R}^2 : (10x + 1)(10y + 7) = p, x \geq 0, y \geq 0\}, \\
  C_4 &= \{(x, y) \in \mathbb{R}^2 : (10x + 1)(10y + 7) = p + 10l, x \geq 0, y \geq 0\},
\end{align*}
\]

subsets of \( \mathbb{R}^2 \) and \( d \) the euclidean metric, then

\[
\begin{align*}
  d(C_1, C_2) &= \inf_{x \in C_1} \inf_{y \in C_2} d(x, y) = \frac{\sqrt{2}}{10} \left( \sqrt{p + 10l} - \sqrt{p} \right), \\
  d(C_3, C_4) &= \inf_{x \in C_3} \inf_{y \in C_4} d(x, y) = \frac{\sqrt{2}}{10} \left( \sqrt{p + 10l} - \sqrt{p} \right).
\end{align*}
\]

**Proof.** Just use the Lagrange multipliers, see \cite{2}.

\[\square\]
Theorem 1. Let \( p \) be a large prime number that ends in 7, then \( p + 10l \) is a prime number, where \( l \in \mathbb{N} \), \( l \) is a fixed natural number, smaller than \( p \), provided that

\[
\frac{p - 27}{90}, \frac{p - 27}{30}, \frac{p + 10l - 27}{90}, \frac{p + 10l - 27}{30}, \frac{p + 10l - 7}{70}, \frac{p - 7}{70},
\]

are not integers, and the \((x, y)\) natural numbers belonging to intervals

\[
\left\langle \frac{p - 27}{30}, \frac{p + 10l - 27}{30} \right\rangle \times \left\langle \frac{p - 27}{90}, \frac{p + 10l - 27}{90} \right\rangle,
\]

are not solutions of the equation \( p + 10l = (10x + 9)(10y + 3) \), in addition, the \((x, y)\) natural numbers belonging to the intervals

\[
\left\langle \frac{p - 7}{70}, \frac{p + 10l - 7}{70} \right\rangle \times \left[ \frac{p - 7}{10}, \frac{p + 10l - 7}{10} \right],
\]

are not integer solutions of the equation \( p + 10l = (10x + 1)(10y + 7) \).

Proof. Suppose that \( p + 10l, l \geq 1, l \in \mathbb{N} \), is not a prime number. Then there exists \((A, B) \in \mathbb{N} \times \mathbb{N}\) such that the next two eqs. (1) or (2) will happen

\[
p + 10l = (10A + 9)(10B + 3) \quad (1)
\]

\[
p + 10l = (10A + 1)(10B + 7) \quad (2)
\]

If it happens to occur (1) we have that the \((x, y)\) natural numbers belonging to the intervals

\[
\left\langle \frac{p - 27}{30}, \frac{p + 10l - 27}{30} \right\rangle \times \left\langle \frac{p - 27}{90}, \frac{p + 10l - 27}{90} \right\rangle,
\]

are not integer solutions of the equation \( p + 10l = (10x + 9)(10y + 3) \), thus

\[
A \in \left\langle 0, \frac{p - 27}{30} \right\rangle, \quad B \in \left\langle 0, \frac{p - 27}{90} \right\rangle.
\]

The straight line \( y = B \) intersects the equation

\[
p = (10x + 9)(10y + 3),
\]

so, we have

\[
p = (10x + 9)(10B + 3). \quad (3)
\]
From relations (1), (3) and lemma (2.2) we have
\[ \frac{\sqrt{2}}{10} \left[ \sqrt{p + 10l} - \sqrt{p} \right] \leq A - x = \frac{l}{10B + 3}. \] (4)

Similar analysis shows us that
\[ \frac{\sqrt{2}}{10} \left[ \sqrt{p + 10l} - \sqrt{p} \right] \leq B - y = \frac{l}{10A + 9}. \] (5)

From the equation (1) we have two possibilities
\[ \sqrt{p + 10l} \leq 10A + 9 \] (6)
\[ \sqrt{p + 10l} \leq 10B + 3. \] (7)

If the relation (6) is true, then from this relation and (5) we have
\[ \frac{\sqrt{2}}{10} \left[ \sqrt{p + 10l} - \sqrt{p} \right] \leq B - y \leq \frac{l}{\sqrt{p + 10l}}. \] (8)

If the relation (7) is true, then from this relation and (4) we have
\[ \frac{\sqrt{2}}{10} \left[ \sqrt{p + 10l} - \sqrt{p} \right] \leq A - x \leq \frac{l}{\sqrt{p + 10l}}. \] (9)

Therefore we have that either (8) or (9) is true.
Since \( p \) is a large prime number, we have that \((A, B) \in \mathbb{N} \times \mathbb{N}\) would be a solution of the equation (3), which is false, since \( p \) is a prime number.
Similarly, if it happens to be the case (2) we will have
\[
A \in \left\langle 0, \frac{p - 7}{70} \right\rangle, \quad B \in \left\langle 0, \frac{p - 7}{10} \right\rangle,
\]
\[ \frac{\sqrt{2}}{10} \left( \sqrt{p + 10l} - \sqrt{p} \right) \leq A - x \leq \frac{l}{\sqrt{p + 10l}} \text{ or } \]
\[ \frac{\sqrt{2}}{10} \left( \sqrt{p + 10l} - \sqrt{p} \right) \leq B - y \leq \frac{l}{\sqrt{p + 10l}}. \]

For large \( p \) we have that \((A, B)\) would be a solution of the equation
\[ p = (10x + 1)(10y + 7) \]
which is false, since \( p \) is a prime number.
**Theorem 2.** Let $k$ be a large natural number ending in 7, $k \neq 3$. If \( \frac{k - 27}{30}, \frac{k - 27}{90}, \frac{k - 17}{90}, \frac{k - 17}{30} \notin \mathbb{N} \) and there exists an unique \((a, b) \in \mathbb{N} \times \mathbb{N}\) such that

$$ k = (10a + 9)(10b + 3), $$

then $k + 10$ is a prime number, provided that the equation

$$ k + 10 = (10x + 1)(10y + 7), \quad x \neq 3 + 2, \quad y \neq 3 + 2, \quad y \neq 7, $$

does not possess an integer solution.

**Proof.** Suppose $k + 10$ is not a prime number, so there exists \((A, B) \in \mathbb{N} \times \mathbb{N}\) such that

$$ k + 10 = (10A + 9)(10B + 3). \quad (10) $$

So one has

$$ \frac{k - 27}{30}, \frac{k - 27}{90}, \frac{k - 17}{90}, \frac{k - 17}{30} \notin \mathbb{N} \Rightarrow 0 < A < \frac{k - 27}{30}, 0 < B < \frac{k - 27}{90}. $$

The straight line $y = B$ intersects the equation

$$ k = (10x + 9)(10y + 3), $$

which implies that

$$ k = (10x + 9)(10B + 3). \quad (11) $$

From (10) and (11) we have

$$ A - x = \frac{1}{10B + 3}. \quad (12) $$

Similarly we have

$$ B - y = \frac{1}{10A + 9}. \quad (13) $$

Given that $\sqrt{k} \leq 10A + 9$ or $\sqrt{k} \leq 10B + 3$ then from (12) and lemma (2.2) or (13) and lemma (2.2) we have that

$$ \frac{\sqrt{2}}{10} \left[ \sqrt{k + 10} - \sqrt{k} \right] \leq A - x \leq \frac{1}{\sqrt{k}} \quad \text{or} \quad \frac{\sqrt{2}}{10} \left[ \sqrt{k + 10} - \sqrt{k} \right] \leq B - y \leq \frac{1}{\sqrt{k}}; $$

so, as $k$ is large we might conclude that \((A, B)\) is another solution of the equation

$$ k = (10x + 9)(10y + 3), $$

which is a contradiction, since there exists an unique solution by hypothesis. \qed
Theorem 3. Let $k$ be a large natural number, ending in 7, $k \neq 3$. If \[ \frac{k - 7}{70}, \frac{k + 3}{70} \notin \mathbb{N} \text{ and there exists an unique } (a, b) \in \mathbb{N} \times \mathbb{N} \text{ such that} \]

\[ k = (10a + 1)(10b + 7), a \geq 1, \]

then $k + 10$ is a prime number, provided that the equation

\[ k + 10 = (10x + 9)(10y + 3), x \neq 3, y \neq 3 - \{0\}, \]

does not possess an integer solution, and $y_0 = \frac{k - 7}{10}$ is not a solution of the equation

\[ k + 10 = (10x + 1)(10y + 7). \]

Proof. Analogous to the above. \qed

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References
